Transmuted exponentiated Chen distribution with application to survival data

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Abstract

This article considers an extension of the exponentiated Chen distribution based on the quadratic rank transmutation map technique. Some structural properties of the transmuted exponentiated Chen distribution are discussed. The estimation procedure is performed using the method of maximum likelihood. Finally, the flexibility of the new distribution is illustrated using strengths of glass fibres data and nicotine in cigarettes data.

Contents

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1 Introduction

In life testing theory, many different families of lifetime distributions have been proposed for modelling lifetime data and phenomenon with monotone failure rates. Due to randomness in real world scenarios, the behaviour of instantaneous failure rates varies from one mechanism to another, depending upon the nature of the component or device. In these real world scenarios, the probabilistic modelling approach plays an important role in explaining the failure mechanisms of a system or process.

Recently, Chaubey and Zhang [3] introduced and studied the exponentiated Chen distribution which extended the two parameter Chen lifetime distribution proposed by Chen [2]. The cumulative distribution function (cdf) of the exponentiated Chen distribution is

$$G(x) = \left\{1 - \exp\left[\alpha\left(1 - e^{x^{\beta}}\right)\right]\right\}^{\theta}, \quad x > 0\,, \tag{1}$$

where $\alpha > 0$ is the scale parameter, and $\beta > 0$ and $\theta > 0$ are the shape parameters of the exponentiated Chen distribution. The corresponding probability density function (pdf) is

$$f(x) = \alpha\beta\theta x^{\beta-1} \exp\left[x^{\beta} + \alpha\left(1 - e^{x^{\beta}}\right)\right] \left\{1 - \exp\left[\alpha\left(1 - e^{x^{\beta}}\right)\right]\right\}^{\theta-1}, \quad (2)$$

For more flexibility, Shaw et al. [10] developed the quadratic rank transmutation map technique, which is a method for adding a new parameter to the baseline distribution. Using this approach, we introduce the four-parameter transmuted exponentiated Chen (TEC) distribution which provides flexibility in modelling lifetime data, such as in reliability engineering, biomedical sciences, and for life testing components or process. The transmuting approach provides a rich family of increasing and decreasing bathtub shaped hazard functions. Section 2 presents the analytical shapes of the probability density and hazard functions of the TEC distribution. Section 3 considers some structural properties, such as moments and probability weighted moments. When using the transmutation technique, a random variable X is said to have a transmuted distribution if its cdf is

$$F(x) = (1+\lambda)G(x) - \lambda G(x)^2, \quad |\lambda| \leqslant 1, \quad (3)$$

and its pdf is

$$f(x) = g(x)[1 + \lambda - 2\lambda G(x)], \quad (4)$$

where λ is the transmuted parameter, and $G(x)$ is the cdf of the base distribution which has pdf $g(x)$. At $\lambda = 0$, $F(x)$ reduces to $G(x)$.

Various different distributions have been used to develop the transmuted distribution. Aryal and Tsokos [1] studied the transmuted Weibull distribution to analyse reliability data. More recently, Khan et al. [7] proposed the transmuted Chen distribution and investigated various structural properties and their applications. Khan et al. [4, 5, 6] proposed the transmuted modified Weibull distribution and the transmuted inverse Weibull distribution, and studied some applications. Merovci [8] proposed and studied the transmuted

Rayleigh distribution, among several other distributions, using the quadratic rank transmutation map technique. For the TEC, Section 4 discusses maximum likelihood estimates (MLE) of the unknown parameters while Section 5 presents results of our simulation study. Applications to real data sets are illustrated in Section 6.

2 Transmuted exponentiated Chen distribution

A random variable X has the TEC distribution, defined through the quadratic rank transmutation map technique [10], with parameters $\alpha, \beta, \theta > 0$ and $|\lambda| \leqslant 1$, and pdf

$$f(x) = \alpha\beta\theta x^{\beta-1} \exp\left[x^{\beta} + \alpha\left(1 - e^{x^{\beta}}\right)\right] \times \left\{1 - \exp\left[\alpha\left(1 - e^{x^{\beta}}\right)\right]\right\}^{\theta-1} u_2(x)\,, \tag{5}$$

$$u_g(x) = 1 + \lambda - g\lambda\left\{1 - \exp\left[\alpha\left(1 - e^{x^{\beta}}\right)\right]\right\}^{\theta}, \quad g = 1, 2\,. \tag{6}$$

The cdf corresponding to (5) is

$$F(x) = \left\{1 - \exp\left[\alpha\left(1 - e^{x^{\beta}}\right)\right]\right\}^{\theta} u_1(x)\,. \tag{7}$$

The parameter α controls the scale of the distribution, whereas the parameters β and θ control its shape. The parameter λ is the transmuting parameter which offers additional flexibility in the TEC distribution. We obtain the baseline model when the transmuting parameter $\lambda = 0$. For $\theta = 1$ the TEC distribution reduces to the transmuted Chen distribution proposed by Khan et al. [7]. The Chen distribution [2] is a special sub-model of equation (5) for $\theta = 1$ and $\lambda = 0$. If X is a random variable with density function (5), then we describe this random variable as $X \sim \mathrm{TEC}(x; \alpha, \beta, \theta, \lambda)$.

Figures 1 and 2 plot the TEC distribution for some selected parameters and suggest that the additional parameter λ provides extra flexibility in the new extended model.

