Bifurcation and periodic points in the l_1-norm minimization problem

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Abstract

We explore an optimization problem which arises naturally in the design of feedback controllers to achieve optimal robustness. Stated mathematically, the problem imposes an l_1-norm objective on the input and output signals of a linear discrete-time dynamic system. Recently I presented an algorithm which systematically determines initial conditions for which exact solutions can be found. The contribution of this article is twofold. Firstly, we illustrate the usefulness of the algorithm in understanding optimal dynamic response for a specific example. Secondly, we investigate the apparent disappearance of an attracting periodic point as an input data parameter is varied. I conjecture that the dynamic evolution of optimal solutions may exhibit chaos.

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1 Introduction

The problem of designing a linear time-invariant controller to optimally reject bounded disturbances for discrete-time linear dynamic systems was introduced in the 1980's by Vidyasagar [5]. A comprehensive early treatment is by Dahleh & Diaz–Bobillo [1]. Since that time an open problem, still unsolved, is to determine whether or not optimal solutions always have a rational Z-transform. In recent work [4, 3], I introduced the idea of describing the time evolution of the solution to the optimization problem as the output of a non-linear dynamic system, and gives an algorithm for the determination of periodic points for the non-linear system describing the evolution of the optimal dual variables. Here we investigate the stability of these periodic points and show that both the number and type—attracting, repelling or saddle—can change abruptly with changes in an input data parameter.

We consider an example for which most of the periodic points are repelling, and introduce the hypothesis that the time evolution of the optimal solution sometimes exhibits chaos.

2 Formulation

2.1 Terminology

Denote by $\mathbb{R}^n$ the n-dimensional real space. A $p \times k$ matrix M will sometimes have its dimension made explicit by the notation $M_{p\times k}$. The $p \times p$ identity matrix is denoted I_p. The set of positive integers is denoted $\mathbb{N}$. The l_1-norm of a vector sequence $e = (e_k)_{k=1}^{\infty}$ is defined as $\|e\|_1 = \sum_{k=1}^{\infty} |e_k|$ whenever the series exists. The Banach space of absolutely summable sequences, equipped with the l_1-norm, is denoted l_1. The space of continuous linear functionals on l_1, that is the dual of l_1, is denoted l_∞; it is the space of bounded sequences with the norm $\|e\|_\infty := \sup_k |e_k|$. The Z-transform of an arbitrary sequence $e = (e_k)_{k=1}^{\infty}$ is defined to be $\hat{e}(z) = \sum_{k=1}^{\infty} e_k z^{k-1}$, where z lies within the radius of convergence of the series. Given a vector e and any $s \in \mathbb{N}$, $t \in \mathbb{N}$ satisfying $s < t$, denote $(e_s, e_{s+1}, \ldots, e_t)$ by $e_{(s:t)}$. If $s, t, q, r \in \mathbb{N}$, $1 < s < t$ and $1 < q < r$ then $M_{(s:t,q:r)}$ is a matrix composed of row s to row t, and of columns q to r, of the matrix M having at least t rows and at least r columns. The concatenation of $(e_1, u_1) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_1}$ and $(e_2, u_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^{n_2}$ is defined to be the vector $(e_3, u_3) \in \mathbb{R}^{n_1+n_2} \times \mathbb{R}^{n_1+n_2}$ for which $e_3 = [e_1^T, e_2^T]^T$ and $u_3 = [u_1^T, u_2^T]^T$. The superscript T denotes transpose.

3 Problem description

The decision vectors in the space l_1 are denoted e and u. The cost function is $K_1 \|e\|_1 + K_2 \|u\|_1$.

The problem we investigate is

$$\mathcal{P}(b): \quad \begin{cases} \min\limits_{e\in l_1, u\in l_1} K_1 \|e\|_1 + K_2 \|u\|_1 \,, \\ \text{subject to} \quad \hat{d}\hat{e} + \hat{n}\hat{u} = \hat{b}\,, \end{cases} \tag{1}$$

where $\hat{n}$, $\hat{d}$ and $\hat{b}$ are polynomials with real coefficients:

$$\begin{aligned} \hat{n}(z) &= n_1 + n_2 z + n_3 z^2 + \cdots + n_{l+1} z^l \,, \\ \hat{d}(z) &= d_1 + d_2 z + d_3 z^2 + \cdots + d_{l+1} z^l \,, \\ \hat{b}(z) &= b_1 + b_2 z + \cdots + b_l z^{l-1} \,, \end{aligned} \tag{2}$$

n_{l+1} and d_{l+1} are not both zero, and $l \geq 1$ is a positive integer. I assume that neither $\hat{n}(z)$ nor $\hat{d}(z)$ have zeros lying on the unit circle in the complex plane, and that $\hat{n}(z)$ and $\hat{d}(z)$ have no zeros in common. The vector b specifies initial conditions for the discrete-time dynamic system represented by the equation $\hat{d}\hat{e} + \hat{n}\hat{u} = \hat{b}$.

