A fractional-order implicit difference approximation for the space-time fractional diffusion equation

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(received 14 October 2005; revised 4 June 2006)

Abstract

We consider a space-time fractional diffusion equation on a finite domain. The equation is obtained from the standard diffusion equation by replacing the second order space derivative by a Riemann–Liouville fractional derivative of order between one and two, and the first order time derivative by a Caputo fractional derivative of order between zero and one. A fractional order implicit finite difference approximation for the space-time fractional diffusion equation with initial and boundary values is investigated. Stability and convergence

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See http://anziamj.austms.org.au/V47EMAC2005/Liu for this article, Published June 23, 2006; amended June 26, 2006. ISSN 1446-8735

results for the method are discussed, and finally, some numerical results show the system exhibits diffusive behaviour.

Contents

1 Introduction

Fractional order partial differential equations have recently found new applications in engineering, physics, finance and hydrology [16]. A physical/mathematical approach to anomalous diffusion [15] may be based on a generalized diffusion equation containing derivatives of fractional order in space or time or space-time. Such evolution equations implies for the flux a fractional Fick's law that accounts for spatial and temporal non-locality [5].

Space fractional diffusion equations were considered by West and Seshadri [19] and more recently by Gorenflo and Mainardi [2, 3]. Time fractional diffusion equations have recently been treated by a number of authors. Typically, the solution is given in closed form in terms of Fox functions [20]. Schneider and Wyss [17] considered the time fractional diffusion and wave equations and derived the corresponding Green's functions in closed form for arbitrary space dimensions in terms of Fox functions. Gorenflo et al. [4] used the similarity method and the method of Laplace transform to obtain the scale-invariant solution of the time-fractional diffusion-wave equation in terms of the Wright function. However, an explicit representation of the Green functions for the problem in a half-space is difficult to determine, except in the special cases $\alpha = 1$ (that is, the first order time derivative) with arbitrary n, or $n = 1$ with arbitrary α (that is, the fractional order time derivative). Huang and Liu [6] considered the time-fractional diffusion equations in an n dimensional whole-space and half-space. They investigated the explicit relationships between the problems in whole-space with the corresponding problems in half-space by the Fourier–Laplace transform. Liu et al. [8] considered a time fractional advection dispersion equation and derived the complete solution. Space-time fractional diffusion equations have been investigated by Mainardi et al. [13] and Gorenflo et al. [5]. In [13] the fundamental solution of the space-time fractional diffusion equation was discussed and in [5] a discrete random walk model for space-time fraction diffusion was proposed.

However, numerical methods and analysis of the fractional order partial differential equations are limited to date. Some different numerical methods for solving the space or time fractional partial differential equations have been proposed. Liu et al. [9, 10] transformed the space fractional partial differential equation into a system of ordinary differential equations (Method of Lines), which was then solved using backward differentiation formulas. Fix and Roop [1] developed a least squares finite element solution of a fractional order two-point boundary value problem. Meerschaert et al. [14] proposed finite difference approximations for fractional advection-dispersion flow equations. Shen et al. [18] proposed an explicit finite difference approximation for the space fractional diffusion equation and gave an error analysis. Liu et al. [12]

discussed an approximation of the Lévy–Feller advection-dispersion process by a random walk and finite difference method. Liu et al. [11] derived an analysis of a discrete non-Markovian random walk approximation for the time fractional diffusion equation. Zhuang and Liu [21] analyzed an implicit difference approximation for the time fractional diffusion equation, stability and convergence of the method were discussed. Lin and Liu [7] proposed the high order (2–6) approximations of the fractional ordinary differential equation (FODE) and discussed the consistency, convergence and stability of these fractional high order methods. However, numerical methods and error analysis for space-time fractional order diffusion equation are quite limited.

