Numerical solution of nonlinear elliptic systems by block monotone iterations

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Abstract

We present numerical methods for solving a coupled system of nonlinear elliptic problems, where reaction functions are quasimonotone nondecreasing. We utilize block monotone iterative methods based on the Jacobi and Gauss–Seidel methods incorporated with the upper and lower solutions method. A convergence analysis and the theorem on uniqueness of solutions are discussed. Numerical experiments are presented.

Contents

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1 Introduction

Several problems in the chemical, physical and engineering sciences are characterized by coupled systems of nonlinear elliptic equations [3]. In this article, we construct block monotone iterative methods for solving the coupled system of nonlinear elliptic equations

$$\begin{aligned} &-L_\alpha u_\alpha(x,y) + f_\alpha(x,y,u) = 0\,, \quad (x,y) \in \omega\,, \quad \alpha = 1,2\,, \\ &\omega = \{(x,y) : 0 < x < 1\,, 0 < y < 1\}\,, \\ &u(x,y) = g(x,y)\,, \quad (x,y) \in \partial\omega\,, \end{aligned} \tag{1}$$

where $u = (u_1, u_2)$, $f = (f_1, f_2)$, $g = (g_1, g_2)$, and $\partial\omega$ is the boundary of ω. The differential operators L_α, $\alpha = 1,2$, are defined by

$$L_\alpha u_\alpha(x,y) \equiv \varepsilon_\alpha(u_{\alpha,xx} + u_{\alpha,yy})\,,$$

where ε_α with $\alpha = 1,2$, are positive constants. It is assumed that the functions f_α and g_α, $\alpha = 1,2$, are smooth in their respective domains.

Block monotone iterative methods, based on the method of upper and lower solutions, have only been used for solving nonlinear scalar elliptic equations [1, 2, 4]. The basic idea of the block monotone iterative methods is to decompose a two dimensional problem into a series of one dimensional two-point boundary value problems. Each of the one dimensional problems can be solved efficiently by a standard computational scheme such as the Thomas algorithm.

In this article we construct and investigate block monotone iterative methods based on the Jacobi and Gauss–Seidel methods for solving coupled systems of nonlinear elliptic equations with quasimonotone nondecreasing reaction functions f_α with $\alpha = 1, 2$.

In Section 2 we consider a nonlinear difference scheme which approximates the nonlinear elliptic problem (1) and describe the construction of the block monotone Jacobi and Gauss–Seidel iterative methods. A convergence analysis of the block monotone Jacobi and Gauss–Seidel iterative methods is discussed. The theorem on uniqueness of a solution to the nonlinear difference scheme is proved. Section 3 presents numerical experiments.

2 Block monotone iterative methods

On $\bar{\omega} = \omega \cup \partial\omega$ we introduce a rectangular mesh $\bar{\omega}^h = \bar{\omega}^{hx} \times \bar{\omega}^{hy} = \omega^h \cup \partial\omega^h$ where $\partial\omega^h$ is the boundary of the mesh ω^h and

$$\begin{aligned} \bar{\omega}^{hx} &= \{x_i, i = 0, 1, \ldots, N_x; \quad x_0 = 0, \quad x_{N_x} = 1; \quad h_x = x_{i+1} - x_i\}, \\ \bar{\omega}^{hy} &= \{y_j, j = 0, 1, \ldots, N_y; \quad y_0 = 0, \quad y_{N_y} = 1; \quad h_y = y_{j+1} - y_j\}. \end{aligned}$$

For a mesh function $U(p_{ij}) = \big(U_1(p_{ij}), U_2(p_{ij})\big)$ with $p_{ij} = (x_i, y_j) \in \bar{\omega}^h$ we use the difference scheme

$$\begin{aligned} &\mathcal{L}_{\alpha,ij} U_\alpha(p_{ij}) + f_\alpha(p_{ij}, U) = 0, \quad p_{ij} \in \omega^h, \quad \alpha = 1, 2, \\ &U(p_{ij}) = g(p_{ij}), \quad p_{ij} \in \partial\omega^h, \end{aligned} \tag{2}$$

The linear difference operators $\mathcal{L}_\alpha$ are defined by

$$\mathcal{L}_{\alpha,ij} U_\alpha(p_{ij}) = -\varepsilon_\alpha \left(D_x^2 U_\alpha(p_{ij}) + D_y^2 U_\alpha(p_{ij})\right),$$

where $D_x^2 U_\alpha(p_{ij})$ and $D_y^2 U_\alpha(p_{ij})$ for $\alpha = 1,2$ are the central difference approximations to the second derivatives:

$$D_x^2 U_\alpha(p_{ij}) = \frac{U_{\alpha,i-1,j} - 2U_{\alpha,ij} + U_{\alpha,i+1,j}}{h_x^2},$$
$$D_y^2 U_\alpha(p_{ij}) = \frac{U_{\alpha,i,j-1} - 2U_{\alpha,ij} + U_{\alpha,i,j+1}}{h_y^2}, \quad U_{\alpha,ij} \equiv U_\alpha(p_{ij}).$$

The vector mesh functions $\widetilde{U}$ and $\widehat{U}$ are ordered upper and lower solutions, respectively, of (2) which satisfy the inequalities

$$\widetilde{U}_\alpha(p_{ij}) \geqslant \widehat{U}_\alpha(p_{ij}), \quad p_{ij} \in \bar{\omega}^h, \tag{3}$$
$$\mathcal{L}_{\alpha,ij}\widehat{U}_\alpha(p_{ij}) + f_\alpha(p_{ij}, \widehat{U}) \leqslant 0 \leqslant \mathcal{L}_{\alpha,ij}\widetilde{U}_\alpha(p_{ij}) + f_\alpha(p_{ij}, \widetilde{U}), \quad p_{ij} \in \omega^h,$$
$$\widehat{U}_\alpha(p_{ij}) \leqslant g_\alpha(p_{ij}) \leqslant \widetilde{U}_\alpha(p_{ij}), \quad p_{ij} \in \partial\omega^h, \quad \alpha = 1,2.$$