A solution to $\mathcal{P}(b)$ with finite cost is guaranteed to exist. There is a stronger conjecture, namely that all optimizing vectors (e, u) for $\mathcal{P}(b)$ have rational Z-transforms. I put forward the hypothesis in this article that this conjecture does not hold in general because of chaos in the time evolution of the optimal response of e and u. We do not prove the existence of chaos, but give reasons for suggesting the possibility of it occurring. This is interesting because any proof of the validity of the conjecture would necessarily imply the absence of chaos.

3.1 Linear programming formulation of problem $\mathcal{P}(b)$

Using block matrix notation, the problem $\mathcal{P}(b)$ can be written as

$$\mathcal{P}(b): \quad \begin{cases} \min\limits_{e \in l_1, u \in l_1} K_1 \|e\|_1 + K_2 \|u\|_1 \\ \text{subject to} \quad \begin{bmatrix} D & N \end{bmatrix} \begin{bmatrix} e \\ u \end{bmatrix} = \begin{bmatrix} b^T, 0, \ldots \end{bmatrix}^T. \end{cases} \tag{3}$$

where D is the infinite-dimensional lower-triangular toeplitz matrix with $(d_1, \ldots, d_{l+1}, 0, 0, \ldots)$ as its first column, and matrix N, defined similarly, has first column $(n_1, \ldots, n_{l+1}, 0, 0, \ldots)$. Also $b := [b_1, \ldots, b_l]^T$.

3.2 Toeplitz and circulant matrix notation

Define

$$D_{\text{UT}} := \begin{bmatrix} d_{l+1} & d_l & \ldots & d_2 \\ 0 & d_{l+1} & \vdots & \vdots \\ 0 & 0 & d_{l+1} & d_l \\ 0 & 0 & 0 & d_{l+1} \end{bmatrix}_{l \times l} \quad \text{and } D_{\text{LT}} := \begin{bmatrix} d_1 & 0 & 0 & 0 \\ \vdots & d_1 & 0 & \vdots \\ d_{l-1} & \vdots & d_1 & 0 \\ d_l & d_{l-1} & \ldots & d_1 \end{bmatrix}_{l \times l}.$$

Then for any integer $p \geq 2l$ the North-West corner submatrix of D with dimension $p \times p$ can be written as

$$D_{(1:p,1:p)} = \begin{bmatrix} D_{\text{LT}} & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & D_{\text{UT}} & D_{\text{LT}} \end{bmatrix}_{p \times p}.$$

In this block matrix representation the first and third rows are each composed of l rows, and the middle block row has $p - 2l$ rows.

Denote by $D_C(p)$ the circulant matrix of dimension $p \times p$ whose first column is $(d_1, d_2, \ldots, d_{l+1}, 0, \ldots, 0)$. For example

$$D_C(3l) := \begin{bmatrix} D_{\mathrm{LT}} & 0 & D_{\mathrm{UT}} \\ D_{\mathrm{UT}} & D_{\mathrm{LT}} & 0 \\ 0 & D_{\mathrm{UT}} & D_{\mathrm{LT}} \end{bmatrix}_{3l \times 3l} .$$

The matrices N_{UT}, N_{LT} and $N_C(p)$ are defined similarly.

We shall also need a matrix related to the Bezoutian of $\hat{d}(z)$ and $\hat{n}(z)$. Define

$$W := [D_{\mathrm{LT}} N_{\mathrm{UT}} - N_{\mathrm{LT}} D_{\mathrm{UT}}]^{-1}.$$

The matrix $D_{\mathrm{LT}} N_{\mathrm{UT}} - N_{\mathrm{LT}} D_{\mathrm{UT}}$, the Bezoutian of $\hat{d}(z)$ and $\hat{n}(z)$ with the order of the rows reversed, is non-singular because $\hat{d}(z)$ and $\hat{n}(z)$ are coprime.

3.3 Matching terminal and initial conditions for subproblems

Any vector pair $(e, u) \in l_1 \times l_1$ satisfying the constraints to (3) will be termed feasible for $\mathcal{P}(b)$. Let (e, u) be feasible for $\mathcal{P}(b)$ and let $M \geq 2l$ be an integer. Define the *initial condition at time M* for (e, u) by

$$b^{(M)}(e, u) := -\left[D_{\mathrm{UT}} e_{(M-l+1:M)} + N_{\mathrm{UT}} u_{(M-l+1:M)}\right]. \tag{4}$$

Then the concatenation $\left(\begin{bmatrix} e^{(M)} \\ e^{(1)} \end{bmatrix}, \begin{bmatrix} u^{(M)} \\ u^{(1)} \end{bmatrix}\right)$ is feasible for $\mathcal{P}(b)$ if and only if $(e^{(M)}, u^{(M)})$ satisfies $D_{(1:M,1:M)} e^{(M)} + N_{(1:M,1:M)} u^{(M)} = \begin{bmatrix} b^T, 0, \ldots, 0 \end{bmatrix}^T$ and $(e^{(1)}, u^{(1)})$ is feasible for $\mathcal{P}(b^{(M)}(e, u))$.