We propose a fractional order implicit difference approximation for the space-time fractional diffusion equation (STFDE) of the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D_x^\beta u(x,t), \quad 0 \le x \le L\,, \quad 0 < t \le T\,, \tag{1}$$

$$u(x,0) = f(x)\,, \quad 0 \le x \le L\,, \tag{2}$$

$$u(0,t) = u(L,t) = 0\,. \tag{3}$$

Here, the Riemann–Liouville fractional derivative of order β $(1 < \beta \le 2)$ is defined by

$$D_x^\beta u(x,t) = \begin{cases} \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_0^x \frac{u(\xi,t)d\xi}{(x-\xi)^{\beta-1}}\,, & 1 < \beta < 2\,, \\ \frac{\partial^2 u(x,t)}{\partial x^2}\,, & \beta = 2\,, \end{cases} \tag{4}$$

and the Caputo fractional derivative of order α $(0 < \alpha < 1)$ is defined by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\eta)}{\partial \eta} \frac{d\eta}{(t-\eta)^\alpha}\,, & 0 < \alpha < 1\,, \\ \frac{\partial u(x,t)}{\partial t}\,, & \alpha = 1\,. \end{cases} \tag{5}$$

When $\alpha = 1$ and $\beta = 2$ we recover in the limit the well-known diffusion equation (Markovian process),

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}\,.$$

In the case $\alpha < 1$, we must consider all previous time levels (non-Markovian process).

A fractional order implicit difference approximation (FOIDA) is presented in Section 2. Stability and convergence analyses of FOIDA are investigated in Sections 3 and 4, respectively. Finally, Section 5 presents some numerical results using a fractional order implicit difference approximation for the space-time fractional diffusion equation to show that the system exhibits diffusive behaviors.

2 A fractional order implicit difference approximation for STFDE

In this section, a fractional order implicit difference approximation for the space-time fractional diffusion equation (1)–(3) is proposed.

Define $t_k = k\tau$, $k = 0, 1, 2, \dots, n$, $x_i = ih$, $i = 0, 1, 2, \dots, m$, where $\tau = T/n$ and $h = L/m$ are the space and time steps, respectively. Let $u(x_i, t_k)$, $i = 1, 2, \dots, m-1$; $k = 1, 2, \dots, n$ be the exact solution of the fractional partial differential equations (1)–(3) at mesh point (x_i, t_k). Let u_i^k be the numerical approximation to $u(x_i, t_k)$.

In the differential equation (1), the time fractional derivative term is approximated by the following scheme:

$$\begin{aligned}
&\frac{\partial^\alpha u(x_i, t_{k+1})}{\partial t^\alpha} \\
&\approx \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{\tau} \int_{j\tau}^{(j+1)\tau} \frac{d\xi}{(t_{k+1} - \xi)^\alpha} \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{\tau} \int_{(k-j)\tau}^{(k-j+1)\tau} \frac{d\eta}{\eta^\alpha}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{u(x_i,t_{k+1-j}) - u(x_i,t_{k-j})}{\tau} \int_{j\tau}^{(j+1)\tau} \frac{d\eta}{\eta^\alpha} \\
&= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k} \frac{u(x_i,t_{k+1-j}) - u(x_i,t_{k-j})}{\tau} [(j+1)^{1-\alpha} - j^{1-\alpha}] \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} [u(x_i,t_{k+1}) - u(x_i,t_k)] \\
&\quad + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k} [u(x_i,t_{k+1-j}) - u(x_i,t_{k-j})][(j+1)^{1-\alpha} - j^{1-\alpha}].
\end{aligned}
$$

Now, let $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$, $j = 0, 1, 2, \ldots, n$, and define

$$
L^\alpha_{h,\tau} u(x_i,t_{k+1}) = \frac{\tau^{-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \sum_{j=0}^{k} b_j [u(x_i,t_{k+1-j}) - u(x_i,t_{k-j})]. \tag{6}
$$

Then we have

$$
\left| \frac{\partial^\alpha u(x_i,t_{k+1})}{\partial t^\alpha} - L^\alpha_{h,\tau} u(x_i,t_{k+1}) \right| \le C_1 \tau \int_0^{t_{k+1}} \frac{d\xi}{(t_{k+1}-\xi)^\alpha} \le C\tau\,, \tag{7}
$$

where C_1 and C are constants.

For every β $(0 \le n-1 < \beta < n)$ the Riemann–Liouville derivative exists and coincides with the Grünwald–Letnikov derivative. The relationship between the Riemann–Liouville and Grünwald–Letnikov definitions also has another consequence which is important for the numerical approximation of fractional order differential equations, formulation of applied problems, manipulation with fractional derivatives and formulation of physically meaningful initial and boundary value problems for fractional order differential equations. This allows the use of the Riemann–Liouville definitions during problem formulation, and then the Grünwald–Letnikov definitions for obtaining the numerical solution.

