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Asymptotic expansion and numerical solution for gravity waves in a porous medium

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Abstract

In this paper, we study wave interaction with a porous structure. A nonlinear diffusive wave equation is used to describe gravity surface wave propagation in a porous media. We solve the equations using asymptotic expansion method and numerically using a staggered finite volume method. We then derive the dispersion relation that holds for gravity waves inside a porous structure. This dispersion relation explains the diffusive mechanism of wave amplitude inside the porous structure. Analysis of the dispersion relation shows that amplitude reduction depends on porous medium parameters such as porosity, friction coefficient, length of the structure, and wave frequency. To validate our numerical scheme, we compare the wave reduction amplitude from the numerical result with the asymptotic solution. A good agreement of the comparison is observed. Furthermore, the numerical model is

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employed to investigate the effectiveness of porous media in dissipating wave energy of an incoming wave. The results from this paper can be used to determine the optimum dimension of the porous medium so that the incoming wave can be reduced as much as possible to protect shoreline.

Subject class: 76S05;81U30;35L05

Keywords: nonlinear diffusive wave equation; asymptotic expansion method; dispersion relation; staggered finite volume method

Contents

1 Introduction

Study on wave propagation passing through an emerged porous media has many practical applications in coastal and ocean engineering. One function of this porous structure is as a breakwater. Nowadays, mangrove forest as an emerged porous media has been planted in several beach areas for coastal protection. An emerged porous structure as a breakwater functions to reduce the amplitude of waves propagating through the structure. There is some research in this area, such as by Sollitt and Cross [9], Madsen [5], and Sulisz [10], where researchers have developed a mathematical model for waves propagation in a porous structure and studied wave reflection and transmission from the porous structures. Some researchers such as Fernando, et al [1], Van Gent [11], Lynett, et al [4], Pengzhi Lin and Karunarathna [2], Liu, et al [3], and Scarlatos and Singh [8] studied wave interaction with a porous structure experimentally and numerically using mild slope equation or potential theory with linear friction. Here, we investigate wave damping inside a porous media by using a simpler model that is based on a shallow water type model. We modified the shallow water equation by including fully non-linear frictional dissipation. However, by including nonlinear terms, the equation becomes relatively difficult to solve analytically. Therefore, we propose an asymptotic expansion approach to solve the equation. Once this is performed, we propose a non-dissipative numerical scheme based on a staggered finite volume method to solve our modified shallow water equations numerically. With this, we can further our study by implementing our numerical scheme to simulate wave in the porous media to investigate the wave reduction by a porous medium. Our relatively simple approach can now provide solutions that could be useful in practice, in either an operational or strategic manner for coastal protection using porous media.

2 Governing equations

In this section, the governing equation of the flow inside a porous medium will be discussed. The model we used here is based on our previous model that has been explained briefly in [6]. In our previous research, we divided the water

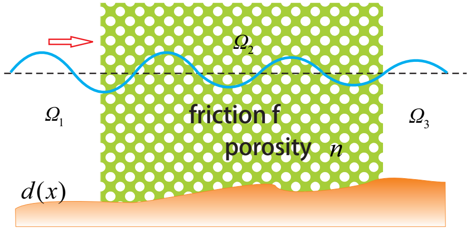

Figure 1: Sketch of the domain

domain into three regions: two free regions Ω_1, Ω_3 and one porous region Ω_2, see Figure 1. In this research, we will focus on free surface elevation inside the porous region Ω_2.

Let η and u denote surface elevation and horizontal fluid velocity, respectively. For the flow in porous media with dimensionless porosity n, the rate of change of the free surface η depends on the filtered velocity $\frac{u}{n}$. The porosity n is $0 < n \leqslant 1$. Porosity $n = 1$ means there is no porous media, and $n = 0$ is for rigid structure. We consider a shallow water type model over a flat bottom topography d that consists of mass conservation and momentum balance. In the momentum equation, we add a resistance term to the porous media. Here, we implement a frictional force as a resistance formulated by Dupuit-Forchheimer $(\alpha + \beta|u|)u$ as written in [11]. Hence, the full governing equations in the porous media are:

$$\eta_t + \frac{d}{n}u_x = 0, \tag{1}$$

$$\frac{1}{n}u_t + g\eta_x = -(\alpha + \beta|u|)u, \tag{2}$$

with g as the gravitational acceleration. The coefficient α expresses the laminar flow resistance whereas β expresses the turbulent flow resistance.

