Monotone alternating direction implicit method for nonlinear integro-parabolic equations

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Abstract

The paper deals with numerical solving nonlinear integro-parabolic problems based on an alternating direction implicit (ADI) scheme. A monotone iterative ADI method is constructed. An analysis of convergence of the monotone iterative ADI method is given.

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Keywords: integro-parabolic problems; alternating direction implicit (ADI) scheme; monotone iterative ADI method

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Contents

1 Introduction

We consider the nonlinear integro-parabolic problem

$$u_t - (u_{x_1x_1} + u_{x_2x_2}) + f(x,t,u) + \int_0^t g^*(x,t,s,u(x,s))ds = 0, \qquad (1)$$
$$(x,t) \in \omega \times (0,T], \quad \omega = \{0 \leqslant x_\nu \leqslant l_\nu, \nu = 1,2\},$$
$$u(x,t) = 0, \quad (x,t) \in \partial\omega \times (0,T], \quad u(x,0) = \psi(x), \quad x \in \overline{\omega},$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $\partial\omega$ is the boundary of ω, and the functions f and g^* satisfy the Lipschitz continuity condition. Various reaction-diffusion-type problems in chemical, physical and engineering sciences are described by problem (1) [5].

Alternating direction implicit (ADI) methods are very efficient methods for solving two or three dimensional parabolic problems. At each time-step, the ADI method reduces two or three dimensional problems to a succession of one dimensional problems, and, usually, one needs only to solve a sequence of tridiagonal systems. We have previously constructed a nonlinear ADI scheme, based on the Douglas–Rachford ADI scheme [2], for solving nonlinear parabolic problems [1]. In this paper, we extend the monotone ADI approach of Boglaev [1] to nonlinear integro-parabolic problems. Our iterative scheme is based on the method of upper and lower solutions and associated monotone iterates. We formulate a nonlinear ADI scheme for the numerical solution of (1). A monotone iterative ADI method for the nonlinear ADI scheme is then given. Convergence analysis of the monotone ADI method is discussed, before finally presenting the results of numerical experiments.

2 The monotone ADI method

2.1 The statement of the iterative ADI method

Let $i = (i_1, i_2)$ be a multiple index with $i_\nu = 0, 1, \ldots, M_\nu$, $\nu = 1, 2$. Introduce the uniform mesh

$$\begin{aligned} &\overline{\omega}^h = \{x_i = (x_{i_1}, x_{i_2}), i_\nu = 0, 1, \ldots, M_\nu, h_\nu = l_\nu / M_\nu, \nu = 1, 2\}, \\ &\overline{\omega}^\tau = \{t_k = k\tau, 0 \leqslant k \leqslant N, t_0 = 0, t_N = T\}, \end{aligned} \tag{2}$$

where h_ν, $\nu = 1, 2$, and τ are, respectively, space and time steps. When no confusion arises, we write $i \in \overline{\omega}^h$ and $k \in \overline{\omega}^\tau$, instead of, respectively, $x_i \in \overline{\omega}^h$ and $t_k \in \overline{\omega}^\tau$. Set

$$u_{i,k} \equiv u(i_1 h_1, i_2 h_2, k\tau), \quad f_{i,k}(u_{i,k}) \equiv f(i_1 h_1, i_2 h_2, k\tau, u_{i,k}).$$

We approximate the integral in (1) by the finite sum g based on the Riemann sum (the rectangular rule)

$$g_{i,k}(u_{i,k}) = \sum_{l=1}^{k} \tau g^*(i_1 h_1, i_2 h_2, k\tau, l\tau, u_{i,l}).$$

For solving (1), consider the nonlinear two-level implicit difference scheme

$$\begin{aligned} &\tau^{-1}\left[U_{i,k} - U_{i,k-1}\right] + \mathcal{L}^h U_{i,k} + \Phi_{i,k}(U_{i,k}) = 0, \quad i \in \omega^h, \quad k \geqslant 1, \\ &U_{i,k} = 0, \quad i \in \partial\omega^h, \quad k \geqslant 1, \quad U_{i,0} = \psi_i, \quad i \in \overline{\omega}^h, \quad \Phi_{i,k} \equiv f_{i,k} + g_{i,k}. \end{aligned} \tag{3}$$

The difference operator $\mathcal{L}^h = \mathcal{L}^h_1 + \mathcal{L}^h_2$ is defined as follows

$$\mathcal{L}^h_\nu U_{i,k} = h_\nu^{-2}[2U(x_i,t_k) - U(x_i + h_\nu e_\nu, t_k) - U(x_i - h_\nu e_\nu, t_k)], \quad \nu = 1,2,$$

where e_ν is the unit vector in the x_ν-direction, $\nu = 1,2$.