For $D_x^\beta u(x,t)$, we adopt the shifted Grünwald formula at all time levels for approximating the second order space derivative [14]:

$$D_x^\beta u(x_i,t_{k+1}) = \frac{1}{h^\beta}\sum_{j=0}^{i+1} g_j u(x_i-(j-1)h,t_{k+1}) + \mathcal{O}(h). \tag{8}$$

Here the normalized Grünwald weights are defined by

$$g_0 = 1 \quad \text{and} \quad g_j = (-1)^j \frac{(\beta)(\beta-1)\cdots(\beta-j+1)}{j!}\,; \quad j=1,2,3,\ldots\,. \tag{9}$$

Thus, we have

$$u_i^{k+1} - u_i^k + \sum_{j=1}^{k} b_j(u_i^{k+1-j} - u_i^{k-j}) = \mu\Gamma(2-\alpha)\sum_{j=0}^{i+1} g_j u_{i+1-j}^{k+1}\,, \tag{10}$$

for $i=1,2,\ldots,m-1\,,\ k=0,1,2,\ldots,n-1\,.$ Let $\mu=\tau^\alpha/h^\beta$ and $r=\mu\Gamma(2-\alpha)\,.$ The resulting equation is written as

$$u_i^{k+1} = u_i^k - \sum_{j=1}^{k} b_j(u_i^{k+1-j} - u_i^{k-j}) + r\sum_{j=0}^{i+1} g_j u_{i-j+1}^{k+1}\,, \tag{11}$$

that is,

$$u_i^1 = u_i^0 - \beta r u_i^1 + r\sum_{j=0,i\neq 1}^{i+1} g_j u_{i-j+1}^1\,,$$
$$u_i^{k+1} = (1-b_1)u_i^k + r\sum_{j=0}^{i+1} g_j u_{i-j+1}^{k+1} + b_k u_i^0 + \sum_{j=1}^{k-1}(b_j-b_{j+1})u_i^{k-j}\,,$$

where $i=1,2,\ldots,m-1\,;\ k=1,2,\ldots,n-1\,.$ Further, we obtain the following fractional order implicit difference approximation (FOIDA) for STFDE (1):

$$(1+\beta r)u_i^1 - r\sum_{j=0,j\neq 1}^{i+1} g_j u_{i-j+1}^1 = u_i^0\,,$$

$$(1+\beta r)u_i^{k+1} - r\sum_{j=0,j\neq 1}^{i+1} g_j u_{i-j+1}^{k+1} = (1-b_1)u_i^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})u_i^{k-j} + b_k u_i^0 \,. \tag{12}$$

The above equation is expressed in matrix form:

$$\left\{\begin{array}{rcl} A\mathbf{u}^1 &=& \mathbf{u}^0\,, \\ A\mathbf{u}^{k+1} &=& (1-b_1)\mathbf{u}^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})\mathbf{u}^{k-j} + b_k\mathbf{u}^0\,, \quad k>1\,, \\ \mathbf{u}^0 &=& \mathbf{f}\,, \end{array}\right. \tag{13}$$

where $A=[A_{i,j}]$ is the matrix of coefficients. These coefficients, for $i=1,2,\ldots,m-1$ and $j=1,2,\ldots,m-1$ are

$$A_{i,j} = \begin{cases} 0\,, & \text{when } j\geq i+1\,, \\ 1+\beta r\,, & \text{when } j=i\,, \\ -rg_{i-j+1}\,, & \text{otherwise,} \end{cases} \tag{14}$$

and $\mathbf{u}^k = \left[u_1^k, u_2^k, \ldots, u_{m-1}^k\right]^T$, $k=1,2,\ldots$; $\mathbf{f}=[f(x_1), f(x_2), \ldots, f(x_{m-1})]^T$.

3 Stability analysis of FOIDA for STFDE

In this section, the stability analysis of the fractional order implicit difference approximation is studied.

From [14], we can prove the following lemma:

Lemma 1 *In (12), the coefficients b_k $(k=0,1,2,\ldots)$ and g_j $(j=0,1,2,\ldots)$ satisfy:*

1. $b_j > b_{j+1}\,,\ j=0,1,2,\ldots;$

2. $b_0 = 1$, $b_j > 0$, $j = 0, 1, 2, \ldots$;
3. $g_1 = -\beta$, $g_j \geq 0$ $(j \neq 1)$, $\sum_{j=0}^{\infty} g_j = 0$;
4. *For any positive integer* n, *we have* $\sum_{j=0}^{n} g_j < 0$.

