Infiltration from irrigation channels into soil with impermeable inclusions

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(Received 15 October 2004; revised 16 September 2005)

Abstract

We consider the effect of impermeable inclusions on the infiltration of water from irrigation channels in a homogeneous soil. An expression for the matric flux potential throughout the soil is obtained in terms of a boundary integral equation. A Green's function derived by Basha [*Water Resources Research*, 30:2105–2118, 1994] is suitable for numerical calculations for this class of problems. This Green's function is employed in the boundary integral equation to obtain numerical values for the matric flux potential for a soil with embedded

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See http://anziamj.austms.org.au/V46/CTAC2004/Lobo for this article, © Austral. Mathematical Soc. 2005. Published October 14, 2005. ISSN 1446-8735

impermeable inclusions of various shapes. The numerical results indicate how impermeable inclusions may be used to effectively direct the flow from irrigation channels to particular regions below the soil surface.

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1 Introduction

A number of researchers undertook analysis of steady infiltration into unsaturated soils. For example, Philip [5, 6] and Batu [3] solved steady infiltration problems from a point, line, strip, and disc sources. These authors considered infiltration through a uniform homogeneous soil.

The present study is concerned with the solution of a class of infiltration problems from one or more irrigation channels into a soil with impermeable inclusions (see Figure 1). We extend the work previously developed by Azis, Clements and Lobo [1] on the use of the boundary element method for steady infiltration from irrigation channels in a soil. Our aim is to determine how

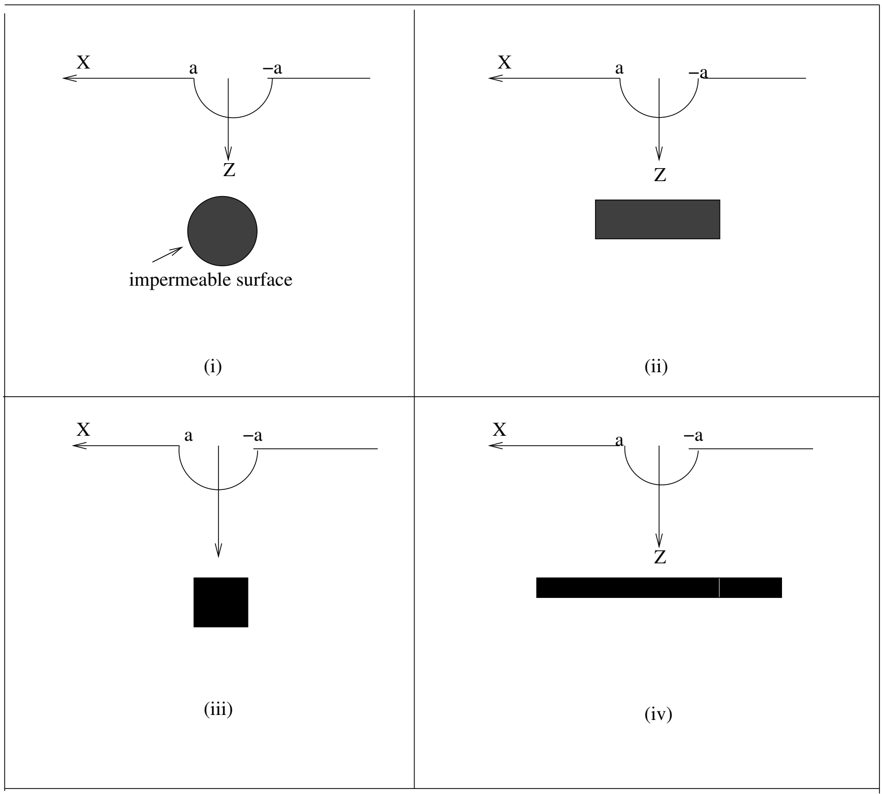

FIGURE 1: Illustration of the physical problem for a single channel.

impermeable inclusions below the soil surface influence the direction of the flow of water through the soil.

The governing equation we solve is the linearized form of the infiltration equation

$$\frac{\partial^2\Theta}{\partial X^2} + \frac{\partial^2\Theta}{\partial Z^2} = \alpha\frac{\partial\Theta}{\partial Z}\,, \tag{1}$$

(see for example Batu [3]) where Θ is the matric flux potential throughout the soil, α is an empirical constant that provides a measure of the relative significance of gravity and capillarity for water movement in the soil (see Philip [5]) and X and Z are Cartesian coordinates (see Figure 1).