On each time level $k \geqslant 1$, introduce the linear difference problem

$$(\mathcal{L}^h + (\tau^{-1} + c_{i,k}))W_{i,k} = \Xi_{i,k}, \quad i \in \omega^h, \quad W_{i,k} = 0, \quad i \in \partial\omega^h, \tag{4}$$

where $c_{i,k} \geqslant 0$, $i \in \overline{\omega}^h$ and $\Xi_{i,k}$, $i \in \overline{\omega}^h$ is an arbitrary mesh function. We formulate the maximum principle and give an estimate to the solution of (4).

Lemma 1. *(i) If a mesh function $W_{i,k}$ satisfies the conditions*

$$(\mathcal{L}^h + (\tau^{-1} + c_{i,k}))W_{i,k} \geqslant 0 \ (\leqslant 0), \quad i \in \omega^h, \quad W_{i,k} \geqslant 0 \ (\leqslant 0), \quad i \in \partial\omega^h,$$

then $W_{i,k} \geqslant 0$ $(\leqslant 0)$ $i \in \overline{\omega}^h$.

(ii) The following estimate to the solution to (4) holds

$$\|W_k\|_{\overline{\omega}^h} \leqslant \max_{i\in\omega^h} \frac{|\Xi_{i,k}|}{c_{i,k} + \tau^{-1}}, \quad \|W_k\|_{\overline{\omega}^h} \equiv \max_{i\in\omega^h} |W_{i,k}|. \tag{5}$$

The proof of the lemma has been previously presented by Samarskii [6].

For solving (3), we use the nonlinear ADI scheme

$$\begin{aligned}
&\mathcal{L}_1 U^*_{i,k} = \tau^{-1} U_{i,k-1}, \quad i \in \omega^h, \\
&U^*_{(0,M_1),i_2,k} = 0, \quad i_2 = 1,\dots,M_2 - 1, \\
&\mathcal{L}_2 U_{i,k} = \tau^{-1} U^*_{i,k} - \Phi_{i,k}(U_{i,k}), \quad i \in \omega^h, \\
&U_{i_1,(0,M_2),k} = 0, \ i_1 = 1,\dots,M_1 - 1, \\
&U_{i,0} = \psi_i, \quad i \in \overline{\omega}^h, \quad k \geqslant 1, \quad \mathcal{L}_\nu \equiv \mathcal{L}^h_\nu + \tau^{-1}, \quad \nu = 1,2.
\end{aligned} \tag{6}$$

We have to solve $M_2 - 1$ linear systems in the x_1-direction and $M_1 - 1$ nonlinear systems in the x_2-direction, for, respectively, $U^*_{i,k}$ and $U_{i,k}$.

Remark 2. The class of ADI methods belongs to the class of splitting methods which have a number of generic forms, e.g., splitting linear terms from nonlinear terms, splitting terms corresponding to different physical processes, splitting x_1-direction from x_2-direction (dimensional splitting is the location of ADI methods), spitting a large domain into smaller pieces (domain decomposition) [3, 4].

Mesh functions $\widetilde{U}_{i,k}$, $\widetilde{U}^*_{i,k}$ and $\widehat{U}_{i,k}$, $\widehat{U}^*_{i,k}$ are ordered upper and lower solutions of (6), if they satisfy $\widetilde{U}_{i,k} \geqslant \widehat{U}_{i,k}$, $\widetilde{U}^*_{i,k} \geqslant \widehat{U}^*_{i,k}$, $i \in \overline{\omega}^h$, $k \geqslant 1$, and