We suppose that $\widetilde{u}_i^j$, $i = 0, 1, 2, \ldots, m$; $j = 0, 1, 2, \ldots, n$ is the approximate solution of (12), the error $\varepsilon_i^j = \widetilde{u}_i^j - u_i^j$, $i = 0, 1, 2, \ldots, m$; $j = 0, 1, 2, \ldots, n$ satisfies

$$
\begin{aligned}
(1+\beta r)\varepsilon_i^1 - r \sum_{j=0, j\neq 1}^{i+1} g_j \varepsilon_{i-j+1}^1 &= \varepsilon_i^0 , \\
(1+\beta r)\varepsilon_i^{k+1} - r \sum_{j=0, j\neq 1}^{i+1} g_j \varepsilon_{i-j+1}^{k+1} & \\
= (1-b_1)\varepsilon_i^k + \sum_{j=1}^{k-1} (b_j - b_{j+1}) \varepsilon_i^{k-j} , &
\end{aligned}
\tag{15}
$$

where $i = 1, 2, \ldots, m-1$; $k = 1, 2, \ldots, n-1$. The above formula can be written in matrix form:

$$
\begin{cases}
A\mathbf{E}^1 = \mathbf{E}^0 , \\
A\mathbf{E}^{k+1} = (1-b_1)\mathbf{E}^k + \sum_{j=1}^{k-1} (b_j - b_{j+1}) \mathbf{E}^{k-j} + b_k \mathbf{E}^0 , \quad k > 1 , \\
\mathbf{E}^0 = \mathbf{0} ,
\end{cases}
\tag{16}
$$

where $\mathbf{E}^k = \begin{bmatrix} \varepsilon_1^k & \varepsilon_2^k & \cdots & \varepsilon_{m-1}^k \end{bmatrix}^T$.

We now analyze the stability via mathematical induction.

Let $\|\mathbf{E}^1\|_\infty = |\varepsilon_l^1| = \max_{1\leq i \leq m-1} |\varepsilon_i^1|$.

When $k = 1$, note that $\sum_{j=0}^{l+1} g_j = \beta + \sum_{j=0,j\neq 1}^{l+1} g_j \leq 0$ and $g_j > 0$ $(j \neq 1)$, we have

$$\begin{aligned}
\|\mathbf{E}^1\|_\infty = |\varepsilon_l^1| &\leq (1+\beta r)|\varepsilon_l^1| - r\sum_{j=1,j\neq 1}^{l+1} g_j|\varepsilon_l^1| \\
&\leq (1+\beta r)|\varepsilon_l^1| - r\sum_{j=1,j\neq 1}^{l+1} g_j|\varepsilon_{l-j+1}^1| \\
&\leq |(1+\beta r)\varepsilon_l^1 - r\sum_{j=1,j\neq 1}^{l+1} g_j\varepsilon_{l-j+1}^1| \\
&= |\varepsilon_l^0| \\
&\leq \|E^0\|_\infty\,.
\end{aligned}$$

Let $\|\mathbf{E}^{k+1}\|_\infty = |\varepsilon_l^{k+1}| = \max_{1\leq i\leq m-1}|\varepsilon_i^{k+1}|$, and assume that $\|\mathbf{E}^j\|_\infty \leq \|\mathbf{E}^0\|_\infty$, $j = 1, 2, \ldots, k$ and using the Lemma 1, we also have

$$\begin{aligned}
\|\mathbf{E}^{k+1}\|_\infty = |\varepsilon_l^{k+1}| &\leq (1+\beta r)|\varepsilon_l^{k+1}| - r\sum_{j=0,j\neq 1}^{l+1} g_j|\varepsilon_l^{k+1}| \\
&\leq (1+\beta r)|\varepsilon_l^{k+1}| - r\sum_{j=0,j\neq 1}^{l+1} g_j|\varepsilon_{l-j+1}^{k+1}| \\
&\leq |(1+\beta r)\varepsilon_l^{k+1} - r\sum_{j=0,j\neq 1}^{l+1} g_j\varepsilon_{l-j+1}^{k+1}| \\
&= |(1-b_1)\varepsilon_l^k + b_k\varepsilon_l^0 + \sum_{j=1}^{k-1}(b_j - b_{j+1})\varepsilon_l^{k-j}| \\
&\leq (1-b_1)\|\mathbf{E}^k\|_\infty + b_k\|\mathbf{E}^0\|_\infty + \sum_{j=1}^{k-1}(b_j - b_{j+1})\|\mathbf{E}^{k-j}\|_\infty \\
&\leq (1-b_1)\|\mathbf{E}^0\|_\infty + b_k\|\mathbf{E}^0\|_\infty + \sum_{j=1}^{k-1}(b_j - b_{j+1})\|\mathbf{E}^0\|_\infty
\end{aligned}$$

$$= \quad \|\mathbf{E}^0\|_\infty \,.$$

Hence, the following theorem holds.

