First eigenvalue of Schrödinger operator of space-like hypersurfaces

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Abstract

We introduce two Schrödinger operators of compact space-like hypersurfaces in a de Sitter space. If the hypersurfaces have constant mean curvature or constant scalar curvature, we obtain some spectral characterisations of totally umbilical space-like hypersurfaces by the first eigenvalue of the Schrödinger operators.

Contents

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1 Introduction

Space-like hypersurfaces with constant mean curvature or constant scalar curvature in an arbitrary Lorentz manifold play an important role in general relativity. Space-like hypersurfaces in a Lorentz manifold have recently been investigated by many researchers from both physical and mathematical points of view. Let $M_1^{n+1}(c)$ be an $(n+1)$-dimensional Lorentzian space form with constant sectional curvature c. According to $c > 0$, $c = 0$ or $c < 0$, it is called a de Sitter space, a Minkowski space or an anti-de Sitter space, respectively, and is denoted by $S_1^{n+1}(c)$, R_1^{n+1} or $H_1^{n+1}(c)$. When $c = 1$, we denote the de Sitter space by S_1^{n+1}. A hypersurface in a Lorentzian manifold is said to be space-like if the induced metric on the hypersurface is positive definite.

We know that hypersurfaces with constant mean curvature in a Riemannian manifold $M^{n+1}(c)$ of constant sectional curvature c are critical points of the area functional under variations that keep constant a certain volume function. Barbosa, do Carmo and Eschenburg [4] studied the stability for hypersurfaces of constant mean curvature in Riemannian manifold. In analogy with the case of constant mean curvature, questions of stability can be considered for hypersurfaces with constant scalar curvature. Alencar, do Carmo and Colares [2] extended the study of stability to hypersurfaces with constant scalar curvature. As researched by C. Wu [16] for minimal submanifolds in a unit sphere, Alías et al. [3] and Cheng [9] studied the first eigenvalue of some Jacobi operator of hypersurfaces with constant mean curvature or constant scalar curvature in a unit sphere and obtained some spectral characterizations of so called $H(r)$-torus $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$ or Riemannian product $S^m(r) \times S^{n-m}(\sqrt{1-r^2})$, $1 \leqslant m \leqslant n-1$.

Comparing the stability for hypersurfaces with constant mean curvature or constant scalar curvature in Riemannian manifolds, Barbosa, Oliker [5], Liu and Deng [12] studied the stability for space-like hypersurfaces with constant mean curvature or constant scalar curvature in Lorentz manifolds. From the results of Barbosa and Oliker [5, 6], we know that constant mean curvature space-like hypersurfaces are solutions to a variational problem. They are critical points of the area functional for variations that leave constant a certain volume function.

We define two Schrödinger operators L_H and L_R by (1) and (2) and obtain some spectral characterizations of totally umbilical space-like hypersurfaces by the first eigenvalue of the Schrödinger operator L_H or L_R.

Since space-like hypersurfaces have particular structure, from the definition of the Schrödinger operator L_H or L_R, we notice that the Schrödinger operators L_H and L_R are different from that of the Riemannian hypersurface which were studied by Alías, Barros and Brasil [3] and Cheng [9]. In the case of space-like hypersurfaces, the first eigenvalues of L_H and L_R have special forms.

In a neighbourhood of a point x of the space-like hypersurface M, we choose an orthonormal frame field $\{e_1, \dots, e_n\}$ such that $h_{ij} = \lambda_i \delta_{ij}$ at x, where h_{ij} are the components of the second fundamental form of M. Let H denote the mean curvature of M. We introduce the operator ϕ by

$$\langle \phi X, Y \rangle = \langle hX, Y \rangle - H\langle X, Y \rangle.$$

Putting $\phi = \sum_{i,j} \phi_{ij} \omega_i \otimes \omega_j$, where $\phi_{ij} = h_{ij} - H\delta_{ij}$, we can see that ϕ is traceless, that the basis $\{e_1, \dots, e_n\}$ also diagonalizes ϕ at x with eigenvalues $\mu_i = \lambda_i - H$, and that

$$|\phi|^2 = \sum_i \mu_i^2 = \frac{1}{2n} \sum_{i,j} (\lambda_i - \lambda_j)^2 = S - nH^2,$$

where S denotes the norm square of the second fundamental form of M. We know that $|\phi|^2 \equiv 0$ if and only if M is totally umbilical.