$$\begin{aligned}
&\mathcal{L}_1 \widehat{U}^*_{i,k} - \frac{1}{\tau}\widehat{U}_{i,k-1} \leqslant 0 \leqslant \mathcal{L}_1 \widetilde{U}^*_{i,k} - \frac{1}{\tau}\widetilde{U}_{i,k-1}, \quad i \in \omega^h, \qquad (7)\\
&\widehat{U}^*_{(0,M_1),i_2,k} \leqslant 0 \leqslant \widetilde{U}^*_{(0,M_1),i_2,k}, \quad i_2 = 1, \ldots, M_2 - 1,\\
&\mathcal{L}_2 \widehat{U}_{i,k} - \frac{1}{\tau}\widehat{U}^*_{i,k} + \Phi_{i,k}(\widehat{U}_{i,k}) \leqslant 0 \leqslant \mathcal{L}_2 \widetilde{U}_{i,k} - \frac{1}{\tau}\widetilde{U}^*_{i,k} + \Phi_{i,k}(\widetilde{U}_{i,k}), \quad i \in \omega^h,\\
&\widehat{U}_{i_1,(0,M_2),k} \leqslant 0 \leqslant \widetilde{U}_{i_1,(0,M_2),k}, \quad i_1 = 1, \ldots, M_1 - 1, \quad k \geqslant 1,
\end{aligned}$$

We note that in some literature upper and lower solutions are called supersolution and subsolution. Assume that f and g^* satisfy the constraints

$$\frac{\partial f}{\partial u}(x,t,u) \leqslant c(x,t), \quad 0 \leqslant -\frac{\partial g^*}{\partial u}(x,t,u), \quad (x,t,u) \in \omega \times [0,T] \times (-\infty, \infty), \qquad (8)$$

where $c(x,t)$ is a nonnegative bounded function in $\omega \times [0,T]$.

For solving (6), we calculate iterates $V^{(n)}_{i,k}$, $n \geqslant 1$, by using the recurrence formulae

$$\mathcal{L}_1 V^*_{i,k} = \tau^{-1} V_{i,k-1}, \quad i \in \omega^h, \quad V^*_{(0,M_1),i_2,k} = 0, \quad i_2 = 1, \ldots, M_2 - 1, \tag{9}$$
$$(\mathcal{L}_2 + c_{i,k}) Z^{(n)}_{i,k} = -\mathcal{R}_{i,k}(V^{(n-1)}_{i,k}), \quad i \in \omega^h,$$
$$\mathcal{R}_{i,k}(V^{(n-1)}_{i,k}) \equiv \mathcal{L}_2 V^{(n-1)}_{i,k} + \Phi_{i,k}(V^{(n-1)}_{i,k}) - \tau_k^{-1} V^*_{i,k},$$
$$Z^{(1)}_{i_1,(0,M_2),k} = -V^{(0)}_{i_1,(0,M_2),k}, \quad Z^{(n)}_{i_1,(0,M_2),k} = 0, \quad i_1 = 1, \ldots, M_1 - 1, \quad n \geqslant 2,$$
$$V^{(n)}_{i,k} = V^{(n-1)}_{i,k} + Z^{(n)}_{i,k}, \quad V_{i,k} = V^{(n_k)}_{i,k}, \quad V_{i,0} = \psi_i, \quad i \in \overline{\omega}^h,$$

where $\mathcal{R}_{i,k}(V^{(n-1)}_{i,k})$ is the residual of the difference scheme (6) on $V^{(n-1)}_{i,k}$, $V_{i,k-1}$ is an approximation of the exact solution on time level $k-1$, n_k is a number of iterative steps on time level k, and $c_{i,k}$ is defined in (8). We note that, if $\frac{\partial f_{i,k}}{\partial u_{i,k}}(V^{(n-1)}_{i,k})$ is in use instead of $c_{i,k}$ in (9), the iterative method becomes Newton's method. In general, Newton's method does not possess monotone property of iterative sequences which is a requirement for their convergence (see Theorem 4 below, for details).

2.2 Monotone property of the ADI method

We introduce the notation

$$F_{i,k}(U_{i,k}) = c_{i,k} U_{i,k} - \Phi_{i,k}(U_{i,k}). \tag{10}$$

Lemma 3. *Let $U_{i,k}$, $V_{i,k}$ be two mesh functions such that $\widehat{U}_{i,k} \leqslant V_{i,k} \leqslant U_{i,k} \leqslant \widetilde{U}_{i,k}$, and let (8) hold. Then for k fixed*

$$F_{i,k}(U_{i,k}) \geqslant F_{i,k}(V_{i,k}), \quad i \in \overline{\omega}^h. \tag{11}$$

We now prove the monotone property of the iterative method (9).