Suppose now we are given an optimal (hence feasible) solution to $\mathcal{P}(b)$, denoted $(e^{\mathcal{P}(b)}, u^{\mathcal{P}(b)})$. That is $(e^{\mathcal{P}(b)}, u^{\mathcal{P}(b)}) \in \arg\min \mathcal{P}(b)$. The initial condition at time M for $(e^{\mathcal{P}(b)}, u^{\mathcal{P}(b)})$, $b^{(M)}(e^{\mathcal{P}(b)}, u^{\mathcal{P}(b)})$, is given by (4).

It follows from the discussion above and Bellman's principle of optimality that $(e^{(2)}, u^{(2)}) \in \arg\min \mathcal{P}(b^{(M)}(e^{\mathcal{P}(b)}, u^{\mathcal{P}(b)}))$ if $\left(\begin{bmatrix} e^{\mathcal{P}(b)}_{(1:M)} \\ e^{(2)} \end{bmatrix}, \begin{bmatrix} u^{\mathcal{P}(b)}_{(1:M)} \\ u^{(2)} \end{bmatrix}\right) \in \arg\min \mathcal{P}(b)$.

Motivated by this observation, consider, as M tends to infinity, the time evolution of $b^{(M)}(e^{\mathcal{P}(b)}, u^{\mathcal{P}(b)})$. For some initial conditions b, either $b^{(M)}(e^{\mathcal{P}(b)}, u^{\mathcal{P}(b)})$ is a decaying periodic vector, or $b^{(M)}(e^{\mathcal{P}(b)}, u^{\mathcal{P}(b)})$ is a finite sum of decaying periodic vectors. For some problem data $b^{(M)}(e^{\mathcal{P}(b)}, u^{\mathcal{P}(b)})$ will exhibit this behaviour for all initial conditions b. However, for some other problems there exist initial conditions which do not seem to produce any form of decaying periodicity in $b^{(M)}(e^{\mathcal{P}(b)}, u^{\mathcal{P}(b)})$. There is the interesting possibility that $b^{(M)}(e^{\mathcal{P}(b)}, u^{\mathcal{P}(b)})$ wanders around for ever on a strange attractor.

The rest of this Section sets up a framework for the analysis of the dynamics of the time evolution of $b^{(M)}(e^{\mathcal{P}(b)}, u^{\mathcal{P}(b)})$ when there *is* periodicity in both the pattern of the locations of the zero values of $e^{\mathcal{P}(b)}$ and $u^{\mathcal{P}(b)}$, and in the sign pattern of the non-zero values of $e^{\mathcal{P}(b)}$ and $u^{\mathcal{P}(b)}$. Such periodicity will be related in Sections 3.4 and 4.3 to basis periodicity, where the term basis is the familiar one used in linear programming theory.

3.4 Notation for a basis

Consider the set of equations $Ae + Bu = b$, or in block matrix notation

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} e \\ u \end{bmatrix} = b\,, \tag{5}$$

where A and B are any real $p \times p$ matrices, and e, u and b are real p-dimensional column vectors. Let $\begin{bmatrix} A & B \end{bmatrix}_{\mathcal{B}}$ be any non-singular $p \times p$ submatrix made up of the columns of the $p \times 2p$ matrix $\begin{bmatrix} A & B \end{bmatrix}$. Thus the p integers $i_1, i_2, \ldots, i_p$ from $1, 2, \ldots, 2p$ identify the columns of $\begin{bmatrix} A & B \end{bmatrix}$ that have been retained in $[\begin{array}{cc} A & B \end{array}]_{\mathcal{B}}$. The set $\{i_1, i_2, \ldots, i_p\}$ determines the basis $[\begin{array}{cc} A & B \end{array}]_{\mathcal{B}}$ and the notation $\mathcal{B} = [i_1, i_2, \ldots, i_p]$ will be used to identify the basis, where $i_1, \ldots, i_p$ have been ordered by increasing size. The vector $\mathcal{B}$ will be referred to as the basis vector. If the p components of $[\begin{array}{cc} e^T & u^T \end{array}]^T$ not associated with columns of $[\begin{array}{cc} A & B \end{array}]_{\mathcal{B}}$ are set equal to zero, the solution to the resulting set of equations is said to be a *basic solution* to (5) with respect to the basis vector $\mathcal{B}$, denoted $\begin{bmatrix} e \\ u \end{bmatrix}_{\mathcal{B}}^{\text{bsol}}$, a $2p \times 1$ vector. We use the notation $(e^{\text{bsol}}(\mathcal{B}), u^{\text{bsol}}(\mathcal{B}))$ to denote the basic solution with respect to $\mathcal{B}$.