Before announcing our main results, we introduce the following Schrödinger operators:

$$L_H = -\Delta + |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\phi|, \tag{1}$$

$$L_R = -\square + \frac{1}{nH}|\phi|^4 + \frac{1-H^2}{H}|\phi|^2, \tag{2}$$

where the differential operator

$$\square f = \sum_{i,j=1}^{n} (nH\delta_{ij} - h_{ij}) f_{ij}\,,$$

for any C^2-function f, which was introduced and used by S. Y. Cheng and Yau [11]. Now we state our results.

Theorem 1 *Let M be an n-dimensional compact orientable space-like hypersurface in an (n+1)-dimensional de Sitter space S_1^{n+1} with constant mean curvature H. Denote by $\lambda_1^{L_H}$ the first eigenvalue of the Schrödinger operator L_H. If $\lambda_1^{L_H} \geqslant -n(1-H^2)$, then M is totally umbilical.*

Theorem 2 *Let M be an n-dimensional compact orientable space-like hypersurface in an (n+1)-dimensional de Sitter space S_1^{n+1} with constant scalar curvature $n(n-1)R$ $(R < 1)$. Denote by $\lambda_1^{L_R}$ the first eigenvalue of the Schrödinger operator L_R. Then*

$$\lambda_1^{L_R} \leqslant \frac{n-2}{\sqrt{n(n-1)}} \max|\phi|^3 \quad \textit{and} \quad \lambda_1^{L_R} = \frac{n-2}{\sqrt{n(n-1)}} \max|\phi|^3,$$

if and only if M is totally umbilical.

2 Preliminaries

Let M be an n-dimensional space-like hypersurface in an $(n+1)$-dimensional de Sitter space S_1^{n+1}. We choose a local field of semi-Riemannian orthonormal frames $\{e_1, \dots, e_{n+1}\}$ in S_1^{n+1} such that at each point of M, $\{e_1, \dots, e_n\}$ span the tangent space of M and form an orthonormal frame there. We use the following convention on the range of indices:

$$1 \leqslant A, B, C, \dots \leqslant n+1\,; \quad 1 \leqslant i, j, k, \dots \leqslant n\,.$$

Let $\{\omega_1, \ldots, \omega_{n+1}\}$ be the dual frame field so that the semi-Riemannian metric of the de Sitter space S_1^{n+1} is $d\bar{s}^2 = \sum_i \omega_i^2 - \omega_{n+1}^2 = \sum_A \epsilon_A \omega_A^2$, where $\epsilon_i = 1$ and $\epsilon_{n+1} = -1$.

Restrict to M,

$$\omega_{n+1} = 0\,. \tag{3}$$

Cartan's Lemma implies that

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j\,, \quad h_{ij} = h_{ji}\,. \tag{4}$$

The Gauss equation is

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{5}$$

where R_{ijkl} are the components of the curvature tensor of M and

$$h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j \tag{6}$$

is the second fundamental form of M. From the above equation,

$$n(n-1)(R-1) = S - n^2H^2, \tag{7}$$

where $n(n-1)R$ is the scalar curvature of M, H is the mean curvature and $S = \sum_{i,j} h_{ij}^2$ is the norm square of the second fundamental form of M.

The Codazzi equation is

$$h_{ijk} = h_{ikj}\,. \tag{8}$$

We consider the differential operator $\Box$ defined by

$$\Box f = \sum_{i,j=1} (nH\delta_{ij} - h_{ij})f_{ij}\,, \tag{9}$$

where $df = \sum_i f_i\omega_i$, $\sum_{i,j} f_{ij}\omega_j = df_i + \sum_j f_j\omega_{ji}$.

We know that the Laplace–Beltrami operator Δ is always elliptic. From the discussion of Cheng and Yau [11], we know that the operator $\square$ is self-adjoint, and from the result of Cheng and Ishikawa [10], we know that if M is an n-dimensional space-like hypersurface in S_1^{n+1} with constant scalar curvature $n(n-1)R$ and $R < 1$, then $\square$ is an elliptic operator (see a Lemma of Cheng and Ishikawa [10]). Let $\lambda_1^{L_H}$ and $\lambda_1^{L_R}$ be the first eigenvalues of the Schrödinger operators L_H and L_R, respectively. Since Δ and $\square$ are elliptic operators, from (1) and (2) we know that L_H and L_R are elliptic operators. We can use the min-max characterisation of $\lambda_1^{L_H}$ and $\lambda_1^{L_R}$, as

$$\lambda_1^{L_H} = \min\left\{\frac{\int_M fL_H(f)\,dv}{\int_M f^2\,dv} : f \in \mathcal{C}^\infty(M),\ f \not\equiv 0\right\}, \tag{10}$$

$$\lambda_1^{L_R} = \min\left\{\frac{\int_M fL_R(f)\,dv}{\int_M f^2\,dv} : f \in \mathcal{C}^\infty(M),\ f \not\equiv 0\right\}. \tag{11}$$