For a given pair of ordered upper and lower solutions $\widetilde{U}$ and $\widehat{U}$ we define the sector

$$\langle \widehat{U}, \widetilde{U} \rangle = \left\{ U(p_{ij}) : \widehat{U}_\alpha(p_{ij}) \leqslant U_\alpha(p_{ij}) \leqslant \widetilde{U}_\alpha(p_{ij}), \quad p_{ij} \in \bar{\omega}^h, \quad \alpha = 1,2 \right\}.$$

We assume that on $\langle \widehat{U}, \widetilde{U} \rangle$ the vector function $f(p_{ij}, U)$ in (2) satisfies the constraints

$$\left(f_\alpha(p_{ij}, U)\right)_{u_\alpha} \leqslant c_\alpha(p_{ij}), \quad U \in \langle \widehat{U}, \widetilde{U} \rangle, \quad \alpha = 1,2, \tag{4}$$
$$-\left(f_\alpha(p_{ij}, U)\right)_{u_{\alpha'}} \geqslant 0, \quad U \in \langle \widehat{U}, \widetilde{U} \rangle, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1,2, \tag{5}$$

for $p_{ij} \in \bar{\omega}^h$ and where $(f_\alpha)_{u_\alpha} \equiv \partial f_\alpha / \partial u_\alpha$, $(f_\alpha)_{u_{\alpha'}} \equiv \partial f_\alpha / \partial u_{\alpha'}$ and c_α are non-negative bounded functions on $\bar{\omega}^h$. The vector function $f(p_{ij}, U)$ is quasimonotone nondecreasing on $\langle \widehat{U}, \widetilde{U} \rangle$ if it satisfies (5).

To construct block iterative methods we write the difference scheme (2) at an interior mesh point $p_{ij} \in \omega^h$ in the form

$$d_{\alpha,ij}U_{\alpha,ij} - l_{\alpha,ij}U_{\alpha,i-1,j} - r_{\alpha,ij}U_{\alpha,i+1,j} - b_{\alpha,ij}U_{\alpha,i,j-1} - t_{\alpha,ij}U_{\alpha,i,j+1} =$$
$$- f_\alpha(p_{ij}, U_{1,ij}, U_{2,ij}) + G^*_{\alpha,ij}, \tag{6}$$

where G^*_α, like the boundary function g_α, describes the boundary mesh points, and

$$l_{\alpha,ij} = r_{\alpha,ij} = \frac{\varepsilon_\alpha}{h_x^2}\,, \quad b_{\alpha,ij} = t_{\alpha,ij} = \frac{\varepsilon_\alpha}{h_y^2}\,,$$
$$d_{\alpha,ij} = l_{\alpha,ij} + r_{\alpha,ij} + b_{\alpha,ij} + t_{\alpha,ij}\,, \quad \alpha = 1,2\,,$$

Define column vectors and diagonal matrices by

$$U_{\alpha,i} = (U_{\alpha,i,0}, \dots, U_{\alpha,i,N_y})^T\,, \quad G^*_{\alpha,i} = (G^*_{\alpha,i,1}, \dots, G^*_{\alpha,i,N_y-1})^T\,,$$
$$F_{\alpha,i}(U_{1,i}, U_{2,i}) = \big(f_{\alpha,i,1}(U_{1,i,1}, U_{2,i,1}), \dots, f_{\alpha,i,N_y-1}(U_{1,i,N_y-1}, U_{2,i,N_y-1})\big)^T\,,$$
$$L_{\alpha,i} = \operatorname{diag}(l_{\alpha,i,1}, \dots, l_{\alpha,i,N_y-1})\,, \quad R_{\alpha,i} = \operatorname{diag}(r_{\alpha,i,1}, \dots, r_{\alpha,i,N_y-1})\,,$$

for $i = 0, 1, \dots, N_x$ and where

$$F_{\alpha,i}(U_{\alpha,i}, U_{\alpha',i}) = \begin{cases} F_{1,i}(U_{1,i}, U_{2,i})\,, & \alpha = 1\,, \\ F_{2,i}(U_{1,i}, U_{2,i})\,, & \alpha = 2\,, \end{cases} \quad \alpha' \neq \alpha\,, \tag{7}$$

with symmetry $F_{\alpha,i}(U_{\alpha,i}, U_{\alpha',i}) = F_{\alpha,i}(U_{\alpha',i}, U_{\alpha,i})$. Thus, $L_{\alpha,1}U_{\alpha,0}$ is on the boundary and in $G^*_{\alpha,1}$, and $R_{\alpha,N_x-1}U_{\alpha,N_x}$ is on the boundary and in G^*_{α,N_x}. Then the difference scheme (2) is written in the form

$$A_{\alpha,i}U_{\alpha,i} - (L_{\alpha,i}U_{\alpha,i-1} + R_{\alpha,i}U_{\alpha,i+1}) = -F_{\alpha,i}(U_{\alpha,i}, U_{\alpha',i}) + G^*_{\alpha,i}\,, \tag{8}$$
$$U_i = (U_{1,i}, U_{2,i})\,, \quad i = 1, 2, \dots, N_x - 1\,, \quad \alpha = 1, 2\,,$$

where $A_{\alpha,i}$ is the tridiagonal matrix with elements $d_{\alpha,ij}$, $l_{\alpha,ij}$ and $r_{\alpha,ij}$ with $j = 0, 1, \dots, N_y$. The elements of the matrices $L_{\alpha,i}$ and $R_{\alpha,i}$ are the coupling coefficients of a mesh point to $U_{\alpha,i-1,j}$ and $U_{\alpha,i+1,j}$ with $j = 1, 2, \dots, N_y - 1$.