Theorem 4. *Assume that f and g^* satisfy (8), where $\widetilde{U}_{i,k}$ and $\widehat{U}_{i,k}$ are ordered upper and lower solutions (7) of (6). Then the sequences $\{\overline{V}^{(n)}_{i,k}\}$ with*

$\overline{V}^{(0)}_{i,k} = \widetilde{U}_{i,k}$ *and* $\{\underline{V}^{(n)}_{i,k}\}$ *with* $\underline{V}^{(0)}_{i,k} = \widehat{U}_{i,k}$ *generated by (9) are, respectively, ordered upper and lower solutions to (6) and converge monotonically*

$$\underline{V}^{(n-1)}_{i,k} \leqslant \underline{V}^{(n)}_{i,k} \leqslant \overline{V}^{(n)}_{i,k} \leqslant \overline{V}^{(n-1)}_{i,k}, \quad i \in \overline{\omega}^h, \quad n \geqslant 1, \quad k \geqslant 1. \tag{12}$$

Proof: Let $W^*_{i,k} = \widetilde{U}^*_{i,k} - V^*_{i,k}$, $k \geqslant 1$. From (7) and (9), it follows that

$$\mathcal{L}_1 W^*_{i,1} \geqslant 0, \quad i \in \omega^h, \quad W^*_{(0,M_1),i_2,1} \geqslant 0, \quad i_2 = 1, \dots, M_2 - 1.$$

By the maximum principle in Lemma 1, it follows that $W^*_{i,1} \geqslant 0$, $i \in \overline{\omega}^h$. From here and $\overline{V}^{(0)}_{i,k} = \widetilde{U}_{i,k}$ is an upper solution, we conclude that $\mathcal{R}_{i,1}(\widetilde{U}_{i,1}) \geqslant 0$, $i \in \omega^h$ in (9). From here and (9), we have

$$(\mathcal{L}_2 + c_{i,1})\overline{Z}^{(1)}_{i,1} \leqslant 0, \quad i \in \omega^h, \quad \overline{Z}^{(1)}_{i_1,(0,M_2),1} \leqslant 0, \quad i_1 = 1, \dots, M_1 - 1.$$

By Lemma 1, it follows that

$$\overline{Z}^{(1)}_{i,1} \leqslant 0, \quad i \in \overline{\omega}^h. \tag{13}$$

Similarly, we conclude that

$$\underline{Z}^{(1)}_{i,1} \geqslant 0, \quad i \in \overline{\omega}^h. \tag{14}$$

From (9), $\overline{V}_{i,0} = \underline{V}_{i,0} = \psi_i$, in the notation $W^{(n)}_{i,k} = \overline{V}^{(n)}_{i,k} - \underline{V}^{(n)}_{i,k}$, $n \geqslant 0$, we have

$$\begin{aligned} &(\mathcal{L}_2 + c_{i,1})W^{(1)}_{i,1} = F_{i,1}(\overline{V}^{(0)}_{i,1}) - F_{i,1}(\underline{V}^{(0)}_{i,1}), \quad i \in \omega^h, \\ &W^{(1)}_{i_1,(0,M_2),1} \geqslant 0, \quad i_1 = 1, \dots, M_1 - 1, \end{aligned}$$

where F is defined in (10). Since $\overline{V}^{(0)}_{i,1} \geqslant \underline{V}^{(0)}_{i,1}$, by Lemma 3, we conclude that the right hand side in the difference equation is nonnegative. The positivity property in Lemma 1 implies $W^{(1)}_{i,1} \geqslant 0$, $i \in \overline{\omega}^h$. From here, (13) and (14), we conclude (12) for $k = 1$, $n = 1$.