In order to observe long waves on shallow water, we introduce the scaled variables

$$\bar{x} = \frac{x}{L}, \quad \bar{t} = \frac{\sqrt{gd}}{L}t, \quad \bar{\eta} = \frac{\eta}{a}, \quad \bar{u} = \frac{d}{a\sqrt{gd}}u, \quad \bar{\alpha} = \frac{L}{\sqrt{gd}}\alpha, \quad \bar{\beta} = L\beta$$

, where L and a represent wavelength and amplitude. When simplifying we omit the overbars, and utilize small parameter $\varepsilon = a/d$. Therefore the governing equation (1) and (2) become

$$\eta_t + \frac{1}{n}u_x = 0, \tag{3}$$

$$\frac{1}{n}u_t + \eta_x = -(\alpha + \varepsilon\beta|u|)u. \tag{4}$$

3 Asymptotic expansion solutions

In this section, we derive the asymptotic solution for the non-linear porous shallow water equations. We assume the wave is a monochromatic wave with wave number k with a certain frequency ω. Now, we seek the solution of equation (3) and (4) in the following perturbation forms:

$$\eta = \eta_0 + \varepsilon\eta_1 + O(\varepsilon^2), \tag{5}$$

$$u = u_0 + \varepsilon u_1 + O(\varepsilon^2). \tag{6}$$

We implement the asymptotic expansion method to solve our governing equations (1, 2). Substituting (5-6) to (1, 2), we obtain

$$\eta_{0t} + \frac{1}{n}u_{0x} + \varepsilon\left(\eta_{1t} + \frac{1}{n}u_{1x}\right) \approx 0, \tag{7}$$

$$\frac{1}{n}u_{0t} + \eta_{0x} + \alpha u_0 + \varepsilon\left(\frac{1}{n}u_{1t} + \eta_{1x} + \alpha u_1 + \beta u_0^2\right) + 2\varepsilon^2\beta u_0 u_1 + \varepsilon^3 u_1^2 \approx 0. \tag{8}$$

Collecting order $O(1)$ terms yields

$$\eta_{0t} + \frac{1}{n}u_{0x} = 0, \tag{9}$$

$$\frac{1}{n}u_{0t} + \eta_{0x} + \alpha u_0 = 0. \tag{10}$$

Solving the corresponding equations starting from the first $O(1)$ terms above and assuming $\eta_0(x,t) = F(x)e^{i\omega t}$ and $u_0(x,t) = G(x)e^{i\omega t}$, the explicit approximate expression of η_0 and u_0 is then obtained as follows:

$$\eta_0(x,t) = a_1 e^{i(kx+\omega t)} + a_2 e^{-i(kx-\omega t)}, \tag{11}$$

$$u_0(x,t) = \frac{n}{\sqrt{1-if}}(a_1 e^{i(kx+\omega t)} - a_2 e^{-i(kx-\omega t)}). \tag{12}$$

Wave number in a porous media k follows this dispersion relation

$$k^2 = \omega^2(1 - i\alpha n/\omega). \tag{13}$$

Using the assumption that was used in Madsen and White (1976) [5], we approximate $\alpha = f\frac{\omega}{n}$. Thus the wave number in equation (13) is

$$\frac{\omega^2}{gk} = \frac{k}{g(1-if)}. \tag{14}$$

Collecting order $O(\epsilon)$ terms, we have

$$\eta_{1t} + \frac{1}{n}u_{1x} = 0, \tag{15}$$

$$\frac{1}{n}u_{1t} + \eta_{1x} + \alpha u_1 + \beta u_0^2 = 0. \tag{16}$$

Let $\eta_1(x,t) = C(x)H(t)$ and $u_1(x,t) = u_{1h}(x,t) + u_{1nh}(x,t)$, where $u_{1h}(x,t) = D_h(x)H_h(t)$ and $u_{1nh}(x,t) = D_{nh}(x)H_{nh}(t)$. Substitute these functions for η_1 and u_1 to equations (15-16). This will yield