The reliability and hazard functions of the TEC distribution are, respectively,

$$R(x) = 1 - \left\{1 - \exp\left[\alpha\left(1 - e^{x^{\beta}}\right)\right]\right\}^{\theta} u_1(x)\,, \tag{8}$$

$$h(x) = \frac{\alpha\beta\theta x^{\beta-1}\exp\left[x^{\beta} + \alpha\left(1 - e^{x^{\beta}}\right)\right]\left\{1 - \exp\left[\alpha\left(1 - e^{x^{\beta}}\right)\right]\right\}^{\theta-1} u_2(x)}{1 - \left\{1 - \exp\left[\alpha\left(1 - e^{x^{\beta}}\right)\right]\right\}^{\theta} u_1(x)}\,. \tag{9}$$

The qth quantile $F(x_q)$ of the TEC random variable is defined by

$$F^{-1}(u) = \left(\log\left\{1 - \frac{1}{\alpha}\log\left[1 - \left(\frac{1 + \lambda - \sqrt{(1+\lambda)^2 - 4\lambda u}}{2\lambda}\right)^{\frac{1}{\theta}}\right]\right\}\right)^{\frac{1}{\beta}}\,. \tag{10}$$

3 Statistical properties

This section presents the kth moment and the probability weighted moments, and also discusses important features of the TEC distribution.

Theorem 1. *If X has the distribution $\mathrm{TEC}(x;\alpha,\beta,\theta,\lambda)$ with $|\lambda| \leqslant 1$, then the kth moment of X is*

$$\acute{\mu}_k = \left[(1+\lambda)\sum_{i,j,m=0}^{\infty}\binom{\theta-1}{i}\nu_{i,j,m} - 2\lambda\sum_{i,j,m=0}^{\infty}\binom{2\theta-1}{i}\nu_{i,j,m}\right]\Gamma\left(\frac{k}{\beta}+1\right).$$

where

$$\nu_{i,j,m} = \frac{\binom{j}{m}(-1)^{i+m+\frac{k}{\beta}}\alpha^{i+1}\theta\,(i+1)^{j}}{j!(m+1)^{\frac{k}{\beta}+1}}\,.$$

Figure 1: Plots of the TEC probability density function for some parameter values.

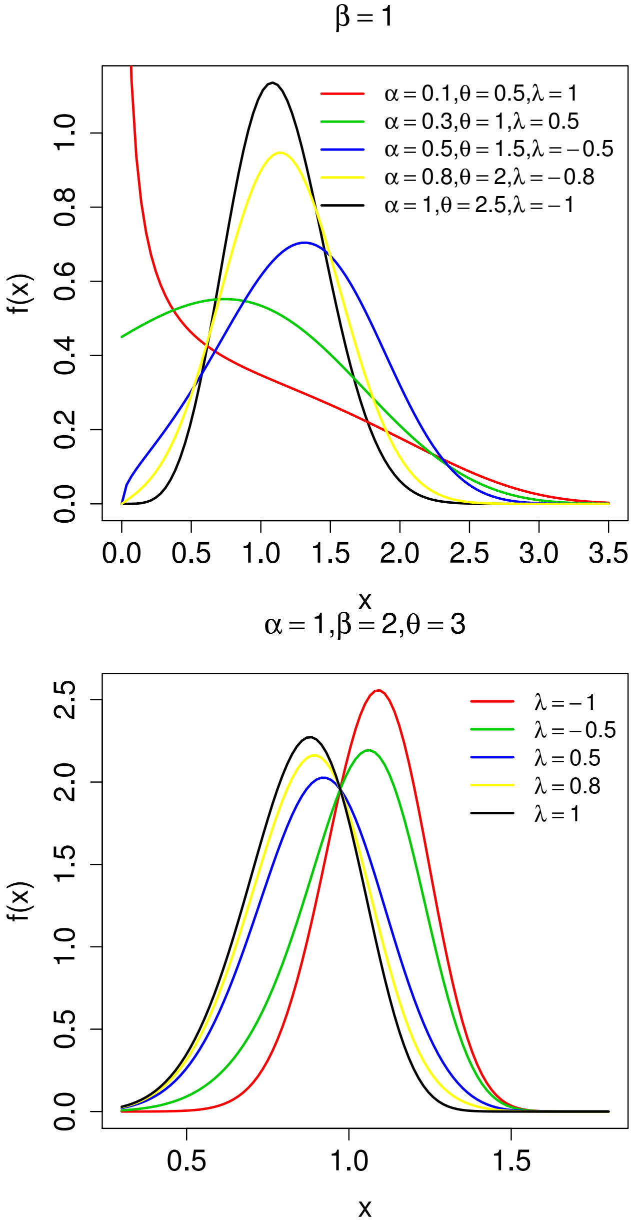
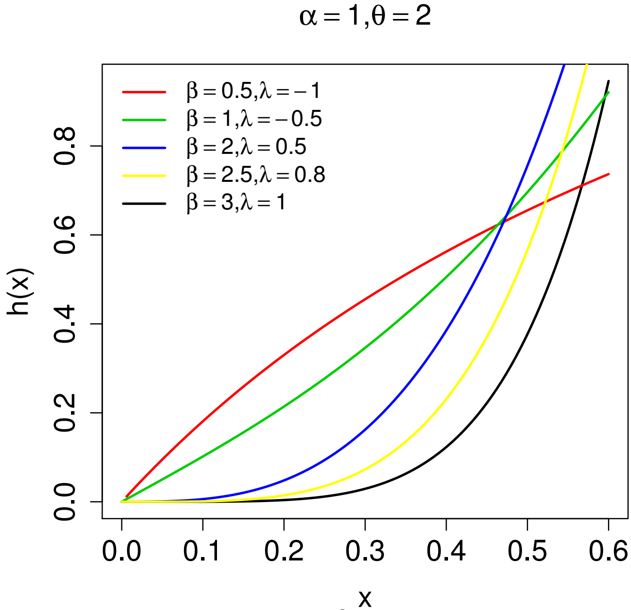
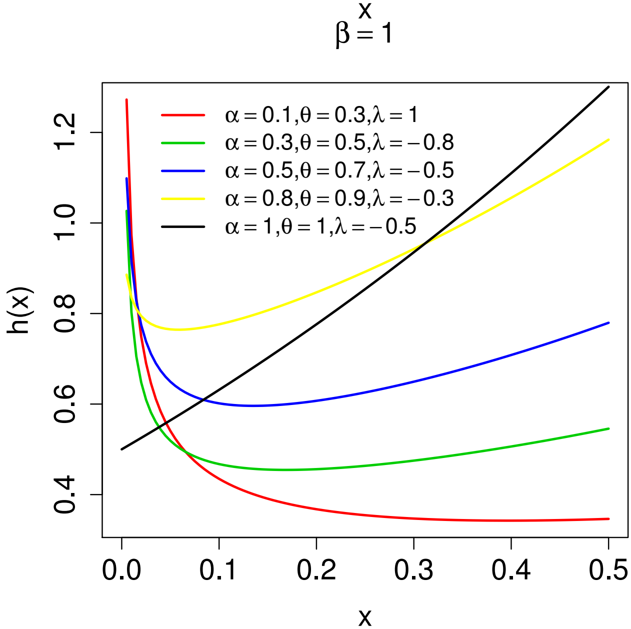