Suppose that, for some integer $p \geq 2l$, a p-dimensional basis vector, $\mathcal{B}$, is given. Define

$$\begin{aligned}
Z(\mathcal{B}) &:= \left[\begin{array}{cc} D_{(1:p,1:p)} & N_{(1:p,1:p)} \end{array}\right]_{\mathcal{B}} , \\
Y(\mathcal{B}) &:= \left[\begin{array}{cc} D_C(p) & N_C(p) \end{array}\right]_{\mathcal{B}} , \\
F(p) &:= \left[\begin{array}{cc} D_{(1:p,1:p)} & N_{(1:p,1:p)} \end{array}\right] - \left[\begin{array}{cc} D_C(p) & N_C(p) \end{array}\right] , \\
H(\mathcal{B}) &:= I_p - YZ^{-1} = [F(p)]_{\mathcal{B}} Z^{-1} , \\
G(\mathcal{B}) &:= H_{(1:l,1:l)} .
\end{aligned} \tag{6}$$

The significance of the $l \times l$ matrix $G(\mathcal{B})$ will now be explained. First recall from (4) the definition, for any (e, u) feasible for $\mathcal{P}(b)$, of the *initial condition at time M*, denoted by $b^{(M)}(e, u)$. For any p-dimensional basis $\mathcal{B}$ there is the basic solution to $D_{(1:p,1:p)}e + N_{(1:p,1:p)}u = \begin{bmatrix} b_{l\times 1} \\ 0_{(p-l)\times 1} \end{bmatrix}$, denoted $(e^{\text{bsol}}(\mathcal{B}), u^{\text{bsol}}(\mathcal{B}))$. Then $G(\mathcal{B})$ maps b to the initial condition at time p for $(e^{\text{bsol}}(\mathcal{B}), u^{\text{bsol}}(\mathcal{B}))$. That is, for any b,

$$G(\mathcal{B}) : b \mapsto b^{(p)}(e^{\text{bsol}}(\mathcal{B}), u^{\text{bsol}}(\mathcal{B})). \tag{7}$$

Let $\left(e^{\text{bsol}(n)}, u^{\text{bsol}(n)}\right)$ denote the basic solution, with respect to $\mathcal{B}$, to the equations $D_{(1:p,1:p)}e + N_{(1:p,1:p)}u = \begin{bmatrix} G^n(\mathcal{B})b_{l\times 1} \\ 0_{(p-l)\times 1} \end{bmatrix}$. Then by (7) and the discussion in Section 3.3, the concatenation

$$\left(\begin{bmatrix} e^{\mathrm{bsol}(0)} \\ e^{\mathrm{bsol}(1)} \\ \vdots \end{bmatrix}, \begin{bmatrix} u^{\mathrm{bsol}(0)} \\ u^{\mathrm{bsol}(1)} \\ \vdots \end{bmatrix} \right) =: (e^{\mathrm{feas}}, u^{\mathrm{feas}})$$

is feasible for $De + Nu = \left[b^T, 0, \ldots\right]^T$. In the terminology of non-linear systems, $G^n(\mathcal{B})b$ acts as a stroboscopic return map of period p for the trajectory $(e^{\mathrm{feas}}, u^{\mathrm{feas}})$. The pattern of zeros in $(e^{\mathrm{feas}}, u^{\mathrm{feas}})$ is necessarily periodic. Periodicity also in the sign pattern of the non-zero elements of $(e^{\mathrm{feas}}, u^{\mathrm{feas}})$ is required for our main optimization result. A sufficient condition for this is that b is an eigenvector of $G(\mathcal{B})$ with corresponding real and positive eigenvalue having magnitude less than one.

The above discussion explains some of the conditions required of so-called *periodic bases,* to be defined in Section 4.3, from which optimal solutions for $\mathcal{P}$ can be constructed. Periodic bases, in addition to their periodicity properties, also by definition optimize certain finite dimensional programs. These will be described next.

4 Duality

The infinite-dimensional dual to $\mathcal{P}(b)$, denoted $\mathcal{D}(b)$, was derived by Dahleh & Pearson [2].

Using the notation of Section 3.2 it can be expressed in the form

$$\mathcal{D}(b) : \quad \begin{cases} \displaystyle\max_{e^* \in l_\infty, u^* \in l_\infty} \quad b^{*T} W b \,, \\ \text{subject to} \quad \|e^*\|_\infty \leq K_1 \,, \quad \|u^*\|_\infty \leq K_2 \\ \text{and} \quad D^T u^* = N^T e^* \,, \end{cases} .$$

where

$$b^* := [N_{\mathrm{LT}}]^T e^*_{(1:l)} - [D_{\mathrm{LT}}]^T u^*_{(1:l)}$$

is the initial condition for the dual.