$$H'C + \frac{d}{n}HD' = 0, \tag{17}$$

$$\frac{1}{n}H'D + gC'H + \alpha DH + \beta u_0^2 = 0. \tag{18}$$

The homogeneous case $u_0^2 = 0$, we use anzat for $H_h(t)$ is $e^{i\omega t}$, then the equation (17) and (18) without βu_0^2 and using definition $\alpha = f\frac{\omega}{n}$ become

$$C(x) = -\frac{d}{i\omega n}D_x, \tag{19}$$

$$D_{xx} + k^2 D = 0. \tag{20}$$

The homogeneous solution of equation (20) is

$$D_h(x) = a_3 e^{-ikx} + a_4 e^{ikx},$$

with k following the dispersion relation (13). Further, we are looking for the non-homogeneous solution for equation (18). For the non-homogeneous case, we assume our $H_{nh}(t) = e^{2i\omega t}$ that correlated with non-homogeneous term u_0^2. Using undetermined coefficient method and substituting solution of u_0 from equation (12) we obtain

$$U_{1_{nh}}(x,t) = (a_5 e^{-2i(kx-\omega t)} + a_6 e^{2i\omega t} + a_7 e^{2i(kx+\omega t)}),$$

where

$$a_5 = \frac{in\beta}{\omega}\frac{1}{-3(1-if)}a_1^2, \tag{21}$$

$$a_6 = \frac{2in\beta}{\omega}\frac{1}{(1-if)}a_1 a_2, \tag{22}$$

$$a_7 = \frac{in\beta}{\omega}\frac{1}{-3(1-if)}a_2^2. \tag{23}$$

Substitute $C(x), u_{1h} + u_{1nh}$, we will obtain solution for η_1 and u_1.

4 A staggered finite volume method

In this section, we will solve equations (3) and (4) numerically. To get the real amplitude reduction by the porous structure quantitatively accurate, we need a numerical scheme that doesn't include error damping. Here, a numerical finite volume method on a staggered grid will be implemented to simulate a diffusive wave in a porous medium. Consider the equation for gravity waves in a porous media (3, 4) in domain $[0, Lx]$. We discretize the porous domain in a staggered way $0 = x_{1/2}, x_1, \ldots, x_{Nx+1/2} = Lx$. Mass conservation (1) is approximated at a cell centered at x_i whereas momentum conservation (2) is approximated at a cell centered at $x_{i+1/2}$, see Figure 2. In this setting, values of η will be computed at every full grid points x_i, with $i = 1, 2, ..., Nx$ using mass conservation (24). Velocity u will be computed at every staggered grid points $x_{i+\frac{1}{2}}$, with $i = 1, 2, ..., Nx - 1$ using momentum equation (25). Approximate equations are then

$$\frac{\eta_i^{n+1} - \eta_i^n}{\Delta t} + \frac{1}{n}\frac{u_{i+1/2}^n - u_{i-1/2}^n}{\Delta x} = 0, \qquad (24)$$

$$\frac{\frac{1}{n}u_{i+1/2}^{n+1} - \frac{1}{n}u_{i+1/2}^n}{\Delta t} + \frac{\eta_{i+1}^{n+1} - \eta_i^{n+1}}{\Delta x} + (\alpha + \beta|u|^n)|_{i+1/2}u_{i+1/2}^{n+1} = 0. \qquad (25)$$

In case a fully nonlinear friction term is used, then discretize the form of $(\alpha + \beta|u|)u$ by implementing the Picard linearization $(\alpha + \beta|u^n)u^{n+1}$.

Further, to simulate the gravity waves in the free water area, the approximate equations are just (24) and (25) with $n = 1$ and $f = 0$.