A boundary integral equation formulation is used to facilitate the numerical solution of the governing differential equation and this is then used to determine the value of the matric flux potential Θ in a soil with impermeable inclusions of various shapes. The solutions obtained are relevant in assessing the influence of an impermeable inclusion in directing the flow from irrigation channels to particular regions below the soil surface.

2 Statement of the problem

Referred to a Cartesian frame $OXYZ$, consider an isotropic homogeneous soil lying in the region $Z > 0$ with OZ vertically downwards. The region contains one or two semi-circular channels and also impermeable inclusions (see Figures 1 and 2). The channels and inclusions are taken to be of infinite length in the OY direction. The channel has surface area $2L$ per unit length in the OY direction where L is a reference length and the channel is filled with water.

Impermeable inclusions of finite width and length are located in the soil in such a way that they do not intersect the boundary or other impermeable inclusions (see Figures 1 and 2). Each semi-circular channel has a radius of

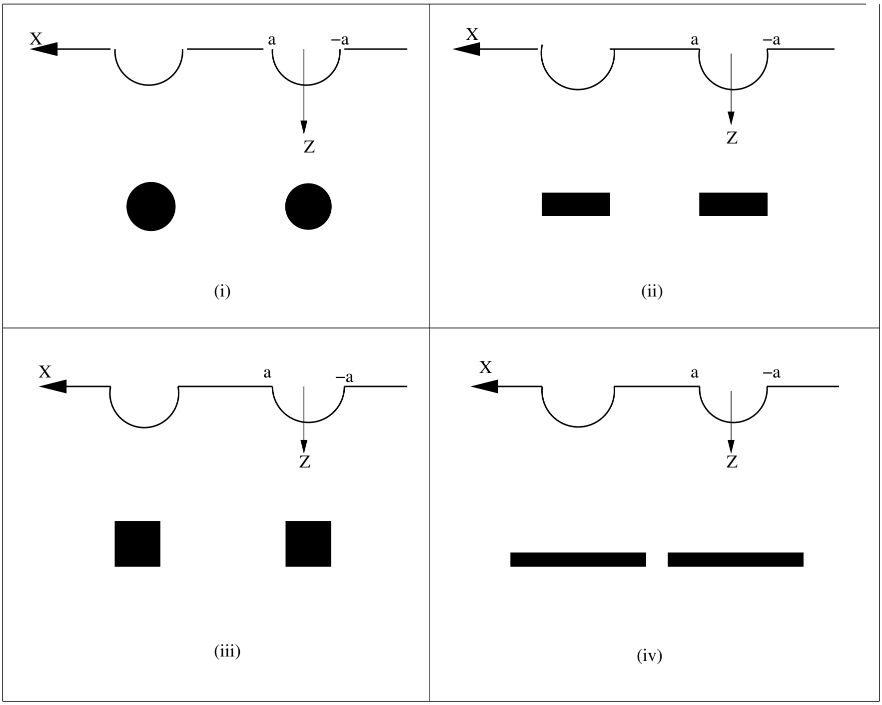

Figure 2: Illustration of the physical problem for two channels.

$a = 2L/\pi$. For the case of two channels (see Figure 2) the distance between the centre of the channels is taken to be $10L$. The inclusions are placed in the soil as shown in Figures 1 and 2 at a depth of $Z = \beta L$ where β is a dimensionless parameter.

The normal flow is taken to be zero over the surface boundary along $Z = 0$ outside the channel. Over the surface of the channel a uniform constant flow is specified normal to the surface of the channel.

We determine the matric flux potential $\Theta(X, Z)$ and the flow throughout the soil $Z > 0$ and observe the effect of the impermeable inclusion on this flux and flow. We assume that the matric flux potential Θ and the derivatives $\partial\Theta/\partial X$ and $\partial\Theta/\partial Z$ vanish as $X^2 + Z^2$ tends to infinity.