From the result of Brasil Jr. et al. [7], we know that if M is a orientable space-like hypersurface with constant mean curvature H in S_1^{n+1}, then

$$\begin{aligned}\frac{1}{2}\Delta|\phi|^2 &= |\nabla\phi|^2 + \left(|\phi|^2\right)^2 - nH\,\mathrm{tr}\,\phi^3 + n(1-H^2)|\phi|^2 \\ &\geqslant |\nabla\phi|^2 + |\phi|^2\left\{|\phi|^2 - \frac{n(n-2)H}{\sqrt{n(n-1)}}|\phi| + n(1-H^2)\right\},\end{aligned} \tag{12}$$

where the following result due to Okumura [14] and Alencar, do Carmo [1] is used:

> Let $\mu_1, \mu_2, \ldots, \mu_n$ be real numbers such that $\sum_i \mu_i = 0$, and $\sum_i \mu_i^2 = \beta^2$, where $\beta =$ constant $\geqslant 0$, then
>
> $$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leqslant \sum_i \mu_i^3 \leqslant \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{13}$$
>
> and equality holds in (13) if and only if at least $(n-1)$ of the numbers μ_i are equal.

From the calculation of Liu [13] or Shu [15], we conclude that for orientable space-like hypersurfaces with constant scalar curvature $n(n-1)R$ in S_1^{n+1},

$$\begin{aligned}\square(nH) &= |\nabla h|^2 - n^2|\nabla H|^2 + \left(|\phi|^2\right)^2 - nH\operatorname{tr}\phi^3 + n(1-H^2)|\phi|^2 \\ &\geqslant |\nabla h|^2 - n^2|\nabla H|^2 + |\phi|^2\left\{|\phi|^2 - \frac{n(n-2)H}{\sqrt{n(n-1)}}|\phi| + n(1-H^2)\right\},\end{aligned} \tag{14}$$

and if $R < 1$, then

$$|\nabla h|^2 \geqslant n^2|\nabla H|^2. \tag{15}$$

3 Proof of theorems

We firstly state a proposition proved by making use of a method similar to that used by C. Wu [16] or A. A. Barros et al. [8] for a Riemannian manifold.

Proposition 3 *Let M be an n-dimensional space-like hypersurface in an $(n+1)$-dimensional de Sitter space S_1^{n+1}. Then*

$$\left|\nabla|\phi|^2\right|^2 \leqslant \frac{4n|\phi|^2}{n+2}|\nabla\phi|^2. \tag{16}$$

Proof: We easily see that $\phi_{ij} = h_{ij} - H\delta_{ij}$ with eigenvalues $\mu_i = \lambda_i - H$ and ϕ is traceless, that is, $\sum_k \phi_{kk} = 0$. By the Cauchy–Schwarz inequality

$$\begin{aligned}\left|\nabla|\phi|^2\right|^2 &= 4\sum_k\left(\sum_{i,j}\phi_{ij}\phi_{ijk}\right)^2 = 4\sum_k\left(\sum_i \mu_i\phi_{iik}\right)^2 \\ &\leqslant 4\sum_i \mu_i^2\sum_{i,k}(\phi_{iik})^2 = 4|\phi|^2\left(\sum_i(\phi_{iii})^2 + \sum_{i,k,k\neq i}(\phi_{iik})^2\right).\end{aligned} \tag{17}$$

Since ϕ is traceless, $\phi_{iii} = -\sum_{k,k\neq i} \phi_{kki}$. Thus

$$\sum_i (\phi_{iii})^2 = \sum_i \left(\sum_{k,k\neq i} \phi_{kki}\right)^2 \leqslant (n-1) \sum_{k,i,k\neq i} (\phi_{iik})^2.$$

From above two inequalities,

$$\left|\nabla|\phi|^2\right|^2 \leqslant 4n|\phi|^2 \sum_{k,i,k\neq i} (\phi_{iik})^2.$$