Figure 2: Like Figure 1, plots of the TEC hazard function for some parameter values.





Proof: The kth moment of the TEC distribution is

$$\acute{\mu}_k = \int_0^\infty \alpha\beta\theta x^{k+\beta-1} \frac{\exp\left[x^\beta + \alpha\left(1 - e^{x^\beta}\right)\right]}{\left\{1 - \exp\left[\alpha\left(1 - e^{x^\beta}\right)\right]\right\}^{1-\theta}} u_2(x)\, dx\,.$$

The above expression expands to

$$\begin{aligned}
\acute{\mu}_k = {} & (1+\lambda) \int_0^\infty \alpha\beta\theta x^{k+\beta-1} \frac{\exp\left[x^\beta + \alpha\left(1 - e^{x^\beta}\right)\right]}{\left\{1 - \exp\left[\alpha\left(1 - e^{x^\beta}\right)\right]\right\}^{1-\theta}}\, dx \\
& - 2\lambda \int_0^\infty \alpha\beta\theta x^{k+\beta-1} \frac{\exp\left[x^\beta + \alpha\left(1 - e^{x^\beta}\right)\right]}{\left\{1 - \exp\left[\alpha\left(1 - e^{x^\beta}\right)\right]\right\}^{1-2\theta}}\, dx \\
= {} & (1+\lambda) \sum_{i,j=0}^\infty \binom{\theta - 1}{i} \frac{(-1)^i \alpha^{i+1}\theta\beta\,(i+1)^j}{j!} \int_0^\infty \frac{x^{k+\beta-1} e^{x^\beta}}{\left(1 - e^{x^\beta}\right)^{-j}}\, dx \\
& - 2\lambda \sum_{i,j=0}^\infty \binom{2\theta - 1}{i} \frac{(-1)^i \alpha^{i+1}\theta\beta\,(i+1)^j}{j!} \int_0^\infty \frac{x^{k+\beta-1} e^{x^\beta}}{\left(1 - e^{x^\beta}\right)^{-j}}\, dx\,.
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
\acute{\mu}_k = {} & (1+\lambda) \sum_{i,j,m=0}^\infty \binom{\theta - 1}{i}\binom{j}{m} \frac{(-1)^{i+m+\frac{k}{\beta}} \alpha^{i+1}\theta\,(i+1)^j}{j!\,(m+1)^{\frac{k}{\beta}+1}} \Gamma\left(\frac{k}{\beta} + 1\right) \\
& - 2\lambda \sum_{i,j,m=0}^\infty \binom{2\theta - 1}{i}\binom{j}{m} \frac{(-1)^{i+m+\frac{k}{\beta}} \alpha^{i+1}\theta\,(i+1)^j}{j!\,(m+1)^{\frac{k}{\beta}+1}} \Gamma\left(\frac{k}{\beta} + 1\right). \qquad (11)
\end{aligned}$$

♠

The important features and characteristics of the TEC distribution are studied through moments. The central moment and the cumulants are, respectively,

$$\mu_n = \sum_{i=0}^n \binom{n}{i} (-1)^i \acute{\mu}_1^n \acute{\mu}_{n-i} \quad \text{and} \quad k_n = \acute{\mu}_n - \sum_{i=1}^{n-1} \binom{n-1}{i-1} k_1 \acute{\mu}_{n-i}\,,$$

Table 1: Moments values of the TEC distribution.