3 Fundamental equations

The matric flux potential Θ is taken to be related to the hydraulic conductivity by the equation (see Gardner [4])

$$\Theta = \int_{-\infty}^{h} K(q)\, dq = \alpha^{-1} K(h)\,, \tag{2}$$

with

$$K(h) = K_s \exp(\alpha h)\,, \tag{3}$$

where h (units L) is the soil water potential, α (units L^{-1}) is an empirical constant and K_s and $K(h)$ denote the hydraulic conductivities in saturated soil and unsaturated soil respectively.

Equation (1) is the linearized form of the steady infiltration equation, with the horizontal and vertical components of the flux

$$U = -\frac{\partial\Theta}{\partial X} \quad \text{and} \quad V = \alpha\Theta - \frac{\partial\Theta}{\partial Z}\,, \tag{4}$$

respectively. The flux normal to a surface with outward pointing unit normal $\mathbf{n} = (n_1, n_2)$ is

$$F = -\frac{\partial \Theta}{\partial X} n_1 + (\alpha\Theta - \frac{\partial \Theta}{\partial Z}) n_2 \,. \tag{5}$$

Dimensionless variables are now defined in the form

$$\begin{aligned} \theta &= \frac{1}{V_0 L}\Theta\,, \quad x = \frac{\alpha}{2}X\,, \quad z = \frac{\alpha}{2}Z\,, \\ u &= \frac{2}{V_0\alpha L}U\,, \quad v = \frac{2}{V_0\alpha L}V\,, \quad f = \frac{2}{V_0\alpha L}F\,, \end{aligned} \tag{6}$$

where V_0 is a reference flux. In terms of these variables equations (1), (4) and (5) may be written in the dimensionless form

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial z^2} = 2\frac{\partial\theta}{\partial z}\,, \tag{7}$$

$$u = -\frac{\partial\theta}{\partial x}\,, \quad v = 2\theta - \frac{\partial\theta}{\partial z}\,, \quad f = -\frac{\partial\theta}{\partial x}n_1 + (2\theta - \frac{\partial\theta}{\partial z})n_2\,. \tag{8}$$

The transformation

$$\theta = e^z\Psi \tag{9}$$

transforms equation (7) to

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial z^2} - \Psi = 0\,. \tag{10}$$

Also the equations (8) transform to

$$u = -e^z\frac{\partial\Psi}{\partial x}\,, \quad v = e^z(\Psi - \frac{\partial\Psi}{\partial z})\,, \quad f = -e^z\left[\frac{\partial\Psi}{\partial n} - \Psi n_2\right]\,, \tag{11}$$

where $\partial\Psi/\partial n = \partial\Psi/\partial x\; n_1 + \partial\Psi/\partial z\; n_2$. Hence

$$\frac{\partial\Psi}{\partial n} = \Psi n_2 - e^{-z} f\,. \tag{12}$$

There is no flow across the soil surface outside the channel so that on the soil surface on $z = 0$

$$\Psi - \frac{\partial \Psi}{\partial z} = 0\,. \tag{13}$$

Over the surface of the channel there is a specified normal flow $f = f_0(x, z)$. Hence

$$-\left[\frac{\partial \Psi}{\partial x} n_1 - (\Psi - \frac{\partial \Psi}{\partial z}) n_2\right] = e^{-z} f_0(x, z) \quad \text{for} \quad (x, z) \in \partial\Omega_1\,, \tag{14}$$

where $\partial\Omega_1$ denotes the boundary of the channel. The boundary condition along the impermeable inclusion is that there is zero flux normal to the boundary.

4 Boundary integral equation

The boundary integral equation for the solution to equation (10) is (see Aziz, Clements and Lobo [1])

$$\lambda \Psi(a, b) = -\int_{\partial\Omega} \left[\frac{\partial \Psi}{\partial n} \phi' - \frac{\partial \phi'}{\partial n} \Psi\right] dS, \tag{15}$$

where $\lambda = 1$ if $(a, b) \in \Omega$ and $\lambda = 1/2$ if $(a, b) \in \partial\Omega$ (the boundary of Ω) and $\partial\Omega$ has a continuously turning tangent. In the case of equation (10) the ϕ' in equation (15) is

$$\phi'(x, z; a, b) = -\frac{1}{2\pi} K_0(r)\,. \tag{16}$$

where $r = ((x - a)^2 + (z - b)^2)^{1/2}$ and K_0 is a modified Bessel function. Substitution of (12) into (15) gives

$$\lambda \Psi(a, b) = -\int_{\partial\Omega} \left[\phi' n_2 - \frac{\partial \phi'}{\partial n}\right] \Psi\, dS + \int_{\partial\Omega} f e^{-z} \phi'\, dS\,. \tag{17}$$