We now prove that $\overline{V}^{(1)}_{i,1}$ and $\underline{V}^{(1)}_{i,1}$ are, respectively, upper and lower solutions (7). Using the mean-value theorem, from (9) we obtain

$$\mathcal{R}_{i,1}(\overline{V}^{(1)}_{i,1}) = -\left(c_{i,1} - \frac{\partial f}{\partial u_{i,1}}(E^{(1)}_{i,1})\right)\overline{Z}^{(1)}_{i,1} + \tau\frac{\partial g_*}{\partial u_{i,1}}(Q^{(1)}_{i,1}),\overline{Z}^{(1)}_{i,1}, \tag{15}$$

where $\overline{V}^{(1)}_{i,1} \leqslant E^{(1)}_{i,1}, Q^{(1)}_{i,1} \leqslant \overline{V}^{(0)}_{i,1}$. From here, (12) for $k = 1$, $n = 1$, (13), (14), it follows that the partial derivatives satisfy (8). From (8), (13) and (15), we conclude that

$$\mathcal{R}_{i,1}(\overline{V}^{(1)}_{i,1}) \geqslant 0, \quad i \in \omega^h, \quad \overline{V}^{(1)}_{i_1,(0,M_2),1} = 0, \quad i_1 = 1, \ldots, M_1 - 1.$$

Thus, $V^{(1)}_{1}(p, t_1)$ is an upper solution. Similarly, we can prove that $V^{(1)}_{-1}(p, t_1)$ is a lower solution. By induction on n, we can prove that $\{\overline{V}^{(n)}_{i,1}\}$ is a monotonically decreasing sequence of upper solutions and $\{\underline{V}^{(n)}_{i,1}\}$ is a monotonically increasing sequence of lower solutions, which satisfy (12) for $k = 1$.

From (12) with $k = 1$, it follows that

$$\widehat{U}_{i,1} \leqslant \underline{V}^{(n_1)}_{i,1} \leqslant \overline{V}^{(n_1)}_{i,1} \leqslant \widetilde{U}_{i,1}, \quad i \in \overline{\omega}^h. \tag{16}$$

From here and by the assumption of the theorem that $\widetilde{U}^*_{i,2}$ and $\widehat{U}^*_{i,2}$ are, respectively, upper and lower solutions (7), we conclude that $\widetilde{U}^*_{i,2}$ and $\widehat{U}^*_{i,2}$ are upper and lower solutions with respect to $\overline{V}_{i,1} = \overline{V}^{(n_1)}_{i,1}$ and $\underline{V}_{i,1} = \underline{V}^{(n_1)}_{i,1}$

$$\mathcal{L}_1\widetilde{U}^*_{i,2} \geqslant \tau^{-1}\overline{V}_{i,1}, \quad \mathcal{L}_1\widehat{U}^*_{i,2} \leqslant \tau^{-1}\underline{V}_{i,1}, \quad i \in \omega^h. \tag{17}$$

From here and (9), in the notation $W^* = \widetilde{U}^* - V^*$, it follows that

$$\mathcal{L}_1 W^*_{i,2} \geqslant 0, \quad i \in \omega^h, \quad W^*_{(0,M_1),i_2,2} \geqslant 0, \quad i_2 = 1, \ldots, M_2 - 1.$$

By the maximum principle in Lemma 1, we have $W^*_{i,2} \geqslant 0$, $i \in \overline{\omega}^h$. From here and $V_1^{(0)} = \widetilde{U}$ is an upper solution, we conclude that $\mathcal{R}_{i,2}(\widetilde{U}_{i,2}) \geqslant 0$, $i \in \omega^h$ in (9). The proofs of the inequalities (compare with (13), (14) and (16))

$$\overline{Z}^{(1)}_{i,2} \leqslant 0 \leqslant \underline{Z}^{(1)}_{i,2}, \quad \underline{V}^{(1)}_{i,2} \leqslant \overline{V}^{(1)}_{i,2}, \quad i \in \overline{\omega}^h,$$

and the fact that $\overline{V}^{(1)}_{i,2}$ and $\underline{V}^{(1)}_{i,2}$ are, respectively, upper and lower solutions are similar to the proofs on time level $k = 1$. By induction on n, we prove that $\{\overline{V}^{(n)}_{i,2}\}$ and $\{\underline{V}^{(n)}_{i,2}\}$ are, respectively, monotonically decreasing and increasing sequences of upper and lower solutions, which satisfy (12) for $k = 2$.