From (8), we know that ϕ_{ijk} are symmetric for the three indices i, j, k. By the above inequality and (17),

$$\begin{aligned}
\left|\nabla|\phi|^2\right|^2 &= |\phi|^2 \sum_{i,j,k} (\phi_{ijk})^2 \\
&= |\phi|^2 \left(\sum_i (\phi_{iii})^2 + 3 \sum_{i,k,i\neq k} (\phi_{iik})^2 + 6 \sum_{i<j<k} (\phi_{ijk})^2\right) \\
&\geqslant |\phi|^2 \left(\sum_i (\phi_{iii})^2 + \sum_{i,k,i\neq k} (\phi_{iik})^2 + 2 \sum_{i,k,i\neq k} (\phi_{iik})^2\right) \\
&\geqslant \frac{1}{4} \left|\nabla|\phi|^2\right|^2 + \frac{1}{2n} \left|\nabla|\phi|^2\right|^2 = \frac{n+2}{4n} \left|\nabla|\phi|^2\right|^2.
\end{aligned}$$

This completes the proof of Proposition 3. ♠

Proof of 1: Since M is orientable, we assume that $H \geqslant 0$. For every $\varepsilon > 0$, from (10), we introduce a smooth function $f_\varepsilon = \sqrt{\varepsilon + |\phi|^2}$ as the test function to estimate $\lambda_1^{L_H}$. Then

$$\Delta f_\varepsilon = \frac{1}{2\sqrt{\varepsilon + |\phi|^2}} \Delta|\phi|^2 - \frac{1}{4(\varepsilon + |\phi|^2)^{3/2}} \left|\nabla|\phi|^2\right|^2. \tag{18}$$

From (12) and (18) and Proposition 3,

$$\begin{aligned} f_\varepsilon \Delta f_\varepsilon &= \frac{1}{2}\Delta|\phi|^2 - \frac{1}{4(\varepsilon+|\phi|^2)}\left|\nabla|\phi|^2\right|^2 \\ &\geqslant |\phi|^2\left\{|\phi|^2 - \frac{n(n-2)H}{\sqrt{n(n-1)}}|\phi| + n(1-H^2)\right\} \\ &\quad + |\nabla\phi|^2 - \frac{1}{4(\varepsilon+|\phi|^2)}\left|\nabla|\phi|^2\right|^2 \\ &\geqslant |\phi|^2\left\{|\phi|^2 - \frac{n(n-2)H}{\sqrt{n(n-1)}}|\phi| + n(1-H^2)\right\} \\ &\quad + \left\{1 - \frac{n|\phi|^2}{(n+2)(\varepsilon+|\phi|^2)}\right\}|\nabla\phi|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} f_\varepsilon L_H f_\varepsilon &= -f_\varepsilon\Delta f_\varepsilon + \left\{|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi|\right\} f_\varepsilon^2 \\ &\leqslant -|\phi|^2\left\{|\phi|^2 - \frac{n(n-2)H}{\sqrt{n(n-1)}}|\phi| + n(1-H^2)\right\} \\ &\quad - \left\{1 - \frac{n|\phi|^2}{(n+2)(\varepsilon+|\phi|^2)}\right\}|\nabla\phi|^2 \\ &\quad + \left\{|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi|\right\}(\varepsilon+|\phi|^2). \end{aligned}$$

Using (10) with f_ε as a test function,

$$\begin{aligned} \lambda_1^{L_H}\int_M(\varepsilon+|\phi|^2)\,dv =& \lambda_1^{L_H}\int_M f_\varepsilon^2\,dv \leqslant \int_M f_\varepsilon L_H(f_\varepsilon)\,dv \\ &\leqslant -\int_M |\phi|^2\left\{|\phi|^2 - \frac{n(n-2)H}{\sqrt{n(n-1)}}|\phi| + n(1-H^2)\right\}dv \end{aligned}$$

$$- \int_M \left\{ 1 - \frac{n|\phi|^2}{(n+2)(\varepsilon + |\phi|^2)} \right\} |\nabla \phi|^2 \, dv$$
$$+ \int_M \left\{ |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\phi| \right\} (\varepsilon + |\phi|^2) \, dv \,. \quad (19)$$

Letting $\varepsilon \to \infty$ in (19),

$$\lambda_1^{L_H} \int_M |\phi|^2 \, dv \leqslant -n(1 - H^2) \int_M |\phi|^2 \, dv - \int_M \frac{2}{n+2} |\nabla \phi|^2 \, dv. \quad (20)$$