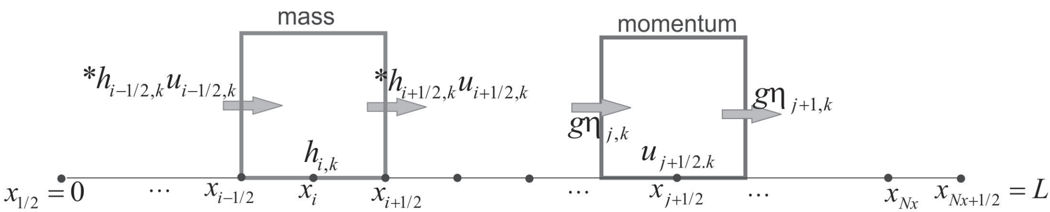

Figure 2: Illustration of staggered grid with cell $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ for mass conservation and cell $[x_{i-1}, x_i]$ for momentum equation.

Here, we derive stability condition for (24, 25). Note that the friction term is calculated implicitly in order to avoid a more restricted stability condition. Let $\eta_j^n = \rho^n e^{iaj}$, $u_{j+\frac{1}{2}}^n = r^n e^{ia(j+\frac{1}{2})}$ and substitute these into (24) and (25). This leads to

$$\begin{pmatrix} 1 & 0 \\ \frac{\Delta t}{\Delta x}2i\sin\frac{a}{2} & 1 \end{pmatrix}\begin{pmatrix} \rho^{n+1} \\ r^{n+1} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\Delta t}{\Delta x}2i\sin\frac{a}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \rho^n \\ r^n \end{pmatrix},$$

or

$$\begin{pmatrix} \rho^{n+1} \\ r^{n+1} \end{pmatrix} = A\begin{pmatrix} \rho^n \\ r^n \end{pmatrix}.$$

The multiplication of matrix A is

$$A = \begin{pmatrix} 1 & -\frac{\Delta t}{\Delta x} 2i \sin\frac{a}{2} \\ -\frac{\Delta t}{\Delta x} 2i \sin\frac{a}{2} & 1 - \left(\frac{\Delta t}{\Delta x}\right)^2 4\sin^2\frac{a}{2} \end{pmatrix}, \tag{26}$$

and eigenvalues λ of matrix A must satisfy

$$(\lambda - 1)(\lambda - 1 + C^2) + C^2 = 0. \tag{27}$$

If $|C| \leqslant 2$ in equation (27) where $C^2 = gd\left(\frac{\Delta t}{\Delta x}\right)^2 4\sin^2\frac{a}{2}$, then λ is a complex number with $|\lambda| = 1$. Thus, the stability condition for (24, 25) is $\frac{\Delta t}{\Delta x} \leqslant 1$. In this case, the complex eigen values λ have norm equal to one, so this scheme is non-dissipative or free from numerical damping error [7].

5 Numerical simulation

In this section, we implement the above scheme to simulate the free surface inside a porous media. Here, we use non-dimensional quantities. For simulation, we take a computational domain $0 < x < 10$. We take $\varepsilon = a/d = 1/10 = 0.1$. The initial condition is still water level $\eta(x, 0) = 0, u(x, 0) = 0$, and for the left wave influx we take a monochromatic wave with amplitude 1

$$\eta(0, t) \;\; = \;\; e^{-i\omega t}, \tag{28}$$

with $\omega = 3$. Along the right boundary, we apply an absorbing boundary. The whole domain is a porous media with parameters $n = 0.9, \alpha = 1, \beta = 0.1$ and for computations we use $\Delta x = 0.1, \Delta t = \Delta x = 0.1$. Figure 3 shows a wave amplitude reduction in the porous medium.