α	β	θ	λ	$\acute{\mu}_1$	$\acute{\mu}_2$	$\acute{\mu}_3$	$\acute{\mu}_4$
0.5	1	1	−1.0	1.249	1.831	2.981	5.249
			−0.5	1.086	1.506	2.381	4.117
			0.5	0.759	0.857	1.181	1.854
			1.0	0.596	0.532	0.581	0.722
1	1	1	−1.0	0.831	0.853	1.004	1.307
			−0.5	0.714	0.692	0.792	1.015
			0.5	0.479	0.371	0.369	0.429
			1.0	0.361	0.211	0.157	0.137
1	1	2	−1.0	1.059	1.252	1.612	2.229
			−0.5	0.945	1.052	1.308	1.769
			0.5	0.717	0.654	0.700	0.846
			1.0	0.603	0.454	0.396	0.385
2	1	3	−1.0	0.766	0.652	0.606	0.609
			−0.5	0.688	0.550	0.493	0.483
			0.5	0.532	0.348	0.266	0.231
			1.0	0.454	0.247	0.153	0.105

where $k_1 = \acute{\mu}_1$, $k_2 = \acute{\mu}_2 - \acute{\mu^2}_1$, $k_3 = \acute{\mu}_3 - 3\acute{\mu}_2\acute{\mu}_1 + 2\acute{\mu}_1^3$, $k_4 = \acute{\mu}_4 - 4\acute{\mu}_3\acute{\mu}_1 + 6\acute{\mu}_2\acute{\mu}_1^2 - 3\acute{\mu}_1^4$, and so on. Some moments and other important features of the TEC distribution are displayed in Tables 1 and 2.

Theorem 2. *If X has the distribution* TEC$(x; \alpha, \beta, \theta, \lambda)$ *with $|\lambda| \leqslant 1$, then the probability weighted moment (*PWM*) is*

$$\xi_{(k,m)} = \alpha\theta \sum_{i,j=0}^{\infty} u_{i,j} \Gamma\left(\frac{k}{\beta} + 1\right) \left[(1+\lambda) \sum_{n,p,q=0}^{\infty} \frac{\nu_{1,n,p,q}}{(q+1)^{\frac{k}{\beta}+1}} \right.$$
$$\left. - 2\lambda \sum_{n,p,q=0}^{\infty} \frac{\nu_{2,n,p,q}}{(q+1)^{\frac{k}{\beta}+1}} \right] .$$

Table 2: Moments based measures of the TEC distribution, including the coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK).

α	β	θ	λ	Mean	Var	CV	CS	CK
0.5	1	1	−1.0	1.249	0.269	0.416	0.139	2.581
			−0.5	1.086	0.326	0.525	0.198	2.420
			0.5	0.759	0.279	0.696	0.713	2.979
			1.0	0.596	0.176	0.704	0.718	2.986
1	1	1	−1.0	0.831	0.162	0.484	0.397	2.753
			−0.5	0.714	0.183	0.599	0.474	2.695
			0.5	0.479	0.142	0.787	1.029	3.778
			1.0	0.361	0.080	0.784	1.009	3.748
1	1	2	−1.0	1.059	0.129	0.339	0.256	2.852
			−0.5	0.945	0.159	0.422	0.209	2.717
			0.5	0.717	0.139	0.519	0.612	3.139
			1.0	0.603	0.090	0.498	0.476	2.899
2	1	3	−1.0	0.766	0.065	0.333	0.442	3.108
			−0.5	0.688	0.077	0.403	0.385	2.989
			0.5	0.532	0.064	0.477	0.754	3.574
			1.0	0.454	0.040	0.441	0.521	3.205

where

$$v_{g,n,p,q} = \binom{g\theta - 1}{n}\binom{p}{q}(-1)^{n+q+\frac{k}{\beta}}\frac{(j+n+1)^p\,\alpha^p}{p!}\,, \qquad g = 1, 2\,.$$

Proof: The PWM of the TEC distribution is

$$\xi_{(k,m)} = \int_0^\infty x^k F(x)^m f(x)\,dx\,.$$

From the above integral, the expansion of the cdf in terms of an infinite weighted sum is

$$F(x)^m = (1+\lambda)^m \sum_{i,j=0}^{\infty} \binom{m}{i} \binom{\theta(m+i)}{j} (-1)^{i+j} \left(\frac{\lambda}{1+\lambda}\right)^i \exp\left[\alpha j \left(1 - e^{x^\beta}\right)\right],$$

and the PWM expands to

$$\begin{aligned}\xi_{(k,m)} &= (1+\lambda)\alpha\beta\theta \sum_{i,j=0}^{\infty} u_{i,j} \int_0^\infty x^{k+\beta-1} \frac{\exp\left[x^\beta + \alpha(j+1)\left(1 - e^{x^\beta}\right)\right]}{\left\{1 - \exp\left[\alpha\left(1 - e^{x^\beta}\right)\right]\right\}^{1-\theta}}\, dx \\ &\quad - 2\lambda\alpha\beta\theta \sum_{i,j=0}^{\infty} u_{i,j} \int_0^\infty x^{k+\beta-1} \frac{\exp\left[x^\beta + \alpha(j+1)\left(1 - e^{x^\beta}\right)\right]}{\left\{1 - \exp\left[\alpha\left(1 - e^{x^\beta}\right)\right]\right\}^{1-\theta}}\, dx\end{aligned}$$

where

$$u_{i,j} = (1+\lambda)^m \binom{m}{i} \binom{\theta(m+i)}{j} (-1)^{i+j} \left(\frac{\lambda}{1+\lambda}\right)^i.$$

Finally, we obtain

$$\begin{aligned}\xi_{(k,m)} = \alpha\theta \sum_{i,j=0}^{\infty} u_{i,j} \Gamma\left(\frac{k}{\beta}+1\right) &\left[(1+\lambda) \sum_{n,p,q=0}^{\infty} \frac{v_{1,n,p,q}}{(q+1)^{\frac{k}{\beta}+1}}\right. \\ &\left. - 2\lambda \sum_{n,p,q=0}^{\infty} \frac{v_{2,n,p,q}}{(q+1)^{\frac{k}{\beta}+1}}\right]. \qquad (12)\end{aligned}$$

♠

The PWMs are useful for finding the L-moments estimators, have less variation and bias compared to the conventional estimator, and are also convenient for the moments of order statistics.