The upper $\{\tilde U^{(n)}_{\alpha,i}\}$ and lower $\{\hat U^{(n)}_{\alpha,i}\}$ sequences of solutions with number of iterations $n \geqslant 1$ are calculated by the following block Jacobi ($\eta = 0$) and Gauss–Seidel ($\eta = 1$) iterative methods:

$$A_{\alpha,i}Z^{(n)}_{\alpha,i} - \eta L_{\alpha,i}Z^{(n)}_{\alpha,i-1} + C_{\alpha,i}Z^{(n)}_{\alpha,i} = -\mathcal{K}_{\alpha,i}(U^{(n-1)}_{\alpha,i}, U^{(n-1)}_{\alpha',i})\,,$$
$$\mathcal{K}_{\alpha,i}(U^{(n-1)}_{\alpha,i}, U^{(n-1)}_{\alpha',i}) = A_{\alpha,i}U^{(n-1)}_{\alpha,i} - L_{\alpha,i}U^{(n-1)}_{\alpha,i-1} - R_{\alpha,i}U^{(n-1)}_{\alpha,i+1}$$
$$+ F_{\alpha,i}(U^{(n-1)}_{\alpha,i}, U^{(n-1)}_{\alpha',i}) - G^*_{\alpha,i}\,,$$

where $\alpha = 1, 2$ and $i = 1, 2, \ldots, N_x - 1$,

$$Z_{\alpha,i}^{(n)} = \begin{cases} g_{\alpha,i} - U_{\alpha,i}^{(0)}, & n = 1, \\ 0, & n \geqslant 2, \end{cases} \qquad i = 0, N_x, \tag{9}$$

$$Z_{\alpha,i}^{(n)} = U_{\alpha,i}^{(n)} - U_{\alpha,i}^{(n-1)}, \quad \eta = 0, 1,$$

where $U_i^{(n-1)} = (U_{1,i}^{(n-1)}, U_{2,i}^{(n-1)})$, $\mathcal{K}_{\alpha,i}(U_{\alpha,i}^{(n-1)}, U_{\alpha',i}^{(n-1)})$ are the residuals of the difference equations (8) on $U_{\alpha,i}^{(n-1)}$, and $\mathbf{0}$ is the zero column vector with $N_x - 1$ components. The matrices $C_{\alpha,i}$ are the diagonal matrices $\operatorname{diag}(c_{\alpha,i,1}, \ldots, c_{\alpha,i,N_y-1})$ where the $c_\alpha = c_\alpha(p_{ij})$ are defined in (4).

The mean-value theorem for vector-valued functions is

$$F_{\alpha,i}(U_{\alpha,i}, U_{\alpha',i}) - F_{\alpha,i}(V_{\alpha,i}, U_{\alpha',i}) = \big(F_{\alpha,i}(Y_{\alpha,i}, U_{\alpha',i})\big)_{u_\alpha}[U_{\alpha,i} - V_{\alpha,i}], \tag{10}$$
$$F_{\alpha,i}(U_{\alpha,i}, U_{\alpha',i}) - F_{\alpha,i}(U_{\alpha,i}, V_{\alpha',i}) = \big(F_{\alpha,i}(U_{\alpha,i}, Y_{\alpha',i})\big)_{u_{\alpha'}}[U_{\alpha',i} - V_{\alpha',i}],$$

where the $Y_{\alpha,i}$ lie between $U_{\alpha,i}$ and $V_{\alpha,i}$, and the $Y_{\alpha',i}$ lie between $U_{\alpha',i}$ and $V_{\alpha',i}$, for $i = 1, 2, \ldots, N_x - 1$, $\alpha' \neq \alpha$, $\alpha, \alpha' = 1, 2$. The partial derivatives are the diagonal matrices

$$(F_{\alpha,i})_{u_\alpha} = \operatorname{diag}\big((f_{\alpha,i,1})_{u_\alpha}, \ldots, (f_{\alpha,i,N_y-1})_{u_\alpha}\big),$$
$$(F_{\alpha,i})_{u_{\alpha'}} = \operatorname{diag}\big((f_{\alpha,i,1})_{u_{\alpha'}}, \ldots, (f_{\alpha,i,N_y-1})_{u_{\alpha'}}\big),$$

where $(f_{\alpha,i,j})_{u_\alpha}$ and $(f_{\alpha,i,j})_{u_{\alpha'}}$, $j = 1, 2, \ldots, N_y - 1$, are calculated at $Y_{\alpha,i}$ and $Y_{\alpha',i}$, respectively.

Theorem 1. *Assume that f_α with $\alpha = 1, 2$ satisfies (4) and (5). Let $\tilde{U} = (\tilde{U}_1, \tilde{U}_2)$ and $\hat{U} = (\hat{U}_1, \hat{U}_2)$ be ordered upper and lower solutions of (2). Then for $i = 0, 1, \ldots, N_x$ the upper sequence $\{\tilde{U}_{\alpha,i}^{(n)}\}$ generated by (9) with $\tilde{U}^{(0)} = \tilde{U}$ converges monotonically from above to a maximal solution $\tilde{V}$, and similarly, the lower sequence $\{\hat{U}_{\alpha,i}^{(n)}\}$ generated by (9) with $\hat{U}^{(0)} = \hat{U}$ converges from below to a minimal solution $\hat{V}$, such that,*

$$\hat{U}_{\alpha,i}^{(n-1)} \leqslant \hat{U}_{\alpha,i}^{(n)} \leqslant \hat{V}_{\alpha,i} \leqslant \tilde{V}_{\alpha,i} \leqslant \tilde{U}_{\alpha,i}^{(n)} \leqslant \tilde{U}_{\alpha,i}^{(n-1)}, \tag{11}$$

where the inequalities between vectors are in a component-wise sense, for example, $U_{\alpha,i} \leqslant V_{\alpha,i}$ implies $U_{\alpha,ij} \leqslant V_{\alpha,ij}$ for all $j = 0, \ldots, N_y$.