Further, we show that our numerical surface profile reduces in the porous media with an envelope that confirms the $|\eta(x, t)|$ from dispersion relation and asymptotic expansion solution. Taking parameter values $\omega = 3$, $n = 0.9, \alpha = 1, \beta = 0.1$ the dispersion relation (13) will give us a complex value wave number $k = 3.0328 - 0.4451\ i$. A monochromatic wave $\exp^{-i(kx - \omega t)}$ with a negative imaginary part

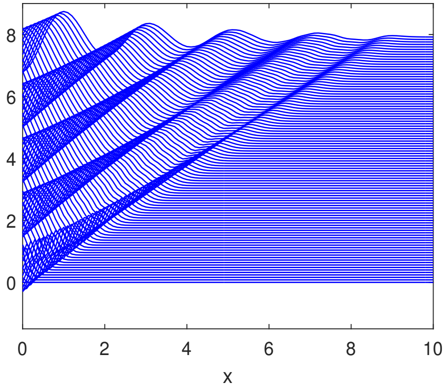

Figure 3: Damping of wave amplitude inside the porous media.

$\mathfrak{I}(k)$ will then undergo an amplitude reduction, see Figure 4. Let $K_T = |\eta(x,t)| = \exp \mathfrak{I}(k)x$, with K_T as the term that denotes amplitude reduction of incident wave as a function of x, that is, the length of a porous media. It is now clear that wave damping depends strongly on the complex wave number k. Parameters involved in (13) are wave frequency ω, porosity n, and linear friction α.

For numerical computations, we take the parameters used before and we use $\Delta x = 0.1$, and $\Delta t = \Delta x$. The surface profile in a porous media is plotted in Figure 4. It is seen that the numerical wave amplitude reduction is in a good agreement with the solution from asymptotic expansion. We also see from Figure 4, that the longer the porous media Lx is, the bigger the wave damping or the smaller the K_T. Wave damping also depends on porosity n, friction coefficient f, and wave number k. We made another comparison amongst the dependance of wave damping with n and $f = \alpha\frac{\omega}{n}$. For the computation, we take $\omega = 12$ and $Lx = 10$. We plot the curve of wave amplitude $|\eta|$ with respect to porosity n for several values of f. From Figure 5 (Left), we conclude that for a certain porosity n, larger friction coefficient will lead to a smaller wave transmission coefficient or larger damping. Larger porosity n will yield larger K_T, see Figure 5 (Right).

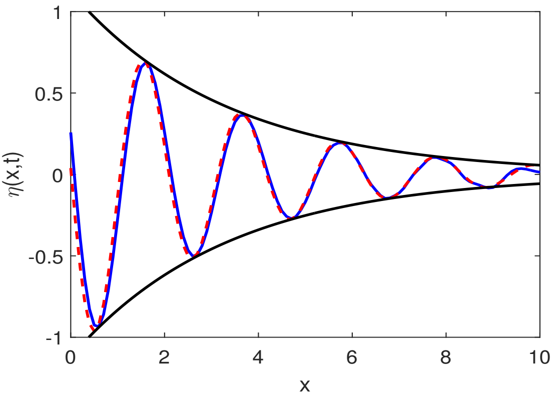

Figure 4: Black line as the curve of $|\eta(x,t)| = |\exp^{-i(kx-\omega t)}|$ at certain time of $k = 3.0328 - 0.4451\, i$. Blue line as the numerical surface elevation in porous media. Red line as the asymptotic expansion solution $\eta = \eta_0 + \epsilon\eta_1$.

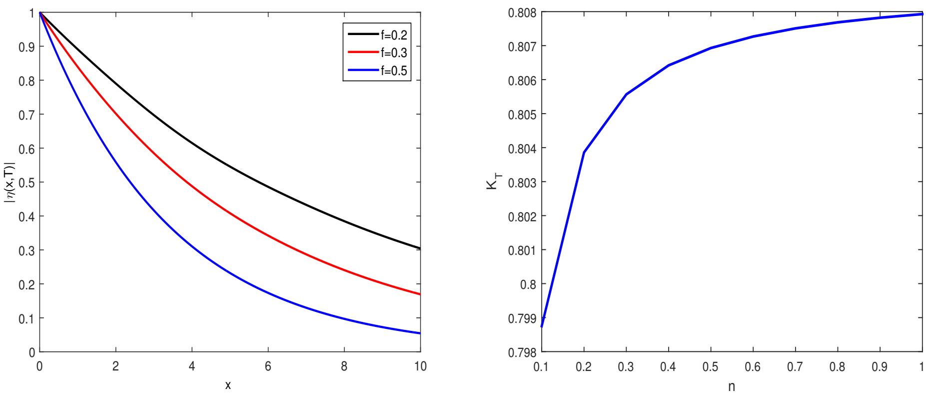

Figure 5: (Left) Curves of $|\eta(x,T)|$ for fixed values of porosity $n = 0.9$ and several friction coefficient. (Right) The curve of K_T w.r.t n for fixed values of friction coefficient.