4 Parameter estimation

Consider the random samples x_1, x_2,...,x_n consisting of n observations from the TEC distribution. Then the log-likelihood function $\ell(\Theta)$=log L of (5) is

$$\begin{aligned}\ell(\Theta) = {} & n\log\alpha + n\log\beta + n\log\theta \\ & + (\beta-1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n} x_i^{\beta} + \alpha\sum_{i=1}^{n}\left(1-e^{x_i^{\beta}}\right) \\ & + (\theta-1)\sum_{i=1}^{n}\log\left\{1-\exp\left[\alpha\left(1-e^{x_i^{\beta}}\right)\right]\right\}, \\ & + \sum_{i=1}^{n}\log\left(1+\lambda-2\lambda\left\{1-\exp\left[\alpha\left(1-e^{x_i^{\beta}}\right)\right]\right\}^{\theta}\right). \end{aligned} \tag{13}$$

By differentiating (13) with respect to α, β, θ and λ, and then equating this derivative to zero, we obtain the components of score vector $U(\Theta)$:

$$\begin{aligned}\frac{\partial\ell(\Theta)}{\partial\alpha} = {} & \frac{n}{\alpha} + \sum_{i=1}^{n}\left(1-e^{x_i^{\beta}}\right) - (\theta-1)\sum_{i=1}^{n}\frac{\exp\left[\alpha\left(1-e^{x_i^{\beta}}\right)\right]}{\left(1-e^{x_i^{\beta}}\right)^{-1}} \\ & + 2\lambda\theta\sum_{i=1}^{n}\frac{\left\{1-\exp\left[\alpha\left(1-e^{x_i^{\beta}}\right)\right]\right\}^{\theta-1}\exp\left[\alpha\left(1-e^{x_i^{\beta}}\right)\right]}{\left(1-e^{x_i^{\beta}}\right)^{-1}\left(1+\lambda-2\lambda\left\{1-\exp\left[\alpha\left(1-e^{x_i^{\beta}}\right)\right]\right\}^{\theta}\right)}, \end{aligned}$$

$$\frac{\partial \ell(\Theta)}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} x_i^{\beta} \log x_i - \alpha \sum_{i=1}^{n} e^{x_i^{\beta}} x_i^{\beta} \log x_i$$
$$+ (\theta - 1) \sum_{i=1}^{n} \frac{\exp\left[\alpha\left(1 - e^{x_i^{\beta}}\right)\right] e^{x_i^{\beta}} x_i^{\beta} \log x_i}{1 - \exp\left[\alpha\left(1 - e^{x_i^{\beta}}\right)\right]}$$
$$- 2\lambda\alpha\theta \sum_{i=1}^{n} \frac{\left\{1 - \exp\left[\alpha\left(1 - e^{x_i^{\beta}}\right)\right]\right\}^{\theta-1} \exp\left[\alpha\left(1 - e^{x_i^{\beta}}\right)\right] \log x_i}{x_i^{-\beta} e^{x_i^{-\beta}} \left(1 + \lambda - 2\lambda\left\{1 - \exp\left[\alpha\left(1 - e^{x_i^{\beta}}\right)\right]\right\}^{\theta}\right)},$$

$$\frac{\partial \ell(\Theta)}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \log\left\{1 - \exp\left[\alpha\left(1 - e^{x_i^{\beta}}\right)\right]\right\}$$
$$- 2\lambda \sum_{i=1}^{n} \frac{\left\{1 - \exp\left[\alpha\left(1 - e^{x_i^{\beta}}\right)\right]\right\}^{\theta} \log\left\{1 - \exp\left[\alpha\left(1 - e^{x_i^{\beta}}\right)\right]\right\}}{\left(1 + \lambda - 2\lambda\left\{1 - \exp\left[\alpha\left(1 - e^{x_i^{\beta}}\right)\right]\right\}^{\theta}\right)},$$

and

$$\frac{\partial \ell(\Theta)}{\partial \lambda} = \sum_{i=1}^{n} \frac{1 - 2\left\{1 - \exp\left[\alpha\left(1 - e^{x_i^{\beta}}\right)\right]\right\}^{\theta}}{\left(1 + \lambda - 2\lambda\left\{1 - \exp\left[\alpha\left(1 - e^{x_i^{\beta}}\right)\right]\right\}^{\theta}\right)}.$$

The asymptotic variance covariance matrix of MLE for the parameter vector $\Theta = (\alpha, \beta, \theta, \lambda)$ is the multivariate normal with the variance covariance matrix and its inverse of the expected information matrix is

$$\left[(\hat{\alpha} - \alpha), (\hat{\beta} - \beta), (\hat{\theta} - \theta), (\hat{\lambda} - \lambda)\right] \sim N_4\left[0, K(\Theta)^{-1}\right],$$

where $K(\Theta)^{-1}$ is the variance covariance matrix of the unknown parameters for the vector $\Theta = (\alpha, \beta, \theta, \lambda)$. The multivariate normal distribution is used to obtain approximate $100(1-\gamma)\%$ confidence intervals for the parameters α, β, θ and λ, determined as, respectively,

$$\hat{\alpha} \pm Z_{\frac{\gamma}{2}} \sqrt{\hat{V}_{11}}, \quad \hat{\beta} \pm Z_{\frac{\gamma}{2}} \sqrt{\hat{V}_{22}}, \quad \hat{\theta} \pm Z_{\frac{\gamma}{2}} \sqrt{\hat{V}_{33}}, \quad \hat{\lambda} \pm Z_{\frac{\gamma}{2}} \sqrt{\hat{V}_{44}},$$

where $Z_{\frac{\gamma}{2}}$ is the upper γth percentile of the standard normal distribution.