Proof: Since $\tilde{u}^{(0)}$ is an initial upper solution (3), from (9) we have

$$A_{\alpha,i}\tilde{Z}^{(1)}_{\alpha,i} - L_{\alpha,i}\tilde{Z}^{(1)}_{\alpha,i-1} + C_{\alpha,i}\tilde{Z}^{(1)}_{\alpha,i} = -\mathcal{K}_{\alpha,i}(\tilde{U}^{(0)}_{\alpha,i}, \tilde{U}^{(0)}_{\alpha',i}), \quad i = 1, 2, \ldots, N_x - 1, \tag{12}$$

$$\tilde{Z}^{(1)}_{\alpha,i} \leqslant 0, \quad i = 0, N_x, \quad \alpha = 1, 2.$$

Since $L_{\alpha,i} \geqslant O$ and $(A_{\alpha,i} + C_{\alpha,i})^{-1} \geqslant O$ (Corollary 3.20, [6]) where O is the $(N_y - 1) \times (N_y - 1)$ null matrix, for $i = 1$ in (12) and $\tilde{Z}^{(1)}_{\alpha,0} \leqslant 0$, we conclude that $\tilde{Z}^{(1)}_{\alpha,1} \leqslant 0$. For $i = 2$ in (12), using $L_{\alpha,2} \geqslant O$ and $\tilde{Z}^{(1)}_{\alpha,1} \leqslant 0$, we obtain $\tilde{Z}^{(1)}_{\alpha,2} \leqslant 0$. Thus, by induction on i we prove that

$$\tilde{Z}^{(1)}_{\alpha,i} \leqslant 0, \quad i = 0, 1, \ldots, N_x, \quad \alpha = 1, 2. \tag{13}$$

Similarly, we can prove that

$$\hat{Z}^{(1)}_{\alpha,i} \geqslant 0, \quad i = 0, 1, \ldots, N_x, \quad \alpha = 1, 2. \tag{14}$$

We now prove that

$$\hat{U}^{(1)}_{\alpha,i} \leqslant \tilde{U}^{(1)}_{\alpha,i}, \quad i = 0, 1, \ldots, N_x, \quad \alpha = 1, 2. \tag{15}$$

Defining $W^{(n)}_{\alpha,i} = \tilde{U}^{(n)}_{\alpha,i} - \hat{U}^{(n)}_{\alpha,i}$ for $i = 0, 1, \ldots, N_x$ and $\alpha = 1, 2$, from (9) with $i = 1, 2, \ldots, N_x - 1$ and $\alpha = 1$ we have

$$\begin{aligned} A_{1,i}W^{(1)}_{1,i} - L_{1,i}W^{(1)}_{1,i-1} + C_{1,i}W^{(1)}_{1,i} \; = \; & C_{1,i}W^{(0)}_{1,i} + R_{1,i}W^{(0)}_{1,i+1} \\ & - \left[F_{1,i}(\tilde{U}^{(0)}_{1,i}, \tilde{U}^{(0)}_{2,i}) - F_{1,i}(\hat{U}^{(0)}_{1,i}, \tilde{U}^{(0)}_{2,i})\right] \\ & - \left[F_{1,i}(\hat{U}^{(0)}_{1,i}, \tilde{U}^{(0)}_{2,i}) - F_{1,i}(\hat{U}^{(0)}_{1,i}, \hat{U}^{(0)}_{2,i})\right], \end{aligned} \tag{16}$$

and for $i = 0, N_x$ we have $W^{(1)}_{1,i} = 0$. By the mean-value theorem (10), for $i = 0, 1, \ldots, N_x$ we have

$$F_{1,i}(\tilde{U}^{(0)}_{1,i}, \tilde{U}^{(0)}_{2,i}) - F_{1,i}(\hat{U}^{(0)}_{1,i}, \tilde{U}^{(0)}_{2,i}) = (F_{1,i}(Q^{(0)}_{1,i}, \tilde{U}^{(0)}_{2,i}))_{u_1}\left[\tilde{U}^{(0)}_{1,i} - \hat{U}^{(0)}_{1,i}\right],$$

$$F_{1,i}(\hat{U}^{(0)}_{1,i}, \tilde{U}^{(0)}_{2,i}) - F_{1,i}(\hat{U}^{(0)}_{1,i}, \hat{U}^{(0)}_{2,i}) = (F_{1,i}(\hat{U}^{(0)}_{1,i}), Q^{(0)}_{2,i})_{u_2}\left[\tilde{U}^{(0)}_{2,i} - \hat{U}^{(0)}_{2,i}\right],$$

where $\hat{u}^{(0)}_{\alpha,i} \leqslant Q^{(0)}_{\alpha,i} \leqslant \tilde{u}^{(0)}_{\alpha,i}$ for $\alpha = 1, 2$, and we conclude that $(F_{1,i})_{u_1}$ and $(F_{1,i})_{u_2}$ satisfy (4) and (5). Now with (16) we have, for $i = 1, 2, \ldots, N_x - 1$,

$$\begin{aligned} A_{1,i}W^{(1)}_{1,i} - L_{1,i}W^{(1)}_{1,i-1} + C_{1,i}W^{(1)}_{1,i} &= \left(C_{1,i} - (F_{1,i})_{u_1}\right) W^{(0)}_{1,i} \\ &\quad - (F_{1,i})_{u_2} W^{(0)}_{2,i} + R_{1,i}W^{(0)}_{1,i+1}, \end{aligned} \tag{17}$$

and $W^{(1)}_{1,i} = 0$ for $i = 0, N_x$. Now with (4) and (5), and since $W^{(0)}_{\alpha,i} \geqslant 0$ for $i = 0, 1, \ldots, N_x$ and $\alpha = 1, 2$, and $R_{1,i} \geqslant O$, we obtain

$$\begin{aligned} &A_{1,i}W^{(1)}_{1,i} + C_{1,i}W^{(1)}_{1,i} \geqslant L_{1,i}W^{(1)}_{1,i-1}, \quad i = 1, 2, \ldots, N_x - 1, \\ &W^{(1)}_{1,i} = 0, \quad i = 0, N_x. \end{aligned} \tag{18}$$