4.1 $MD(b, p)$: a finite-dimensional modification of $\mathcal{D}(b)$

Let $p \geq 2l$ be an integer. We construct a finite dimensional convex programming problem related to $\mathcal{D}(b)$ that has $2p$ variables and p equality constraints. The constraints for $MD(b, p)$, the problem $\mathcal{D}$ constrained in a manner consistent with its variables e^* and u^* being periodic of period p, can be written $D_C^T(p)u^* = N_C^T(p)e^*$.

$$MD(b,p): \quad \begin{cases} \displaystyle\max_{e^* \in \mathbb{R}^p, u^* \in \mathbb{R}^p} \; b^{*T} W b \\ \text{subject to} \quad D_C^T(p)u^* = N_C^T(p)e^* \\ \text{and} \quad \|e^*\|_\infty \leq K_1, \quad \|u^*\|_\infty \leq K_2 . \end{cases}$$

4.2 The dual of $MD(b, p)$, denoted $DMD(b, p)$

For $p \geq 2l$, a dual of $MD(b, p)$ can be constructed in the form

$$DMD(b,p): \quad \begin{cases} \displaystyle\min_{e \in \mathbb{R}^p, u \in \mathbb{R}^p} \sum_{k=1}^{p} K_1 |e_k| + K_2 |u_k| \\ \text{subject to} \quad D_C(p)e + N_C(p)u = \begin{bmatrix} b_{l\times 1} \\ 0_{(p-l)\times 1} \end{bmatrix} . \end{cases}$$

The optimal values of $DMD(b, p)$ and $MD(b, p)$ are equal.

In the next section we give a definition of special basis vectors, which we term *periodic basis vectors*. Associated with every periodic basis vector, denoted $\tilde{\mathcal{B}}$, there is a so-called *periodic initial condition,* denoted $\tilde{b}$. Periodic initial conditions are important because basic feasible solutions constructed from the periodic basis vector $\tilde{\mathcal{B}}$ are optimal for the problem $\mathcal{P}(\tilde{b})$. This is the content of Theorem 5 in Section 4.3 [3].

4.3 Periodic points

Definition 1 *A vector* $\widetilde{\mathcal{B}} = [i_1, i_2, \ldots, i_p]$, *whose elements are* p *integers chosen from* $1, 2, \ldots, 2p$, *is said to be a* p*-dimensional periodic basis vector for the problem* $\mathcal{P}(\cdot)$ *if the following three conditions are satisfied:*

1. $Z(\widetilde{\mathcal{B}}) := \begin{bmatrix} D_{(1:p,1:p)} & N_{(1:p,1:p)} \end{bmatrix}_{\widetilde{\mathcal{B}}}$ *is non-singular;*
2. *there is an eigenvector,* $\tilde{b}$, *of* $G(\widetilde{\mathcal{B}}) := \left[I_p - Y(\widetilde{\mathcal{B}})[Z(\widetilde{\mathcal{B}})]^{-1}\right]_{(1:l,1:l)}$ *with corresponding simple eigenvalue* $\lambda \in [0, 1)$; *and*
3. $Y(\widetilde{\mathcal{B}})$ *is an optimal basis for* $DMD(\tilde{b}, p)$.

An algorithm based on these conditions can determine if a candidate basis exhibits p-dimensional periodicity. Further details on the third condition, which involves testing for satisfaction of a complementarity condition between primal and dual basic solutions, is given elsewhere [3]. All three conditions can be tested exactly using symbolic software.

Definition 2 *The* $l \times 1$ *vector* $\tilde{b}$ *in Definition 1, associated with* $\widetilde{\mathcal{B}}$, *is termed a periodic initial condition of order* p, *or simply a periodic point, for the program* $\mathcal{P}(\cdot)$.

Definition 3 *Suppose that* $\tilde{b}$ *is a periodic initial condition of order* p *for* $\mathcal{P}(\cdot)$. *An optimal solution for* $MD(\tilde{b}, p)$ *will be termed a periodic dual vector corresponding to* $\tilde{b}$, *and will be denoted* $(\tilde{e}^*, \tilde{u}^*)$.

Thus $(\tilde{e}^*, \tilde{u}^*) \in \arg\max MD(\tilde{b}, p)$. *The dual initial condition associated with* $(\tilde{e}^*, \tilde{u}^*)$ *is* $\tilde{b}^* := [N_{LT}]^T \tilde{e}^*_{(1:l)} - [D_{LT}]^T \tilde{u}^*_{(1:l)}$.

In the following Definition $\rho(G(\widetilde{\mathcal{B}}))$ denotes the spectral radius of $G(\widetilde{\mathcal{B}})$. The eigenvalue λ is the Perron–Frobenius eigenvalue of $G(\widetilde{\mathcal{B}})$.

Definition 4 *If a periodic point $\tilde{b}$ of order p satisfies the additional property that $\rho(G(\tilde{\mathcal{B}})) = \lambda$, where $\lambda \in (0,1)$ is the eigenvalue of $G(\tilde{\mathcal{B}})$ associated with $\tilde{b}$, then $\tilde{b}$ is said to be an attracting periodic point of period p for the program $\mathcal{P}$.*

The following theorem has been proved [3]. It shows that for every periodic initial condition $\tilde{b}$ there is an optimal solution for $\mathcal{P}(\tilde{b})$ which satisfies a pth order recurrence relation, where p is the order of the periodic point.