5 Simulation

In this section we evaluate the performance of MLE using simulations for the four parameters of the TEC distribution. Using (10), we generate samples from the TEC distribution of different sizes $n = 50, 100, 200, 400, 800$ for a fixed choice of the parameters $\alpha = 2$, $\beta = 3$, $\theta = 1$ and $\lambda = 0.5$. We fit the TEC distribution for these samples using MLE. Table 3 shows the estimated values of the parameters $\alpha, \beta, \theta, \lambda$ with their corresponding standard error, bias and mean square error. Table 3 shows that the simulated results are quite promising with respect to the sample sizes.

6 Applications

This section provides two data analyses in order to assess the goodness-of-fit of the proposed model.

6.1 Strengths of glass fibres data

The first application that we consider in this study relates to the strengths of glass fibres. The data set was obtained from Smith and Naylor [9] and represents the strengths of 1.5 cm glass fibres, measured at the National

Table 3: Mean estimates, standard errors (SE), biases and mean square errors (MSE) for the TEC distribution from Monte Carlo simulation results based on MLE.

n	Parameter	Mean	SE	Bias	MSE
	α	2.580	0.761	0.580	0.915
50	β	2.733	1.471	−0.267	2.235
	θ	1.107	0.786	0.107	0.628
	λ	0.300	0.602	−0.199	0.402
	α	1.968	0.795	−0.032	0.634
100	β	3.174	0.766	0.174	0.618
	θ	0.929	0.298	−0.071	0.094
	λ	0.599	0.517	0.099	0.277
	α	2.012	0.767	0.012	0.588
200	β	3.058	0.913	0.058	0.837
	θ	1.007	0.404	0.007	0.163
	λ	0.576	0.509	0.076	0.265
	α	1.949	0.693	−0.050	0.482
400	β	3.034	0.629	0.034	0.397
	θ	1.027	0.292	0.027	0.086
	λ	0.586	0.471	0.086	0.229
	α	1.828	0.342	−0.172	0.146
800	β	3.089	0.413	0.089	0.178
	θ	0.956	0.170	−0.044	0.031
	λ	0.608	0.231	0.108	0.065

Table 4: MLE of the parameters for the strengths of glass fibres data with the corresponding standard error given in parentheses below the estimate. Also shown are the Kolmogorov–Smirnov (KS), Cramer–von Mises (W), and Anderson–Darling (A) goodness-of-fit statistics.

Model	α	β	θ	λ	KS	W	A
TEC	0.089	1.789	1.679	0.841	0.126	0.142	0.801
	(0.050)	(0.210)	(0.547)	(0.184)			
GPW	0.618	5.100	1.849	—	0.149	0.209	1.159
	(0.409)	(0.914)	(0.342)				
EW	0.020	7.212	0.681	—	0.147	0.201	1.117
	(0.023)	(1.522)	(0.231)				
EC	0.173	1.683	1.949	—	0.133	0.172	0.965
	(0.076)	(0.177)	(0.664)				

Physical Laboratory, England. The data set is:

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64	1.68	1.73
1.81	2.00	0.74	1.04	1.27	1.39	1.49	1.53	1.59	1.61	1.66
1.68	1.76	1.82	2.01	0.77	1.11	1.28	1.42	1.50	1.54	1.60
1.62	1.66	1.69	1.76	1.84	2.24	0.81	1.13	1.29	1.48	1.50
1.55	1.61	1.62	1.66	1.70	1.77	1.84	0.84	1.24	1.30	1.48
1.51	1.55	1.61	1.63	1.67	1.70	1.78	1.89			

We fitted the TEC, generalized power Weibull (GPW), exponentiated Weibull (EW), and exponentiated Chen (EC) distributions with MLE. The required numerical evaluations are implemented using R language. The MLE and the values of maximized log-likelihoods for the four distributions are shown in Table 4. This table gives the MLE of the unknown parameters (with their standard errors) and the Kolmogorov–Smirnov, Cramer–von Mises and Anderson–Darling goodness-of-fit statistics.

The goodness-of-fit measures in the last three columns of Table 4 indicate that the TEC distribution provides the best fit of the four tested distributions.

Figure 3: Fitted distributions for the strengths of glass fibres data.