Since $(A_{1,i} + C_{1,i})^{-1} \geqslant O$ for $i = 1, 2, \ldots, N_x - 1$, with $i = 1$ in (18) and $W^{(1)}_{1,0} = 0$, we conclude that $W^{(1)}_{1,1} \geqslant 0$. For $i = 2$ in (18), and using $L_{1,2} \geqslant 0$ and $W^{(1)}_{1,1} \geqslant 0$, we obtain $W^{(1)}_{1,2} \geqslant 0$. Thus, by induction on i we prove that

$$W^{(1)}_{1,i} \geqslant 0, \quad i = 0, 1, \ldots, N_x.$$

By following a similar argument we can prove (15) for $\alpha = 2$.

We now prove that $\tilde{u}^{(1)}_{\alpha,i}$ and $\hat{u}^{(1)}_{\alpha,i}$ with $i = 0, 1, \ldots, N_x$ and $\alpha = 1, 2$ are upper and lower solutions to (9), respectively. From (9) with $\alpha = 1$ and using the mean-value theorem (10), we conclude that for $i = 1, 2, \ldots, N_x - 1$,

$$\begin{aligned} \mathcal{K}_{1,i}(\tilde{u}^{(1)}_{1,i}, \tilde{u}^{(1)}_{2,i}) &= -\left(C_{1,i} - \frac{\partial F_{1,i}(\tilde{E}^{(1)}_{1,i}, \tilde{u}^{(0)}_{2,i})}{\partial u_1}\right) \tilde{Z}^{(1)}_{1,i} + \frac{\partial F_{1,i}(\tilde{u}^{(0)}_{1,i}, \tilde{E}^{(1)}_{2,i})}{\partial u_2} \tilde{Z}^{(1)}_{2,i} \\ &\quad - R_{1,i}\tilde{Z}^{(1)}_{1,i+1}, \end{aligned} \tag{19}$$

where

$$\tilde{u}^{(1)}_{\alpha,i} \leqslant \tilde{E}^{(1)}_{\alpha,i} \leqslant \tilde{u}^{(0)}_{\alpha,i}, \quad i = 0, 1, \ldots, N_x, \quad \alpha = 1, 2.$$

From (13), (14) and (15) we conclude that $\partial F_{1,i}/\partial u_1$ and $\partial F_{1,i}/\partial u_2$ satisfy (4) and (5). From (4), (5), (13) and since $R_{1,i} \geqslant O$ we conclude that

$$\mathcal{K}_{1,i}(\tilde{u}^{(1)}_{1,i}, \tilde{u}^{(1)}_{2,i}) \geqslant 0, \quad i = 1, 2, \ldots, N_x - 1. \tag{20}$$

Similarly,

$$\mathcal{K}_{2,i}(\tilde{U}_{2,i}^{(1)}, \tilde{U}_{1,i}^{(1)}) \geqslant 0\,, \quad i = 1, 2, \ldots, N_x - 1\,. \tag{21}$$

From (3), (20) and (21) we conclude that $(\tilde{U}_{1,i}^{(1)}, \tilde{U}_{2,i}^{(1)})$ for $i = 0, 1, \ldots, N_x$ is an upper solution to (2). In a similar manner we obtain

$$\mathcal{K}_{1,i}(\hat{U}_{1,i}^{(1)}, \hat{U}_{2,i}^{(1)}) \leqslant 0\,, \quad \mathcal{K}_{2,i}(\hat{U}_{2,i}^{(1)}, \hat{U}_{1,i}^{(1)}) \leqslant 0\,, \quad i = 1, 2, \ldots, N_x - 1\,,$$

which means $(\hat{U}_{1,i}^{(1)}, \hat{U}_{2,i}^{(1)})$ for $i = 0, 1, \ldots, N_x$ is a lower solution to (2). By induction on n we can prove that $\{\tilde{U}_{\alpha,i}^{(n)}\}$ and $\{\hat{U}_{\alpha,i}^{(n)}\}$ with $i = 0, 1, \ldots, N_x$ and $\alpha = 1, 2$ are, respectively, monotone decreasing upper and monotone increasing lower sequences of solutions.

Now we prove that the limiting functions of the upper sequence $\{\tilde{U}_{\alpha,i}^{(n)}\}$ and lower sequence $\{\hat{U}_{\alpha,i}^{(n)}\}$ with $i = 0, 1, \ldots, N_x$ and $\alpha = 1, 2$ are, respectively, maximal and minimal solutions of (2). From (11) we conclude that $\lim_{n\to\infty} \tilde{U}_{\alpha,i}^{(n)} = \tilde{U}_{\alpha,i}$ exists and

$$\lim_{n\to\infty} \tilde{Z}_{\alpha,i}^{(n)} = 0\,, \quad i = 0, 1, \ldots, N_x\,, \quad \alpha = 1, 2\,. \tag{22}$$

Similar to (19), we have

$$\begin{aligned}\mathcal{K}_{1,i}(\tilde{U}_{1,i}^{(1)}, \tilde{U}_{2,i}^{(1)}) &= -\left(C_{1,i} - \frac{\partial F_{1,i}(\tilde{E}_{1,i}^{(n)}, \tilde{U}_{2,i}^{(n-1)})}{\partial u_1}\right) \tilde{Z}_{1,i}^{(n)} - R_{1,i}\tilde{Z}_{1,i+1}^{(n)} \\ &\quad + \frac{\partial F_{1,i}(\tilde{U}_{1,i}^{(n-1)}, \tilde{E}_{2,i}^{(n)})}{\partial u_2} \tilde{Z}_{2,i}^{(n)}\,, \quad i = 1, 2, \ldots, N_x - 1\,,\end{aligned} \tag{23}$$

where

$$\tilde{U}_{\alpha,i}^{(n)} \leqslant \tilde{E}_{\alpha,i}^{(n)} \leqslant \tilde{U}_{\alpha,i}^{(n-1)}\,, \quad i = 0, 1, \ldots, N_x\,, \quad \alpha = 1, 2\,.$$