6 Conclusions

In our study, first, we derived the dispersion relation that holds for waves inside a porous media where it explains the diffusive mechanism of the porous media itself. Analysis of dispersion relation gives us a damping effect of a specific emerged porous media. Emerged porous media with a particular characteristic and length has a strong influence on the reduction of wave amplitude. Furthermore, the solution for wave amplitude inside the porous media is obtained from asymptotic expansion. The finite volume method on a staggered grid is a stable and free damping error method for simulating wave damping passing through a porous media. Moreover, our numerical scheme results in numerical wave damping that confirms the asymptotic expansion solution.

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References

[1] H.J.S. Fernando, S.P. Samarawickrama, S. Balasubramanian, S.S.L. Hettiarachchi, and S. Voropayev. Effects of porous barriers such as coral reefs on coastal wave propagation. *Journal of Hydro-environment Research*, 1(3):187 – 194, 2008. doi:10.1016/j.jher.2007.12.003 C288

[2] Pengzhi Lin and S. A. Karunarathna. Numerical study of solitary wave interaction with porous breakwaters. *Journal of Waterway, Port, Coastal, and Ocean Engineering*, 133(5):352–363, 2007. doi:10.1061/(ASCE)0733-950X(2007)133:5(352) C288

[3] Philip L.-F. Liu, Pengzhi Lin, Kuang-An Chang, and Tsutomu Sakakiyama. Numerical modeling of wave interaction with porous structures. *Journal of*

Waterway, Port, Coastal, and Ocean Engineering, 125(6):322–330, 1999. doi:10.1061/(ASCE)0733-950X(1999)125:6(322) C288

[4] Patrick J. Lynett, Philip L.-F. Liu, Inigo J. Losada, and Cesar Vidal. Solitary wave interaction with porous breakwaters. *Journal of Waterway, Port, Coastal, and Ocean Engineering*, 126(6):314–322, 2000. C288

[5] O.S. Madsen. Wave transmission through porous structures. *Journal of Waterways, Harbors, Coastal Engineering*, 100:169–188, 1974. C288, C292

[6] I. Magdalena, S. R. Pudjaprasetya, and L. H. Wiryanto. Wave interaction with an emerged porous media. *Advances in Applied Mathematics and Mechanics*, 6(5):680–692, 06 2015. C289

[7] S.R. Pudjaprasetya and I. Magdalena. Momentum conservative scheme for dam break and wave run up simulations. *East Asia Journal on Applied Mathematics*, 4(2):152–165, 2014. doi:10.4208/eajam.290913.170314a C294

[8] Panagiotis D. Scarlatos and Vijay P. Singh. Long-wave transmission through porous breakwaters. *Coastal Engineering*, 11(2):141 – 157, 1987. doi:10.1016/0378-3839(87)90004-4 C288

[9] C.K. Sollitt and R.H. Cross. Long-wave transmission through porous breakwaters. *Proc. 13th Coastal Eng. Conf. 3*, pages 1827–1846, 1972. doi:10.1016/0378-3839(87)90004-4 C288

[10] W. Sulisz. Wave reflection and transmission at permeable breakwaters of arbitrary cross-section. *Coastal Engineering*, 9:371–386, 1985. doi:10.1016/0378-3839(85)90018-3 C288

[11] M. R. A. Van Gent. Wave interaction with permeable coastal structures. *International Journal of Rock Mechanics and Mining Sciences and Geomechanics Abstracts*, 33(6):277A, 1996. C288, C290

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