Theorem 2 *The fractional implicit difference defined by (12) is unconditionally stable.*

4 Convergence analysis of FOIDA for STFDE

In this section, the convergence analysis of FOIDA is discussed.

Let u_i^k, $i = 1,2,\ldots,m-1\,;\ k = 1,2,\ldots,n$ be the numerical solution (FOIDA) of the fractional partial differential equations (1)–(3) at mesh point (x_i,t_k). Define $e_i^k = u(x_i,t_k) - u_i^k\,,\ i = 1,2,\ldots,m-1\,;\ k = 1,2,\ldots,n$ and $\mathbf{e}^k = (e_1^k, e_2^k, \ldots, e_{m-1}^k)^T$. Using $\mathbf{e}^0 = 0$ and $u_i^k = u(x_i,t_k) - e_i^k$, substitution into (12) leads to

$$(1+\beta r)e_i^1 - r\sum_{j=0,j\neq1}^{i+1} g_j e_{i-j+1}^1 = R_i^1\,,$$

$$(1+\beta r)e_i^{k+1} - r\sum_{j=0,j\neq1}^{i+1} g_j e_{i-j+1}^{k+1}$$
$$= (1-b_1)e_i^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})e_i^{k-j} + R_i^{k+1}\,,$$

where $i = 1,2,\ldots,m-1\,;\ k = 1,2,\ldots,n-1\,.$ Also, we have

$$|R_i^k| \le C(\tau^{1+\alpha} + \tau^\alpha h), \quad i = 1,2,\ldots,m-1\,; \quad k = 1,2,\ldots,n.$$

Using mathematical induction and Lemma 1, we give the convergence analysis as follows: For $k = 1$, let $\|\mathbf{e}^1\|_\infty = |e_l^1| = \max_{1\le i\le m-1} |e_i^1|$, we have

$$\begin{aligned} |e_l^1| &\le (1+\beta r)|e_l^1| - r\sum_{j=0,j\ne 1}^{l+1} g_j|e_l^1| \\ &\le (1+\beta r)|e_l^1| - r\sum_{j=0,j\ne 1}^{l+1} g_j|e_{l-j+1}^1| \\ &\le |(1+\beta r)e_l^1 - r\sum_{j=0,j\ne 1}^{l+1} g_j e_l^1 \\ &= |e_l^0 + R_l^1|\,. \end{aligned} \tag{17}$$

Using $\mathbf{e}^0 = \mathbf{0}$ and $|R_l^1| \le C(\tau^{1+\alpha} + \tau^\alpha h)$, we obtain

$$\|\mathbf{e}^1\|_\infty \le C(\tau^{1+\alpha} + \tau^\alpha h)\,.$$

Suppose that $\|\mathbf{e}^j\|_\infty \le C b_{j-1}^{-1}(\tau^{1+\alpha} + \tau^\alpha h^2)$, $j = 1, 2, \ldots, k$, and $|e_l^{k+1}| = \max_{1\le i\le m-1} |e_i^{k+1}|$. Note that $b_j^{-1} \le b_k^{-1}$, $j = 0, 1, \ldots, k$, we have

$$\begin{aligned} |e_l^{k+1}| &\le (1+\beta r)|e_l^{k+1}| - r\sum_{j=0,j\ne 1}^{l+1} g_j|e_l^{k+1}| \\ &\le (1+\beta r)|e_l^{k+1}| - r\sum_{j=0,j\ne 1}^{l+1} g_j|e_{l-j+1}^{k+1}| \\ &\le |(1+\beta r)e_l^{k+1} - r\sum_{j=0,j\ne 1}^{l+1} g_j e_{l-j+1}^{k+1}| \\ &= |(1-b_1)e_l^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})e_l^{k-j} + R_l^{k+1}| \\ &\le (1-b_1)\|\mathbf{e}^k\|_\infty + \sum_{j=1}^{k-1}(b_j - b_{j+1})\|\mathbf{e}^{k-j}\|_\infty + |R_l^{k+1}| \end{aligned}$$

$$\leq \left\{(1-b_1)b_{k-1}^{-1} + \sum_{j=1}^{k-1}(b_j - b_{j+1})b_{k-j-1}^{-1}\right\} C(\tau^{1+\alpha} + \tau^\alpha h) + |R_l^{k+1}|.$$