By taking the limit of both sides of (23), and using (13), it follows that

$$\mathcal{K}_{1,i}(\tilde{U}_{1,i}^{(1)}, \tilde{U}_{2,i}^{(1)}) = 0\,, \quad i = 1, 2, \ldots, N_x - 1\,. \tag{24}$$

Similarly, we obtain

$$\mathcal{K}_{2,i}(\tilde{U}_{2,i}, \tilde{U}_{1,i}) = 0\,, \quad i = 1, 2, \ldots, N_x - 1\,. \tag{25}$$

From (24) and (25) we conclude that $(\tilde{U}_{1,i}, \tilde{U}_{2,i})$ with $i = 0, 1, \ldots, N_x$ is a maximal solution to the nonlinear difference scheme (2). In a similar manner, we can prove that

$$\mathcal{K}_{1,i}(\hat{U}_{1,i}, \hat{U}_{2,i}) = 0\,, \quad \mathcal{K}_{2,i}(\hat{U}_{2,i}, \hat{U}_{1,i}) = 0\,, \quad i = 1, 2, \ldots, N_x - 1\,,$$

which means that $(\hat{U}_{1,i}, \hat{U}_{2,i})$ with $i = 0, 1, \ldots, N_x$ is a minimal solution to the nonlinear difference scheme (2). ♠

2.1 Convergent analysis

Assume that the reaction functions f_α with $\alpha = 1, 2$ satisfy the assumptions

$$0 < \hat{c}_\alpha(x, y) \leqslant \big(f_\alpha(x, y, u)\big)_{u_\alpha} \leqslant \tilde{c}_\alpha(x, y)\,, \tag{26}$$

$$0 \leqslant -\big(f_\alpha(x, y, u)\big)_{u_{\alpha'}} \leqslant q_{\alpha\alpha'}(x, y)\,, \quad \alpha' \neq \alpha\,, \quad \alpha, \alpha' = 1, 2\,, \tag{27}$$

$$\rho = \min_{\alpha=1,2}\left\{\min_{(x,y)\in\tilde{\omega}} \hat{c}_\alpha(x, y)\right\} > 0\,, \tag{28}$$

$$0 < \beta = \max_{\alpha=1,2}\left[\max_{(x,y)\in\tilde{\omega}}\left(\frac{q_{\alpha\alpha'}(x, y)}{\hat{c}_\alpha(x, y)}\right)\right] < 1\,, \quad \alpha' \neq \alpha\,, \quad \alpha, \alpha' = 1, 2\,. \tag{29}$$

A stopping test for the block monotone iterative methods (9) is chosen to be

$$\max_{\alpha=1,2} \|\mathcal{K}_\alpha(U^{(n-1)})\|_{\omega^h} \leqslant \delta\,, \quad \|\mathcal{K}_\alpha(U^{(n-1)})\|_{\omega^h} = \max_{1\leqslant i\leqslant N_x-1} |K_{\alpha,i}(U_i^{(n)})|\,, \tag{30}$$

where δ is a prescribed accuracy.

The linear version of problem (2) is

$$\mathcal{L}_{\alpha,ij} W_\alpha(p_{ij}) + c^*_\alpha(p_{ij}) W_\alpha(p_{ij}) = \Phi_\alpha(p_{ij})\,, \quad p_{ij} \in \omega^h\,, \tag{31}$$
$$W(p_{ij}) = g(p_{ij})\,, \quad p_{ij} \in \partial\omega^h\,, \quad \alpha = 1, 2\,,$$

where $W = (W_1, W_2)$ and the c_α^* with $\alpha = 1, 2$ are positive bounded functions. We give an estimate of the solution to (31) in the following lemma.

Lemma 2. *The solution to (31) satisfies*

$$\|W_\alpha\|_{\bar{\omega}^h} \leqslant \max\{\|g_\alpha\|_{\partial\omega^h}, \|\Phi_\alpha/c_\alpha^*\|_{\omega^h}\}\,, \quad \alpha = 1, 2\,, \tag{32}$$

where

$$\|g_\alpha\|_{\partial\omega^h} = \max_{p_{ij}\in\partial\omega^h} |g_\alpha(p_{ij})|\,, \quad \left\|\frac{\Phi_\alpha}{c_\alpha^*}\right\|_{\omega^h} = \max_{p_{ij}\in\omega^h} \left|\frac{\Phi_\alpha(p_{ij})}{c_\alpha^*(p_{ij})}\right|.$$

Samarskii [5] proves this lemma.

Theorem 3. *Let assumptions (26)–(29) be satisfied. Then for the sequence $\{U^{(n)}\}$ generated by the block monotone iterative methods (9) we have*

$$\|U^{(n_\delta)} - U^*\|_{\bar{\omega}^h} \leqslant \frac{1}{(1-\beta)\rho}\delta\,, \tag{33}$$

where U^* is a solution of the nonlinear difference scheme (2) and n_δ is the minimal number of iterations subject to (30).