If the flux is zero across large sections of the soil surface on $z = 0$ then in place of the fundamental solution (16) it is convenient to use the Green's function derived by Basha [2]:

$$\begin{aligned}\phi'(x,z;a,b) &= -\frac{1}{2\pi}(K_0(r)+K_0(\bar r)) \\ &\quad + \frac{1}{\pi}e^z\int_z^\infty e^{-\mu}K_0\left(\left[(x-a)^2+(z+\mu)^2\right]^{1/2}\right)d\mu,\end{aligned} \qquad (18)$$

where $\bar r = ((x-a)^2 + (z+b)^2)^{1/2}$. With this choice of Green's function $\phi' - \partial\phi'/\partial z = 0$ on $z = 0$ so that equation (17) reduces to

$$\lambda\Psi(a,b) = -\int_{\partial\Omega_1}\left[\phi' n_2 - \frac{\partial\phi'}{\partial n}\right]\Psi\, dS + \int_{\partial\Omega_1} f e^{-z}\phi'\, dS\,. \qquad (19)$$

An alternative boundary integral equation formulation which directly relates the potential θ and the flux f may be obtained as follows. From equation (9)

$$\Psi(x,z) = e^{-z}\theta(x,z)\,. \qquad (20)$$

Now let $\bar\phi = e^{b-z}\phi'$, then

$$\frac{\partial\bar\phi}{\partial n} = \frac{\partial(e^{b-z}\phi')}{\partial n} = e^{b-z}\left(\frac{\partial\phi'}{\partial n} - \phi' n_2\right)\,. \qquad (21)$$

Substitution of equations (20) and (21) into (17) gives

$$\lambda\theta = \int_{d\Omega}\left(\frac{\partial\bar\phi}{\partial n}\theta + f\bar\phi\right)ds\,. \qquad (22)$$

Now $\partial\bar\phi/\partial n = 0$ on $z = 0$ and $f = 0$ on the surface of the inclusion so (22) may be written in the form

$$\lambda\theta = \int_{\partial\Omega_1}\left(\frac{\partial\bar\phi}{\partial n}\theta + f\bar\phi\right)ds + \int_D \frac{\partial\bar\phi}{\partial n}\theta\, ds\,, \qquad (23)$$

where D denotes the boundary of the impermeable inclusion.

5 Results and discussion

In this section some numerical values of the matric flux potential θ associated with infiltration from one or two semi-circular channels with inclusions configured as shown in Figures 1 and 2 are presented. The normal flux over the surface of the channel is chosen to be constant, $F = -V_0$. Hence when the reference length L is chosen to be 100 cm and the empirical constant $\alpha = 0.002\,\text{cm}^{-1}$ (see Philip [6]) then from equation (6) the dimensionless value of the normal flux over the channel surface is

$$f = \frac{2}{V_0 \alpha L} F = -\frac{2}{0.2} = -10 \tag{24}$$

and the dimensionless value of the radius of the semi-circular channel is

$$r = \frac{\alpha L}{\pi} = \frac{0.2}{\pi} = 0.0636\,. \tag{25}$$

The value of β is chosen to be 2.5 so that the dimensionless value of the depth z where the inclusion is placed is

$$z = \frac{\alpha Z}{2} = \frac{\alpha \beta L}{2} = 0.25\,. \tag{26}$$

The dimensionless value of the area of each of the inclusions is identical and is taken to be $\pi/100$. For the single channel case the inclusion is centered at $(x, z) = (0.0, 0.25)$, with the radius of the circle 0.1 and the sides of the square 0.1771, while the rectangle is 0.2 in length and 0.1570 in width. For the double channel case the distance between the centre of the inclusions is equal to the distance between the centre of the channels. The narrower rectangle is twice the length and half the width of the original rectangle.