By induction on k, $k \geqslant 1$, we can prove that $\{\overline{V}^{(n)}_{i,k}\}$ and $\{\underline{V}^{(n)}_{i,k}\}$ are, respectively monotonically decreasing and monotonically increasing sequences of upper and lower solutions, which satisfy (12). We prove the theorem. ♠

2.3 Convergence analysis of the ADI method

We assume that f and g^* satisfy the two-sided constraints

$$0 < c_* \leqslant \frac{\partial f}{\partial u}(x,t,u) \leqslant c^*, \quad 0 \leqslant -\frac{\partial g^*}{\partial u}(x,t,u) \leqslant q^*, \qquad (18)$$
$$(x,t,u) \in \omega \times [0,T] \times (-\infty, \infty),$$

where c_*, c^* and q^* are positive constants. We also assume that

$$\tau < \min(\sqrt{1/q^*}, c_*/q^*). \qquad (19)$$

Lemma 5. *Assume that f, g^* satisfy (18) and τ satisfies (19). Then the nonlinear ADI scheme (6) has a unique solution.*

We choose the stopping criterion of the ADI method (9) in the form

$$\|\mathcal{R}_k(V^{(n)}_{i,k})\|_{\omega^h} \leqslant \delta, \qquad (20)$$

where δ is a prescribed accuracy, and set up $V_{i,k} = V_{i,k}^{(n_k)}$, $i \in \overline{\omega}^h$, such that n_k is minimal subject to (20).

We state Gronwall's inequality from [7] in the following form.

Lemma 6. *Let $\{w_k\}$ be a sequence on nonnegative real numbers satisfying*

$$w_k \leqslant a_k + \sum_{l=1}^{k} b_l w_l, \quad k \geqslant 1,$$

where $\{a_k\}$ is a nondecreasing sequence of nonnegative numbers, and $b_l \geqslant 0$. Then

$$w_k \leqslant a_k \exp\left(\sum_{l=1}^{k} b_l\right), \quad k \geqslant 1.$$

Theorem 7. *Under the assumptions of Lemma 5, for the sequence $\{V_{i,k}^{(n)}\}$, generated by (9), (20), the following estimate holds:*

$$\max_{k \geqslant 1} \|V_k - U_k\|_{\overline{\omega}^h} \leqslant C(T)\delta, \tag{21}$$

where $U_{i,k}$ is the unique solution to (6).

Proof: We present the difference problem for $V_{i,k} = V_{i,k}^{(n_k)}$ in the form

$$\mathcal{L}_2 V_{i,k} + \Phi_{i,k}(V_{i,k}) - \tau^{-1} V_{i,k}^{*} = \mathcal{R}_{i,k}(V_{i,k}^{(n_k)}), \quad i \in \omega^h,$$
$$V_{i_1,(0,M_2),k} = 0, \quad i_1 = 1, \ldots, M_1 - 1.$$

From here, (6) and using the mean-value theorem, we get the following difference problems for $W^*_{i,k} = V^*_{i,k} - U_{i,k}$ and $W_{i,k} = V_{i,k} - U_{i,k}$:

$$\mathcal{L}_1 W^*_{i,k} = \tau^{-1} W_{i,k-1}, \quad i \in \omega^h, \quad W^*_{(0,M_1),i_2,k} = 0, \quad i_2 = 1, \dots, M_2 - 1, \tag{22}$$

$$\left(\mathcal{L}_2 + \frac{\partial f}{\partial u_{i,k}}(E_{i,k})\right) W_{i,k} = \mathcal{R}_{i,k}(V_{i,k}) + \frac{1}{\tau} W^*_{i,k} - \tau \sum_{l=1}^{k} \frac{\partial g^*}{\partial u}(Q_{i,l}) W_{i,l},$$
$$i \in \omega^h, \quad W_{i_1,(0,M_2),k} = 0, \quad i_1 = 1, \dots, M_1 - 1,$$

where the partial derivatives are calculated at intermediate points, which lie between $U_{i,l}$ and $V_{i,l}$, $1 \leqslant l \leqslant k$. From here, by using (5), we have $\|W^*_k\|_{\overline{\omega}^h} \leqslant \|W_{k-1}\|_{\overline{\omega}^h}$. From here, (18), by using (5) and taking into account that according to Theorem 4 the stopping criterion (20) can always be satisfied, we estimate $w_k \equiv \|W_k\|_{\overline{\omega}^h}$ from (22) in the form