Proof: The existence of a solution U^* to the nonlinear difference scheme (2) is established in Theorem 1. From (2), for $U_\alpha^{(n_\delta)}$ and U_α^*, we have

$$\begin{aligned}
&\mathcal{L}_{\alpha,ij} U_\alpha^{(n_\delta)}(p_{ij}) + f_\alpha(p_{ij}, U^{(n_\delta)}) = \mathcal{K}_{\alpha,ij}\big(U_{\alpha,ij}^{(n_\delta-1)}, U_{\alpha',ij}^{(n_\delta-1)}\big)\,, \quad p_{ij}\in\omega^h\,,\\
&U_{\alpha,ij}^{(n_\delta)}(p_{ij}) = g_\alpha(p_{ij})\,, \quad p_{ij}\in\partial\omega^h\,, \quad \alpha = 1, 2\,,\\
&\mathcal{L}_{\alpha,ij} U_\alpha^*(p_{ij}) + f_\alpha(p_{ij}, U^*) = 0\,, \quad p_{ij}\in\omega^h\,,\\
&U_\alpha^*(p_{ij}) = g_\alpha(p_{ij})\,, \quad p_{ij}\in\partial\omega^h\,, \quad \alpha = 1, 2\,.
\end{aligned}$$

Letting $W_\alpha^{(n)} = U_\alpha^{(n)} - U_\alpha^*$ for $\alpha = 1, 2$ and using the mean-value theorem, we obtain

$$\begin{aligned}
&\mathcal{L}_{\alpha,ij} W_\alpha^{(n_\delta)}(p_{ij}) + \big(f_\alpha(p_{ij}, H_\alpha^{(n_\delta)})\big)_{u_\alpha} W_\alpha^{(n_\delta)}(p_{ij}) =\\
&-\big(f_\alpha(p_{ij}, H_{\alpha'}^{(n_\delta)})\big)_{u_{\alpha'}} W_{\alpha'}^{(n_\delta)}(p_{ij}) + \mathcal{K}_{\alpha,ij}(U_{\alpha,ij}^{(n_\delta-1)}, U_{\alpha',ij}^{(n_\delta-1)})\,, \quad p_{ij}\in\omega^h\,,\\
&W_{\alpha,ij}^{(n_\delta)}(p_{ij}) = 0\,, \quad p_{ij}\in\partial\omega^h\,, \quad \alpha' \neq \alpha\,, \quad \alpha, \alpha' = 1, 2\,,
\end{aligned}$$

where $H_\alpha^{(n_\delta)}$ lies between $U_\alpha^{(n_\delta)}$ and U_α^* for $\alpha = 1, 2$. Using the maximum principle (32) we conclude that

$$\begin{aligned}\|W_\alpha^{(n_\delta)}\|_{\bar\omega^h} \leqslant\ & \|\mathcal{K}_\alpha(U^{(n_\delta)})\big[\big(f_\alpha(H_\alpha^{(n)})\big)_{u_\alpha}\big]^{-1}\|_{\omega^h} \\ & + \|\big(f_\alpha(H_{\alpha'}^{(n_\delta)})\big)_{u_{\alpha'}}/\big(f_\alpha(H_\alpha^{(n_\delta)})\big)_{u_\alpha}\|_{\omega^h}\|W_{\alpha'}^{(n_\delta)}\|_{\omega^h}\,.\end{aligned}$$

Letting $\mathcal{W}^{(n_\delta)} = \max_{\alpha=1,2}\|\mathcal{W}_\alpha^{(n_\delta)}\|_{\bar\omega^h}$ and with (28) and (29) we obtain

$$\mathcal{W}^{(n_\delta)} \leqslant (\max_{\alpha=1,2}\|\mathcal{K}_\alpha(U^{(n_\delta)})\|)\rho^{-1} + \beta\mathcal{W}^{(n_\delta)}\,.$$

Now with (30) we have (33). Thus, we prove the theorem. ♠

2.2 Uniqueness of a solution

In this section we prove uniqueness of a solution of the discrete problem (2).

Theorem 4. *Let assumptions (26)–(29) be satisfied. Then the nonlinear difference scheme (2) has a unique solution.*

Proof: To prove the uniqueness of a solution to the nonlinear difference scheme (2), because of (11), it suffices to prove that $\hat V_\alpha = \tilde V_\alpha$, where $\hat V_\alpha$ and $\tilde V_\alpha$ are the minimal and maximal solutions. Substituting $W_\alpha = \tilde V_\alpha - \hat V_\alpha$ into (2) we have

$$\begin{aligned}&\mathcal{L}_{\alpha,ij}W_\alpha(p_{ij}) + f_\alpha(p_{ij},\tilde V) - f_\alpha(p_{ij},\hat V) = 0\,, \quad p_{ij}\in\omega^h\,,\\ &W_\alpha(p_{ij}) = 0\,, \quad p_{ij}\in\partial\omega^h\,, \quad \alpha = 1,2\,.\end{aligned}$$

Using the mean-value theorem we obtain

$$\begin{aligned}&\Big(\mathcal{L}_{\alpha,ij} + \big(f_\alpha(p_{ij},Q_\alpha)\big)_{u_\alpha}\Big)W_\alpha(p_{ij}) = -\big(f_\alpha(p_{ij},Q_{\alpha'})\big)_{u_{\alpha'}}W_{\alpha'}(p_{ij})\,,\\ &p_{ij}\in\omega^h\,, \quad W_\alpha(p_{ij}) = 0\,, \quad p_{ij}\in\partial\omega^h\,, \quad \alpha'\neq\alpha\,, \quad \alpha,\alpha' = 1,2\,,\end{aligned}$$

where $\hat{V}_\alpha(p_{ij}) \leqslant Q_\alpha(p_{ij}) \leqslant \tilde{V}_\alpha(p_{ij}$ for $\alpha = 1, 2$. Using the maximum principle (32) we conclude that

$$\begin{aligned} \|W_\alpha\|_{\bar{\omega}^h} &\leqslant \|(f_\alpha(Q_{\alpha'}))_{u_{\alpha'}} W_{\alpha'}[(f_\alpha(Q_\alpha))_{u_\alpha}]^{-1}\|_{\omega^h} \\ &\leqslant \|(f_\alpha(Q_{\alpha'}))_{u_{\alpha'}}[(f_\alpha(Q_\alpha))_{u_\alpha}]^{-1}\|_{\omega^h}\|W_{\alpha'}\|_{\omega^h} . \end{aligned}$$

Using (29) we obtain

$$\|W_\alpha\|_{\bar{\Omega}^h} \leqslant \beta \|W_{\alpha'}\|_{\omega^h} .$$

Let $W = \max_{\alpha=1,2} \|W_\alpha\|_{\bar{\omega}^h}$ so that

$$W(1-\beta) \leqslant 0 .$$

From (28) and since $W \geqslant 0$ we conclude that $W = 0$. Thus, we prove the theorem. ♠

As follows from Theorems 1 and 4, under assumptions (26)–(29), the sequences of solutions generated by the block Jacobi and Gauss–Seidel methods converge to the unique solution of the nonlinear difference scheme (2).