Using $b_j^{-1} \leq b_k^{-1}$, $j = 0, 1, \ldots, k$ and $|R_l^{k+1}| \leq C(\tau^{1+\alpha} + \tau^\alpha h)$, we obtain

$$\begin{aligned} \|\mathbf{e}^{k+1}\|_\infty &\leq b_k^{-1}\left\{1 - b_1 + \sum_{j=1}^{k-1}(b_j - b_{j+1}) + b_k\right\} C(\tau^{1+\alpha} + \tau^\alpha h) \\ &= b_k^{-1} C(\tau^{1+\alpha} + \tau^\alpha h) . \end{aligned}$$

Because

$$\begin{aligned} \lim_{k\to\infty} \frac{b_k^{-1}}{k^\alpha} &= \lim_{k\to\infty} \frac{k^{-\alpha}}{(k+1)^{1-\alpha} - k^{1-\alpha}} \\ &= \lim_{k\to\infty} \frac{k^{-1}}{(1+\frac{1}{k})^{1-\alpha} - 1} \\ &= \lim_{k\to\infty} \frac{k^{-1}}{(1-\alpha)k^{-1}} \\ &= \frac{1}{1-\alpha} . \end{aligned} \tag{18}$$

Hence, there is a constant C,

$$\|\mathbf{e}^k\|_\infty \leq C k^\alpha (\tau^{1+\alpha} + \tau^\alpha h) .$$

If $k\tau \leq T$ is finite, the convergence of FOIDA is given by the following theorem.

Theorem 3 *Let u_i^k be the approximate value of $u(x_i, t_k)$ computed by using* FOIDA *(12). Then there is a positive constant C, such that*

$$|u_i^k - u(x_i, t_k)| \leq C(\tau + h) , \quad i = 1, 2, \ldots, m-1 ; \quad k = 1, 2, \ldots, n . \tag{19}$$

5 Numerical results

To demonstrate the effectiveness of the implicit difference approximation for solving the space-time fractional diffusion equation, consider the equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D_x^\beta u(x,t)\,, \quad 0 \le x \le 2\,, \quad t > 0\,, \tag{20}$$

with boundary conditions $u(0,t) = u(2,t) = 0$ and initial condition

$$u(x,0) = f(x) = \begin{cases} 2x\,, & 0 \le x \le \frac{1}{2}\,, \\ \frac{4-2x}{3}\,, & \frac{1}{2} \le x \le 2\,. \end{cases} \tag{21}$$

The function $f(x)$ represents the temperature distribution in a bar generated by a point heat source maintained at $x = \frac{1}{2}$ for long enough.

The evolution results for the FOIDA when $t = 0.4$, $\alpha = 0.5$, $0 \le x \le 2$, $1 < \beta \le 2$ and $t = 0.4$, $\beta = 1.5$, $0 \le x \le 2$, $0 < \alpha < 1$ are shown in Figures 1 and 2, respectively. The evolution results for the FOIDA when $x = 1.5$, $\alpha = 0.5$, $0 \le t \le 1$, $1 < \beta \le 2$ and $x = 1.5$, $\beta = 1.5$, $0 \le t \le 1$, $0 < \alpha < 1$ are shown in Figures 3 and 4, respectively. Figures 1–4 show the system exhibits diffusive behaviors. From Figures 1–4, conclude that the solution continuously depends on the space-time fractional derivatives.

6 Conclusions

In this paper, we propose a fractional order implicit difference approximation for the space-time fractional diffusion equation in a bounded domain. We have proved that the fractional order implicit difference approximation is unconditionally stable and convergent. The proposed method and analysis can be applied to solve and analyze other kinds of fractional order partial differential equations.