The boundary integral equation (23) was used to compute the dimensionless values of θ along the line $x = 0.5$ (the line equidistant from the two

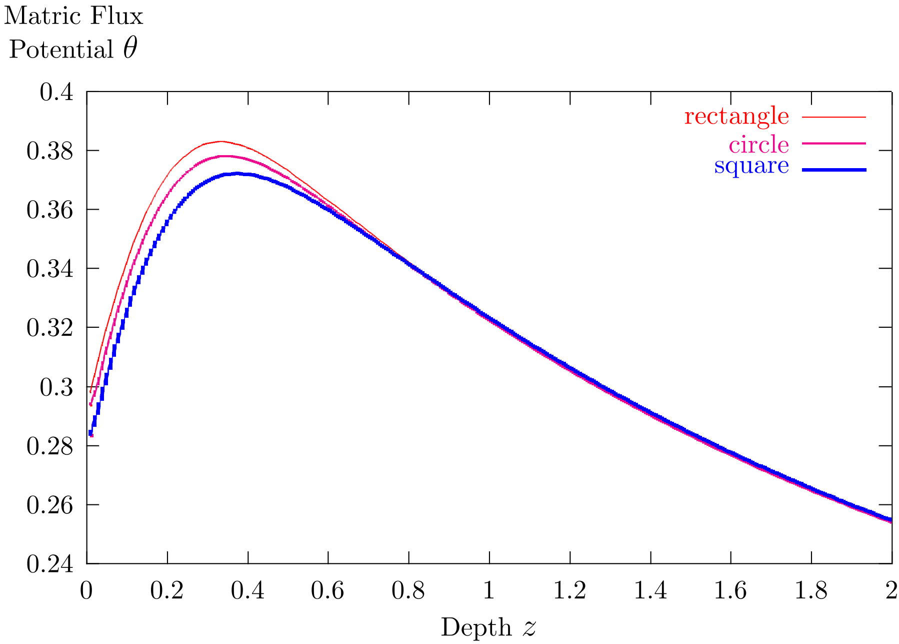

Figure 3: The value of the matric flux potential θ at $x = 0.5$ for a single semi-circular channel with impermeable inclusions.

channels in the two channel case). This particular line segment was chosen since a segment of it would pass through the region in which plant roots would be most likely to be evident. The channel surface was divided into N equally spaced segments and the boundary of the impermeable inclusions into M equally spaced segments. Standard boundary element techniques were then used to transform the integral equation (23) to a system of linear algebraic equations for the unknown function $\theta(a, b)$ and hence facilitate the calculation of θ both on the boundary and then at points along the line $x = 0.5$. The values of M and N was then doubled and the values of θ recalculated. This procedure was repeated until convergence of the values of θ to four decimal places was achieved with $N = 30$ and $M = 40$.

Figure 3 illustrates the dimensionless value of the matric flux potential θ as a function of dimensionless depth z for a single semi-circular channel with

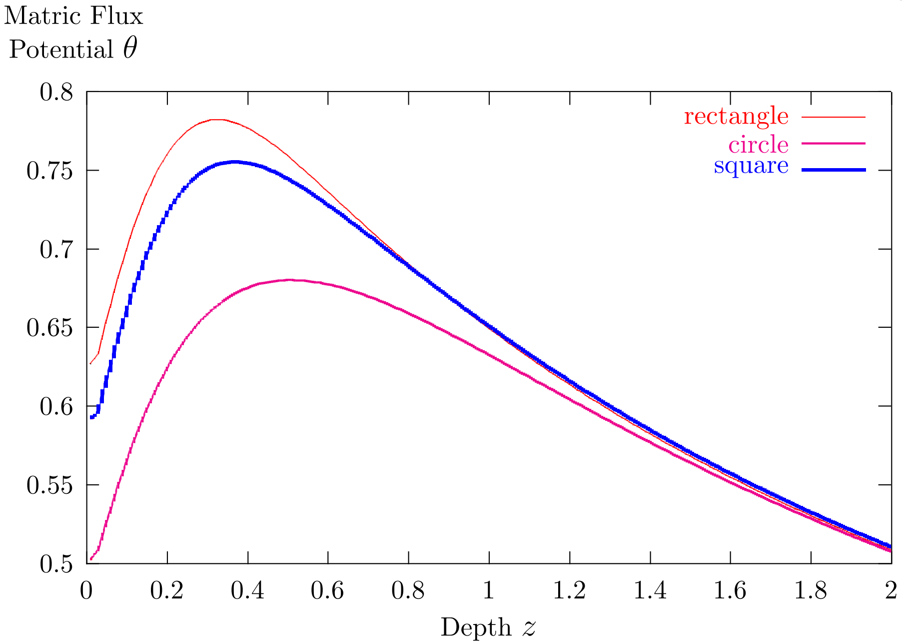

Figure 4: The value of the matric flux potential θ at $x = 0.5$ for double semi-circular channels with various shapes of impermeable inclusions.