Theorem 5 *Suppose $\tilde{b}$ is a periodic initial condition of order p for the program $\mathcal{P}(\cdot)$, with corresponding periodic basis $\tilde{\mathcal{B}}$, and corresponding eigenvalue $\lambda \in [0,1)$. The optimizing solution vector for $\mathcal{D}(\tilde{b})$ is $(\tilde{e}^*_{ext}, \tilde{u}^*_{ext})$, the infinite periodic extension of the periodic dual vector corresponding to $\tilde{b}$. The optimal values for the programs $\mathcal{P}(\tilde{b})$ and $\mathcal{D}(\tilde{b})$ are equal to $\tilde{b}^* W \tilde{b}$, which is also the optimal value for the program $MD(\tilde{b}, p)$. Denote by (e^{bsol}, u^{bsol}) the basic solution, with respect to the basis $\tilde{\mathcal{B}}$, to the equations $D_{(1:p,1:p)} e + N_{(1:p,1:p)} u = \begin{bmatrix} \tilde{b} \\ 0_{(p-l)\times 1} \end{bmatrix}$. Then an optimizing vector for $\mathcal{P}(\tilde{b})$ is*

$$e^{(opt)} = \begin{bmatrix} e^{bsol} \\ \lambda e^{bsol} \\ \lambda^2 e^{bsol} \\ \vdots \end{bmatrix}, \quad u^{(opt)} = \begin{bmatrix} u^{bsol} \\ \lambda u^{bsol} \\ \lambda^2 u^{bsol} \\ \vdots \end{bmatrix}. \tag{8}$$

5 Example

We illustrate the results in this article for the problem $\mathcal{P}(\cdot)$ having the following given data:

- $\widehat{d} = (1 + z/2)(1 + 2z/9)(1 - z/5) = 1 + 47/90z - 1/30z^2 - 1/45z^3$

- $\widehat{n} = (1 - z/3)(1 - 2z/7)(1 - 2z/5) = 1 - 107/105z + 12/35z^2 - 4/105z^3$
- $K_1 = 1$.

We initially take $K_2 = 3/2$ and show that there is an attracting periodic basis of period 7. Putting $K_2 = 7/5$ there is no longer any attracting periodic point having period 15 or less. However, there are many repelling periodic points, for example about 100 having period 15 or less. The question arises: where has the attracting periodic point gone? Visual inspection of a solution of length 50 shows an apparent periodicity of 30 in the optimal solution. However, although there is indeed a periodic point of period 30, it is not an attractor; the orbit will move away from this point eventually. For some values of K_2 the evidence so far available is consistent with the existence of an infinite number of points having associated unstable manifolds, and no periodic attracting periodic points. The well known metaphor of the pin-ball machine may be applicable, with the orbit of the dynamically evolving dual optimal variables being repelled in the manner in which the pins in a pin-ball machine repel the motion of the ball. Such a situation is often taken as being suggestive of chaos.

5.1 Periodic bases for the example

I test for periodic bases by testing exhaustively all possible selections of p from $2p$ integers. This involves, for a given p and a given candidate basis vector $\mathcal{B}$, testing for satisfaction of Conditions 1, 2 and 3 of Definition 1. It is found that, for K_2 lying between 1.4182 and 1.5040 , there are 12 periodic basis vectors of period 7, with only one of these being an attractor.

The periodic basis vector with corresponding attracting periodic initial condition is $\tilde{\mathcal{B}}_1 = [1, 2, 4, 6, 8, 10, 12]$ with

$$\tilde{b}_1 = \left[\frac{28655197 + \sqrt{203277932802849}}{4606468}, -1, 0 \right]^T .$$

Then $H(\tilde{\mathcal{B}}_1) := I_7 - YZ^{-1} = [F(7)]_{\tilde{\mathcal{B}}_1} Z^{-1}$ so

$$G(\tilde{\mathcal{B}}_1) = \left[H(\tilde{\mathcal{B}}_1)\right]_{(1:3,1:3)} = \begin{bmatrix} \frac{85\,550\,647}{5604\,802\,627\,715} & \frac{13\,412\,497}{1120\,960\,525\,543} & \frac{94\,742\,181}{1120\,960\,525\,543} \\ -\frac{2303\,234}{5604\,802\,627\,715} & \frac{11\,379\,090}{1120\,960\,525\,543} & \frac{53\,634\,490}{1120\,960\,525\,543} \\ 0 & 0 & 0 \end{bmatrix}$$

and one of the eigenvectors of $G(\tilde{\mathcal{B}}_1)$ is $\tilde{b}_1$ with corresponding eigenvalue

$$\lambda_1 = \frac{1}{11\,209\,605\,255\,430}\sqrt{203\,277\,932\,802\,849} + \frac{7497\,163}{589\,979\,223\,970}\,.$$

Furthermore $DMD(\tilde{b}_1, 7)$ has $\tilde{\mathcal{B}}_1$ as an optimal basis vector. Hence $\tilde{\mathcal{B}}_1$ is a periodic basis vector for $\mathcal{P}(\cdot)$.