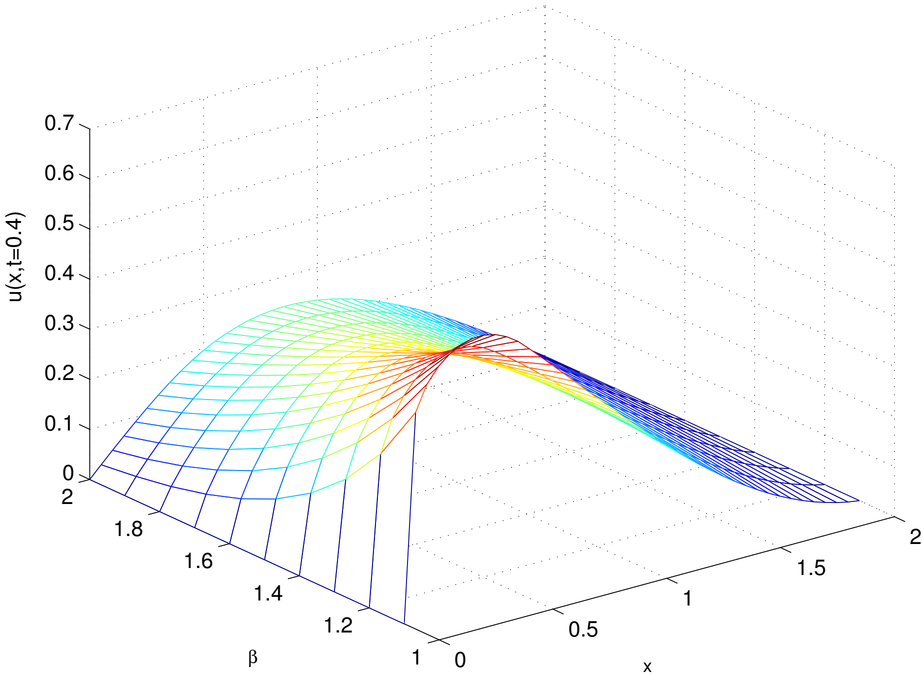

Figure 1: The numerical approximation of $u(x, t)$ when $\alpha = 0.5$ and $t = 0.4$.

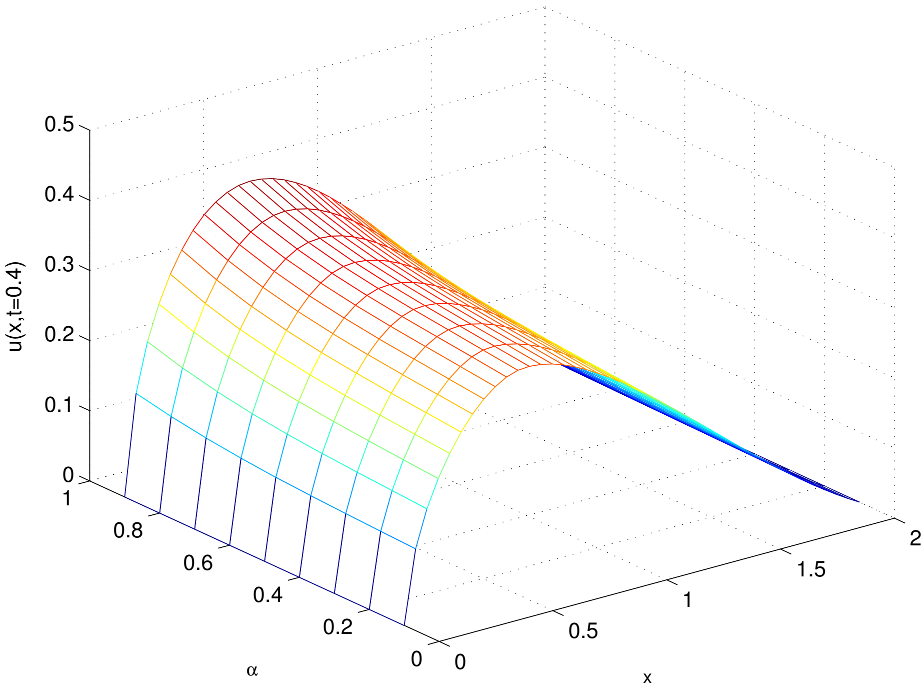

FIGURE 2: The numerical approximation of $u(x,t)$ when $\beta = 1.5$ and $t = 0.4$.