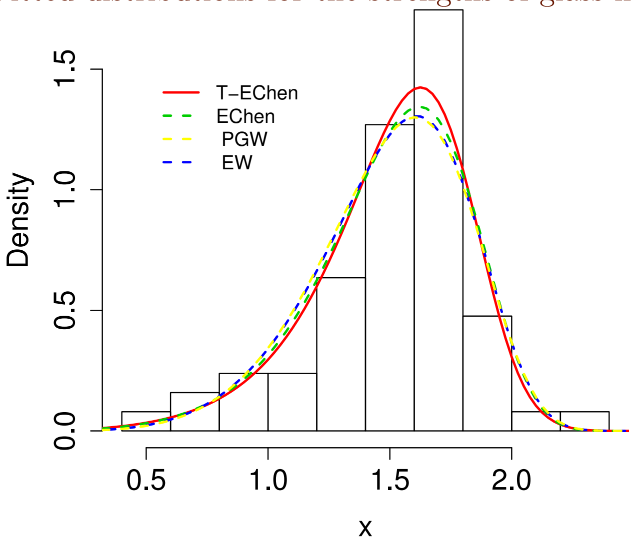


Figure 3 displays a histogram which supports the claim that the TEC is the best fitting distrubition. Therefore, the TEC distribution is the best model of the four for fitting lifetime data. Using the maximum likelihood estimates of the unknown parameters, the approximately 95% two sided confidence intervals for the parameters α, β, θ and λ are $[-0.0095, 0.1879]$, $[1.3774, 2.2006]$, $[0.6066, 2.7509]$ and $[0.4806, 1.2015]$, respectively.

6.2 Nicotine in cigarettes data

The second application is the nicotine content in cigarettes of several brands with $n = 396$ observations. The Federal Trade Commission [11] provided data sets and information about the source of data, smoker behaviour and beliefs about nicotine, tar and carbon monoxide contents in cigarettes. We examine the accuracy of the TEC distribution for modelling the nicotine in cigarettes data. To see the full flexibility of the proposed model, we fit

Table 5: MLE of the parameters for the nocotine in cigarettes data with the corresponding standard error given in parentheses below the estimate. Also shown are the Kolmogorov–Smirnov (KS), Cramer–von Mises (W), and Anderson–Darling (A) goodness-of-fit statistics.

Model	α	β	θ	λ	KS	W	A
TEC	0.869	1.479	1.797	0.696	0.119	0.737	3.904
	(0.152)	(0.169)	(0.324)	(0.235)			
TC	0.524	2.034	—	0.754	0.136	0.961	4.938
	(0.053)	(0.076)		(0.118)			
EC	1.299	1.339	2.095	—	0.121	0.773	4.082
	(0.122)	(0.126)	(0.382)				
C	0.840	1.877	—	—	0.142	1.061	5.464
	(0.043)	(0.068)					

the TEC, exponentiated Chen (EC), transmuted Chen (TC) and Chen (C) distributions.

Table 5 displays the MLE and the corresponding standard errors of the parameters, as well as the Kolmogorov–Smirnov, Cramer–von Mises and the Anderson–Darling goodness-of-fit statistics for the four distributions. Comparing the four fitted lifetime distributions, the preferred model based on these goodness-of-fit measures is the TEC distribution. The lower values of the Kolmogorov–Smirnov, Cramer–von Mises and Anderson–Darling goodness-of-fit statistics indicate that the TEC distribution could be chosen as the best model for the nicotine in cigarettes data.

To assess whether the TEC distribution is an appropriate model, Figure 4 (top left) plots the histogram of the nicotine in cigarettes with the fitted TEC density function. Furthermore, Figure 4 also plots the empirical and estimated survival function (top right), pp-plot (bottom left) and estimated hazard rate function for the TEC distribution (bottom right). All these plotss suggest that the TEC distribution provides a good fit for nicotine in cigarettes.

Figure 4: Estimated fitted TEC distribution for nicotine in cigarettes data showing: (top left) TEC distrbition with histogram of nicotine data; (top right) estimated survival distribution and the empirical survival curve; (bottom left) pp-plot; and (bottom right) estimated hazard rate function.