$$w_k \leqslant w_{k-1} + \tau^2 q^* \sum_{l=1}^{k} w_l + \tau\delta.$$

From here and $w_0 = 0$, by induction on k, we prove the inequality

$$w_k \leqslant k\tau\delta + \tau^2 q^* \sum_{l=1}^{k} (k - l + 1) w_l.$$

By Lemma 6 with $a_k = k\tau\delta$, $k \geqslant 1$ and $b_l = \tau^2\rho(k - l + 1)$, $1 \leqslant l \leqslant k$, we get

$$w_k \leqslant (k\tau\delta) \exp\left(\tau^2 q^* \sum_{l=1}^{k} l\right).$$

From here and taking into account that $\sum_{l=1}^{k} l \leqslant k^2/2$, $k\tau \leqslant T$, we prove (21) with $C(T) = T\exp(q^* T^2/2)$. ♠

3 Numerical experiments

As a test problem, we consider (1) in the form

$$f = u^2, \quad g^* = -u/(1+u), \quad \psi = \sin(\pi x_1)\sin(\pi x_2),$$

where $T = 1$ and $l_\nu = 1$, $\nu = 1, 2$. The following functions $\widetilde{U}_{i,k} = 1$, $\widehat{U}_{i,k} = 0$, $i \in \overline{\omega}^h$, $k \geqslant 1$, are, respectively upper and lower solutions. From here, we have

$$0 \leqslant f_u = 2u \leqslant 2, \quad -g_u^* = 1/(1+u)^2 > 0, \quad 0 \leqslant u \leqslant 1.$$

Thus, we choose $c_{i,k} = 2$, $i \in \overline{\omega}^h$, $k \geqslant 1$, in the iterative method (9).

We discretize the differential problem by the finite difference approximation on an uniform space mesh with the step size $h_1 = h_2 = h$ ($N = 1/h$) and $\delta = 10^{-6}$ in (20). We compare the monotone iterative ADI method with the iterative method, where we employ the conjugate grADIent method with the preconditioner based on the incomplete LU factorization (ILUCG). In Table 1, for different values of N, we present execution times (CPU times) of the monotone iterative ADI and iterative ILUCG methods, where $\tau = h$. The data in the table indicate that the monotone iterative method executes much faster than the iterative ILUCG method.

Table 1: Numerical results for the test problem.

N	32	64	128	256	512
$T_{ADI}(s)$	1.61E-1	2.55E-1	1.38E0	1.21E1	9.61E1
$T_{ILUCG}(s)$	8.11E-1	5.78E0	4.76E1	6.10E2	6.41E3
T_{ILUCG}/T_{ADI}	5.04E0	2.27E1	3.45E1	5.04E1	6.67E1

References

[1] I. Boglaev, Monotone alternating direction implicit scheme for nonlinear parabolic problems. *BIT Numerical Mathematics*, **55**, 2015, 647-676. doi:10.1007/s10543-014-0529-6 C66

[2] J. Douglas, H. H. Rachford, On the numerical solution of heat condition problems in two and three variables. *Trans. Amer. Math. Soc.*, **82**, 1956, 421–439. C65

[3] G. I. Marchuk, *Splitting and alternating direction methods, Handbook of Numerical Analysis,* **1**, *197–462.* (Eds P. Ciarlet, J. Lions). North-Holland, Amsterdam, 1990. C68

[4] R. McLachlan, G. Quispel, Splittings methods. *Acta Numer.*, **11**, 2002, 341–434. doi:10.1017/S0962492902000053 C68

[5] C. V. Pao, *Nonlinear Parabolic and Elliptic Equations*. Plenum Press, New York, 1992. C65

[6] A. Samarskii, *The Theory of Difference Schemes*. Marcel Dekker, New York–Basel, 2001. C67

[7] I. H. Sloan, V. Thomee, Time distretization of an integro-differential equation of parabolic type. *SIAM J. Numer. Anal.*, **23**, 1986, 1052–1061. doi:10.1137/0723073 C73

[8] A. H. Stroud, *Approximate Calculation of Multiple Integrals*. Prentice–Hall, Englewood Cliffs, NJ, 1971.

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