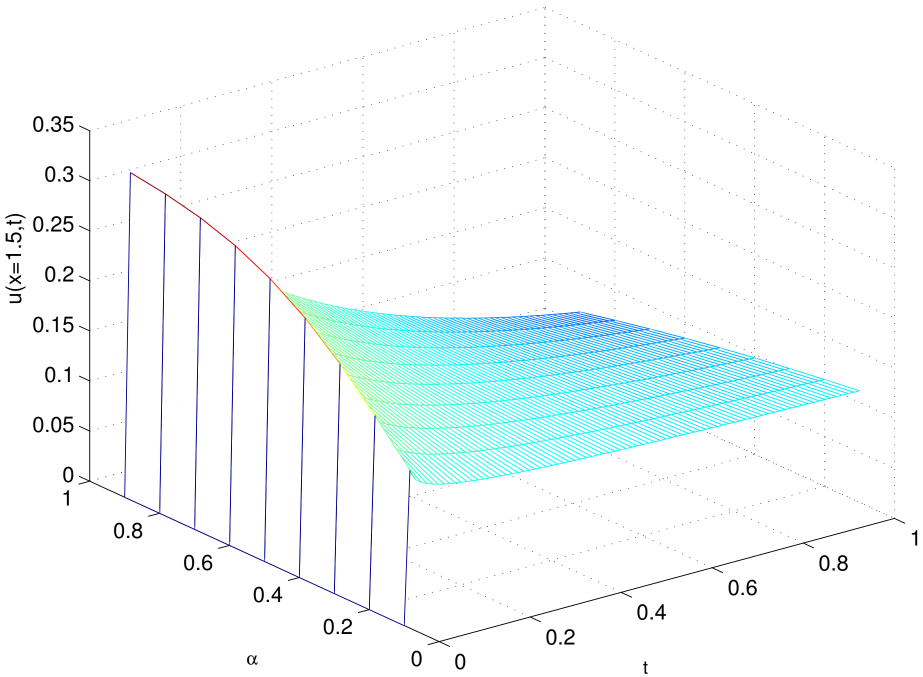

Figure 3: The numerical approximation of $u(x,t)$ when $\alpha = 0.5$ and $x = 1.5$.

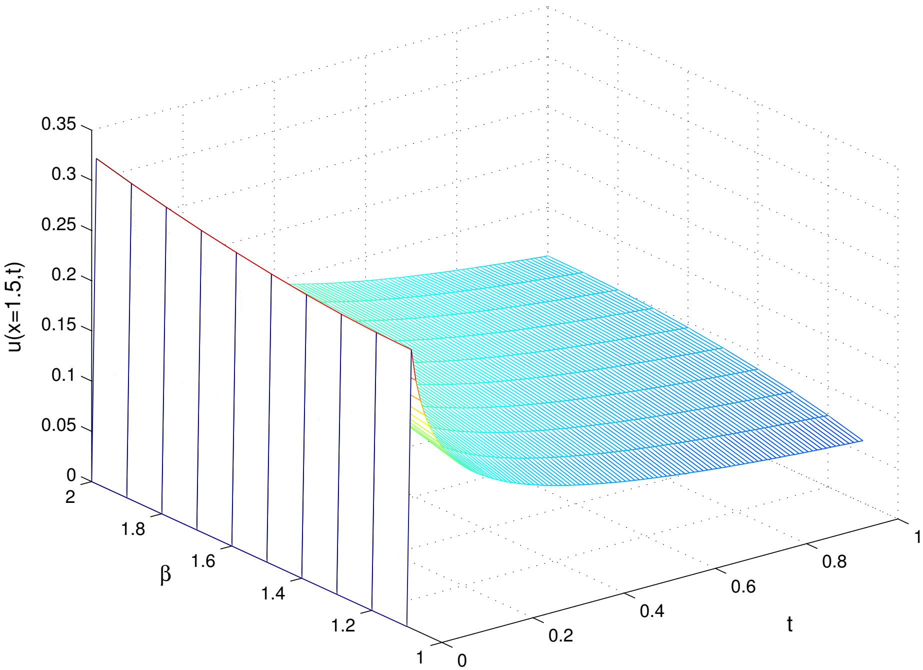

Figure 4: The numerical approximation of $u(x,t)$ when $\beta = 1.5$ and $x = 1.5$.

Acknowledgments: This research was supported by the National Natural Science Foundation of China grant 10271098, Natural Science Foundation of Fujian province grant (Z0511009) and the Australian Research Council grant LP0348653.

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