Since $\lambda_1^{L_H} \geqslant -n(1 - H^2)$, from (20), $|\nabla \phi|^2 = 0$. Proposition 3 implies that $\nabla |\phi|^2 = 0$, that is, $|\phi|^2$ is constant. Therefore, we know that $|\phi|^2 - (n(n-2)/\sqrt{n(n-1)}) H|\phi|$ is constant. From (1), we obtain that $\lambda_1^{L_H} = |\phi|^2 - (n(n-2)/\sqrt{n(n-1)}) H|\phi|$. So

$$-n(1 - H^2) \leqslant |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\phi| \,,$$

that is

$$|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\phi| + n(1 - H^2) \geqslant 0 \,.$$

Therefore, we know that the equalities in (12) and (13) hold and

$$|\phi|^2 \left\{ |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\phi| + n(1 - H^2) \right\} = 0 \,.$$

This implies that $|\phi|^2 = 0$, that is, M is totally umbilical, or

$$|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\phi| + n(1 - H^2) = 0 \,.$$

In this case, the equalities hold in (13) and it follows that M has at most two distinct constant principal curvatures. We conclude that M is totally umbilical from the compactness of M. This completes the proof of Theorem 1. ♠

Proof of 2: Since $R<1$, from the assertion in section 2 and the Gauss equation (7), we know that the operator $\Box$ is elliptic and $n^2H^2=S+n(n-1)(1-R)>0$. Hence, $H\neq 0$. Since M is orientable, we can assume that $H>0$. Thus, from (11), we introduce a smooth function $f=H$ as the test function to estimate $\lambda_1^{L_R}$. By (2) and (14),

$$\begin{aligned} L_R(H) &= -\Box(H)+\frac{1}{n}|\phi|^4+(1-H^2)|\phi|^2 \\ &\leqslant -\left\{\frac{1}{n}|\nabla h|^2-n|\nabla H|^2+\frac{1}{n}|\phi|^4+(1-H^2)|\phi|^2-\frac{n-2}{\sqrt{n(n-1)}}H|\phi|^3\right\} \\ &\quad +\frac{1}{n}|\phi|^4+(1-H^2)|\phi|^2 \\ &= -\left(\frac{1}{n}|\nabla h|^2-n|\nabla H|^2\right)+\frac{n-2}{\sqrt{n(n-1)}}H|\phi|^3. \end{aligned} \tag{21}$$

From (11) and (15),

$$\begin{aligned} \lambda_1^{L_R}\int_M H^2\,dv &\leqslant \int_M HL_R(H)\,dv \\ &= -\int_M H\left(\frac{1}{n}|\nabla h|^2-n|\nabla H|^2\right)\,dv+\int_M\frac{n-2}{\sqrt{n(n-1)}}H^2|\phi|^3\,dv \\ &\leqslant \int_M\frac{n-2}{\sqrt{n(n-1)}}H^2|\phi|^3\,dv\leqslant\frac{n-2}{\sqrt{n(n-1)}}\max|\phi|^3\int_M H^2\,dv\,. \end{aligned} \tag{22}$$

Thus,

$$\lambda_1^{L_R}\leqslant\frac{n-2}{\sqrt{n(n-1)}}\max|\phi|^3.$$

If $\lambda_1^{L_R}=\left[(n-2)/\sqrt{n(n-1)}\right]\max|\phi|^3$, then the equalities in (22), (21), (15), (14) and (13) hold. Since the operator $\Box$ is self-adjoint and M is compact, from (14), we obtain that

$$\int_M|\phi|^2\left\{|\phi|^2-\frac{n(n-2)H}{\sqrt{n(n-1)}}|\phi|+n(1-H^2)\right\}\,dv=0\,.$$

This implies that $|\phi|^2 = 0$ and M is totally umbilical, or

$$|\phi|^2 - \frac{n(n-2)H}{\sqrt{n(n-1)}}|\phi| + n(1-H^2) = 0\,. \tag{23}$$

By the Gauss equation (7), (23) is equivalent to

$$\begin{aligned}|\phi|^2 - &\frac{n-2}{n-1}|\phi|\sqrt{|\phi|^2 - n(n-1)(R-1)} \\ &+ n\left(1 - \frac{1}{n(n-1)}|\phi|^2 + (R-1)\right) = 0\,. \end{aligned}\tag{24}$$

Since the scalar curvature $n(n-1)R$ is constant, from (24), $|\phi|$ is constant. By the equalities of (13), we know that M has at most two distinct constant principal curvatures. We conclude that M is totally umbilical from the compactness of M. This completes the proof of Theorem 2. ♠

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