Furthermore $\tilde{b}_1$ is an *attracting* periodic initial condition of period 7 for $\mathcal{P}(\cdot)$. This follows from the fact that the magnitude λ_1 is greater than the magnitude of the other eigenvalues of $G(\tilde{\mathcal{B}}_1)$. The periodic dual vectors corresponding to $\tilde{b}_1$ are

$$\tilde{e}^* = \begin{bmatrix} 1 \\ -1 \\ -\frac{2919970251055}{7472880243352} \\ 1 \\ -\frac{2328578260779}{7472880243352} \\ -1 \\ \frac{11039165480207}{11209320365028} \end{bmatrix}, \quad \tilde{u}^* = \begin{bmatrix} 3/2 \\ \frac{535046818800}{934110030419} \\ -3/2 \\ \frac{2517453809853}{26155080851732} \\ 3/2 \\ -\frac{24958186655649}{26155080851732} \\ -\frac{15166976335361}{13077540425866} \end{bmatrix},$$

the dual periodic initial condition is

$$\begin{aligned} b^* &:= [N_{\text{LT}}]^T \tilde{e}^*_{(1:3)} - [D_{\text{LT}}]^T \tilde{u}^*_{(1:3)} \\ &= \left[-\frac{1}{210}, \frac{58022561938213}{130775404258660}, -\frac{168276016373615}{78465242555196}\right], \end{aligned}$$

and the optimal cost is therefore

$$\begin{aligned} J_{\mathcal{D}}(\tilde{b}_1) &= b^* W \tilde{b}_1 \\ &= \tfrac{3628076767334478994 5}{4302947963604150092} + \tfrac{1185853695115\sqrt{203277932802849}}{4302947963604150092}. \end{aligned} \tag{9}$$

Now consider the program $\mathcal{P}(\bar{b})$ where $\bar{b} = (1, 0, 0)$. It can be shown that, because $\tilde{b}_1$ is an attracting periodic initial condition, there is an open neighbourhood surrounding $\tilde{b}_1$ for which $(\tilde{e}^*, \tilde{u}^*) \in \arg\max \mathcal{D}(\bar{b})$, and furthermore that $\bar{b}$ belongs to this neighbourhood. In other words, the optimizing vectors for $\mathcal{D}(\tilde{b})$ and $\mathcal{D}(\bar{b})$ are the same. The optimal value for the program $\mathcal{P}(\bar{b})$ is $J_{\mathcal{D}}(\bar{b}) = b^* W[1, 0, 0]^T = 1185853695115/934110030419$.

If the initial condition b moves sufficiently far away from $\tilde{b}_1$, then $(\tilde{e}^*, \tilde{u}^*)$ will no longer be optimal for $\mathcal{D}(b)$. Nevertheless, after an initial aperiodic transient, the optimizing vectors for $\mathcal{D}(b)$ will eventually be identical with $(\tilde{e}^*, \tilde{u}^*)$; see Figure 1. This is again a consequence of the stability of $\tilde{b}_1$.

5.2 Disappearance of the attracting periodic point

Now consider the effect of putting K_2 equal to a value outside the interval $[1.4182,\ 1.5040]$. Consider the problem $\mathcal{P}(\cdot)$ defined by keeping $\widehat{d}$, $\widehat{n}$ and K_1 the same as in the Example, but changing K_2 to $7/5$. An exhaustive search failed to find any attracting periodic initial conditions for this problem. However, there are plenty of repelling periodic points of all periods so far tested, which is up to about 15. The total number of periodic points so far found is more than one hundred. Whether the total number is finite or not is an open question.

A typical plot is show in Figure 2, for which $b = (1, 0, 0)$. Although b is not apparently a periodic initial condition, it is close to a repelling periodic initial condition of period 30.