one inclusion. The results indicate that at $x = 0.5$ the value of θ is not significantly affected by the different shapes of the impermeable inclusion of the same area and in each case is only marginally higher than the corresponding maximum value of $\theta = 0.372$ for a soil with no inclusions. However, the rectangular inclusion produces slightly higher values of θ compared to the other shapes, particularly in the region between the planes $z = 0.1$ and $z = 0.4$.

Figure 4 presents graphs of dimensionless values of θ for two semi-circular channels with various inclusions. From the graphs observe that the value of θ is approximately twice the value of θ for the single channel. This is to be expected since the flux from both channels has contributed to the increase in the value of θ at $x = 0.5$. The results also indicate that the rectangular inclusion produces the highest value of θ ($\theta = 0.7822$ at $z = 0.33$) followed by the square inclusion ($\theta = 0.7553$ at $z = 0.37$) and circular inclusion

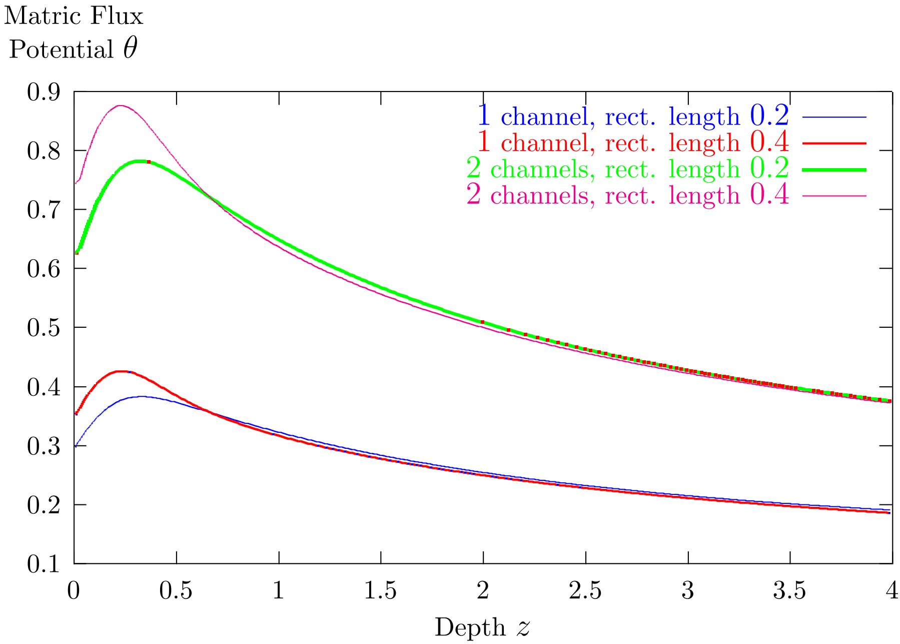

FIGURE 5: The value of the matric flux potential θ at $x = 0.5$ for single and double semi-circular channels with rectangular impermeable inclusion.

($\theta = 0.6802$ at $z = 0.50$).

Figure 5 describes the value of θ from single and double semi-circular channels with rectangular inclusions of the same area but different widths and lengths. The results suggest that an increase in the length of the rectangular inclusion will increase the value of the matric flux potential along the line $x = 0.5$.

The graphs for θ in Figure 4 and Figure 5 together with the equation for the vertical flux v in (11) provide information on the vertical flux through regions in which the plant roots will occur. Specifically the value of the flux along the line $x = 0.5$ increases from zero at $z = 0$ to a maximum value at the depth z at which $\theta(0.5, z)$ is a maximum ($v = \theta$ at this point since $\partial\theta/\partial z = 0$) and then steadily decreases as z increases.

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