3 Numerical experiments

As a test problem we consider the gas-liquid interaction model [3] where reaction functions are

$$f_1(u_1, u_2) = -\sigma_1(1-u_1)u_2 , \quad f_2(u_1, u_2) = \sigma_2(1-u_1)u_2 , \tag{34}$$

where $u_1 \geqslant 0$ and $u_2 \geqslant 0$ are concentrations of the gas and liquid, respectively, and $\sigma_\alpha = \text{const} > 0$ with $\alpha = 1, 2$ are reaction rates.

We choose $\varepsilon_1 = 1$, $\varepsilon_2 = 0.1$, the boundary conditions $g_1(x, y) = 0$ and $g_2(x, y) = 1$, $(x, y) \in \partial\omega$ in (1), and $\sigma_\alpha = 1$ for $\alpha = 1, 2$. The pairs $(\tilde{U}_1, \tilde{U}_2) = (1, 1)$ and $(\hat{U}_1, \hat{U}_2) = (0, 0)$ are ordered upper and lower solutions. From (34) we conclude that

$$\begin{aligned} (f_1)_{u_1} &= u_2 \leqslant 1 , \quad -(f_1)_{u_2} = 1 - u_1 \geqslant 0 , \\ (f_2)_{u_2} &= 1 - u_1 \leqslant 1 , \quad -(f_2)_{u_1} = u_2 \geqslant 0 . \end{aligned}$$

Table 1: Numerical error and order of convergence of the nonlinear scheme (2).

N	8	16	32	64	128
E	0.0071	0.0017	4.47×10^{-4}	1.06×10^{-4}	2.13×10^{-5}
γ	1.97	2.01	2.06	2.32	

Table 2: Number of iterations and CPU time for the block methods.

N	8	16	32	64	128
	block Jacobi method				
# of iterations	101	397	1577	6299	25189
CPU (s)	0.02	0.11	0.91	14.17	225.99
	block Gauss–Seidel method				
# of iterations	51	180	762	3084	12370
CPU (s)	0.01	0.06	0.47	7.34	117.62

It follows that f_α with $\alpha = 1,2$ satisfy (4) with $c_\alpha = 1$ and (5). Since the exact solution of the test problem is unavailable, we define the numerical error and the order of convergence of the numerical solution, respectively, as

$$E(N) = \max_{\alpha=1,2} \left[\max_{p_{ij} \in \bar{\omega}_h} \left| U_\alpha^{(n_\delta)}(p_{ij}) - U_\alpha^{(n_\delta)r}(p_{ij}) \right| \right], \quad \gamma(N) = \log_2 \left(\frac{E(N)}{E(2N)} \right),$$

where $U_\alpha^{(n_\delta)}(p_{ij})$ with $\alpha = 1,2$ are the approximate solutions generated by (9), n_δ is the minimal number of iterations subject to (30), and $U_\alpha^{(n_\delta)r}(p_{ij})$ with $\alpha = 1,2$ are reference solutions with number of mesh points $N = 512$.

Table 1 presents the error $E(N)$ and order of convergence $\gamma(N)$ for different values of $N_x = N_y = N$. This table indicates that the numerical solution of the nonlinear difference scheme (2) converges to the reference solution with second-order accuracy. The numerical and reference solutions are calculated by the block Jacobi or Gauss–Seidel methods. Tables 2 and 3 show that the block Gauss–Seidel method converges faster than the block Jacobi method, and the block monotone methods (Table 2) converge faster than the corresponding monotone Gauss–Seidel and Jacobi methods (Table 3).

Table 3: Number of iterations and CPU time for the Jacobi and Gauss–Seidel methods.

N	8	16	32	64	128
	Jacobi method				
# of iterations	190	771	3092	12378	49520
CPU (s)	0.08	0.11	1.09	16.15	261.28
	Gauss–Seidel method				
# of iterations	97	388	1548	6191	24762
CPU (s)	0.12	0.40	0.53	8.58	141.37

References

[1] Boglaev, I., A block monotone domain decomposition algorithm for a semilinear convection-diffusion problem, J. Comput. Appl. Math., 173(2005), 259–277. doi:10.1016/j.cam.2004.03.011 C80

[2] Boglaev, I., Monotone iterates for solving systems of semilinear elliptic equations and applications, ANZIAM J, Proceedings of the 8th Biennial Engineering Mathematics and Applications Conference, EMAC-2007, 49(2008), C591–C608. doi:10.21914/anziamj.v49i0.311 C80

[3] Pao, C. V., Nonlinear parabolic and elliptic equations, Springer-Verlag (1992). doi:10.1007/978-1-4615-3034-3 C80, C91

[4] Pao, C. V., Block monotone iterative methods for numerical solutions of nonlinear elliptic equations, Numer. Math., 72(1995), 239–262. doi:10.1007/s002110050168 C80

[5] Samarskii, A., The theory of difference schemes, CRC Press (2001). `https://www.crcpress.com/The-Theory-of-Difference-Schemes/Samarskii/p/book/9780824704681` C89

[6] Varga, R. S., Matrix iterative analysis, Springer-Verlag (2000). doi:10.1007/978-3-642-05156-2 C85

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