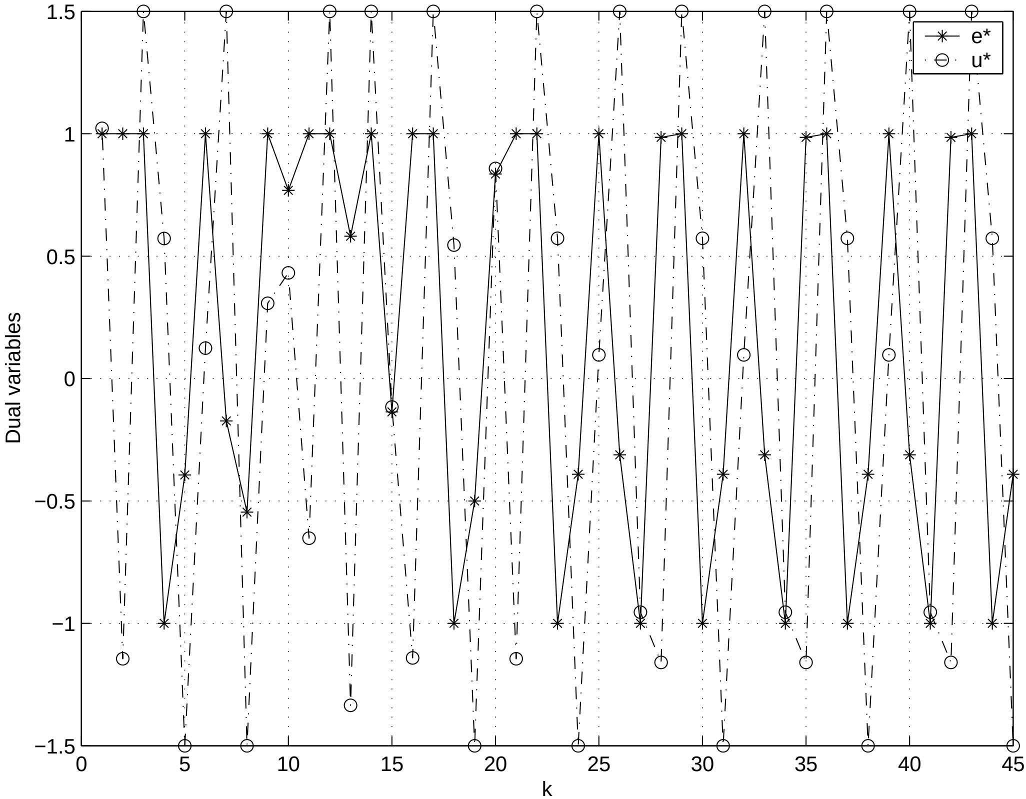

FIGURE 1: The initial condition is $b = (1, 1, 1)$ and $K_2 = 3/2$. There is a stable fixed point initial condition of period 7. After an initial aperiodic transient, the dual optimal variables are indeed periodic with period 7.

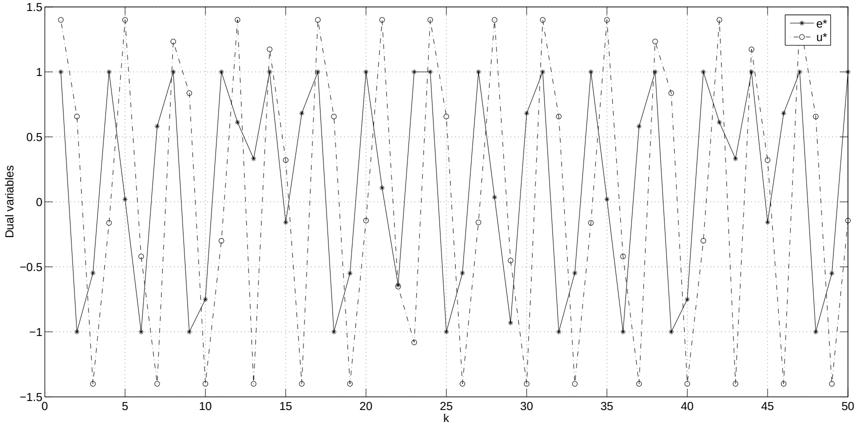

FIGURE 2: $b = (1, 0, 0)$ and $K_2 = 7/5$. The initial condition is near an unstable fixed point of period 30. Although the optimal solution is initially periodic with period 30, it cannot remain so because the eigenvalue associated with the basis vector of period 30 is not a Perron–Frobenius eigenvalue.

It can be verified that the basis of period 30 implied by the first 30 values of the response in Figure 2, denoted $\mathcal{B}_{30}$, is a periodic basis satisfying the conditions of Definition 1. Thus there is an eigenvector of $G(\mathcal{B}_{30})$ which is a repelling periodic point. The eigenvalue corresponding to this periodic initial condition is (approximately) 1.04×10^{-21}. The other two eigenvalues (corresponding to the other eigenvectors of $G(\mathcal{B}_{30})$, neither of which are periodic points) are zero and -4.44×10^{-20}. Thus, unless the initial condition lies exactly in the subspace spanned by the eigenvectors corresponding to the eigenvalues zero and 1.04×10^{-21}(and for this Example they do not lie exactly in this subspace), evolution of e and u according to $\mathcal{B}_{30}$ implies that the response must eventually align itself with the eigenvector associated with the eigenvalue -4.44×10^{-20}. But the optimal solution cannot align itself with this eigenvector because it is not a periodic initial condition. If the plot in Figure 2 were to be continued sufficiently far, the periodicity of $\mathcal{B}_{30}$ would necessarily be lost. For the time being the ultimate alignment of $b^{(M)}(e^{\mathcal{P}(b)}, u^{\mathcal{P}(b)})$ for this example remains unknown.

6 Conclusions

For a specific l_1-norm minimization problem, having cubic polynomials as given problem data, a weighting on the cost function has been found for which the optimal solution displays features suggestive of chaos. For initial conditions b in the neighbourhood of an attracting periodic initial condition $\tilde{b}$ the mapping from b to $\arg\min \mathcal{P}(b)$ is linear; it is given explicitly by the results described in this article. For other initial conditions the mapping from b to $\arg\min \mathcal{P}(b)$ is non-linear. Very little is known about this non-linear map. Characterizing it in special cases is a topic of current research.

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