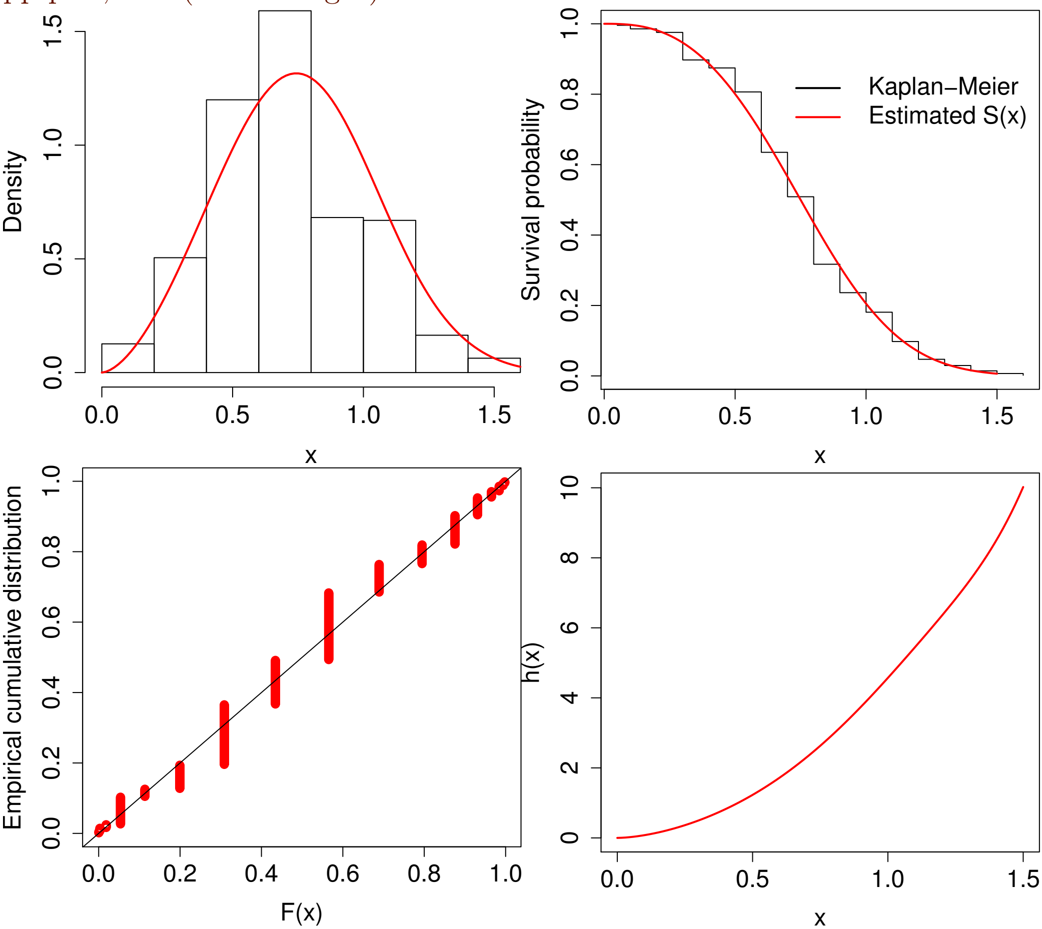


Figure 5 shows the log-likelihood functions for the parameters of the TEC distribution. The log-likelihood functions are fairly peaked and the maximum points are determined by the analytical second partial derivatives which are well approximated numerically. The log-likelihood functions suggest that all the parameters of the TEC distribution have a single mode or maximum point. From these plots we conclude that the estimated results from the TEC distribution are efficient.

7 Conclusion

We introduced a new distribution, the transmuted exponentiated Chen distribution and examined some of its statistical properties with application to survival data. We obtained the analytical shapes of the density and hazard functions. The TEC distribution has increasing and decreasing hazard function for the lifetime data. We conclude that the transmuting parameter λ provides more flexibility in the TEC distribution for fitting the strengths of glass fibres data and the nicotine in cigarettes data. In conclusion, the TEC distribution could be chosen as the best model for fitting survival data.

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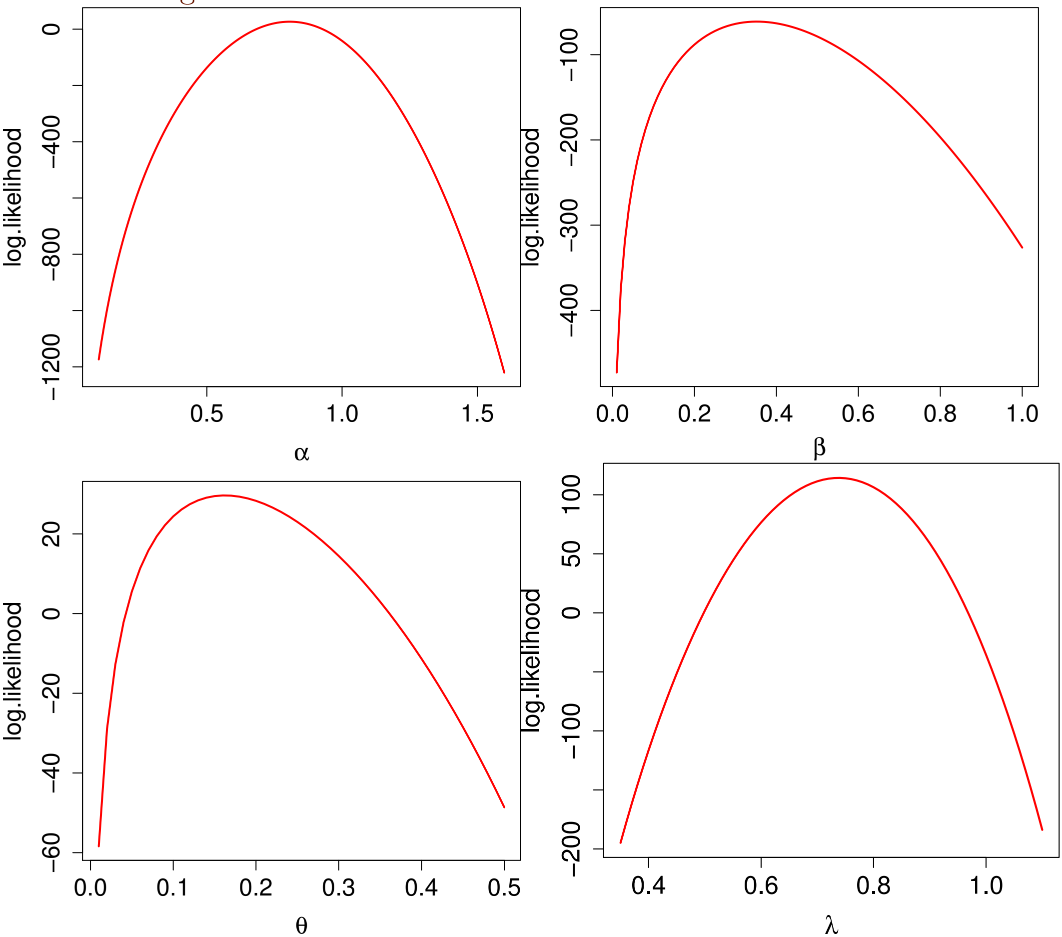
Figure 5: The profile of log-likelihood functions for α, β, θ and λ for the nicotine in cigarettes data.



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