Using influence diagrams as a tool for decision making

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Abstract

This mathematics in industry project explores influence diagrams as tools for decision making. Multi-link connections and transmission of influence are identified as important. Computer simulations prove valuable for understanding the process. Dynamical system models of evolving variable states relate influence diagrams to algebraic concepts.

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1 Introduction

An influence diagram is a tool used in decision making. It consists of a network of nodes representing events joined by directed links that represent the direct influence between events. The overall purpose of the Mathematics and Statistics in Industry Study Group (MISG) project was to explore such structures, to analyse them and to consider their interpretation. The industry representatives from the Defence Science and Technology Organisation (DSTO) encouraged open ended discussion. The initial approach was open minded, and not too focused, in order to think about a variety of ideas and generic concepts. Later in the week some basic assumptions were made to explore an illustrative system; however, this can be readily extended to more general situations within the context of influence diagrams.

Section 2 begins by defining what we mean by an influence diagram. Considering influence diagrams, and relating to the work of DSTO, a number of potential uses are identified. These are discussed in Section 3. The group conducted new investigations of its own involving computer simulations. These were instructive and insightful and suggested new tools and approaches for interpreting influence diagrams. Section 5 gives information on this investigation. Partly inspired by the computational simulations, influence diagrams are shown to relate to dynamical systems and are given an algebraic underpinning in Section 6. Section 7 concludes.

2 Definition and background

An influence diagram is a tool used in decision making [7]. The general structure is a set of nodes linked together by directed paths (Figure 1). As such they are immediately comparable (as a mathematical structure) to a graph or flow diagram.

The nodes represent events, in a very general sense, and a directed arc linking two events indicates that one event influences the other other event. For each event there is an associated value indicating a strength or state. This is modified by the influence of other events. The influence passed along links between events may be known to differing degrees of precision. It may be identified qualitatively or quantitatively. Likewise, the value associated with an event (event value) may be known qualitatively or quantitatively.

For illustration, the influence diagram shown in Figure 1 could be associated with the sport of rugby. Event 1 is the outcome that the All Blacks beat the Wallabies in the current Bledisloe Cup match between New Zealand and Australia. Event 2 is that the Wallabies get demoralised following the Bledisloe Cup and Event 3 is the outcome that the Wallabies win the Webb Ellis Rugby World Cup. Events 1 and 3 are then (individually) either true or false, which might be assigned the discrete values ± 1. These events are presumed to be easy to measure. Event 2, the degree of demoralisation of the Wallabies, is a continuous quantity. It might be expressed qualitatively (for example, quite dejected, miserable, or even elated if they won) and is harder to measure or quantify.

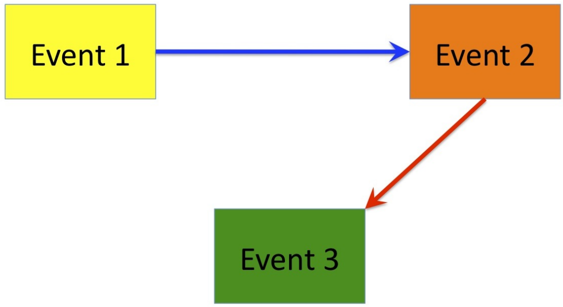

Figure 1: A simple influence diagram connecting three events. The links between events are coloured blue if the influence passed down the link is positive and red if it is negative. The links indicate that Event 1 influences Event 2 positively. In turn Event 2 influences Event 3 negatively. Event 1 does not directly influence Event 3 but does affect this event indirectly by transmission of influence through Event 2.

The influence between two events may be positive, for which an improvement in the first event's state tends to improve the recipient event's state. For example, there is positive influence between Events 1 and 2 in Figure 1 (the All Blacks winning the Bledisloe Cup tends to demoralise the Wallabies afterwards). The influence may instead be negative and have the reverse effect, as between Events 2 and 3 in Figure 1 (if the Wallabies are demoralised after the Bledisloe Cup, then they are less likely to win the Webb Ellis Rugby World Cup).

3 Potential uses of influence diagrams

In general, and in particular in the context of DSTO, a variety of uses of influence diagrams can be identified. Some decisions need to be made rapidly: an influence diagram can be used to provide insight. For other projects, weeks or longer may be taken in planning and a suitable influence diagram may evolve during development. Generating an influence diagram could be an iterative process with "bad guesses" identified by some kind of validation process and then improved upon. It may be possible to isolate parts of a diagram as a sub-network. Diagrams can be constructed and considered as hierarchical structures in some cases, perhaps hiding some details for an easy overview but maintaining the opportunity to call upon them if required.

In order to perform meaningful analysis, the information available on events and influences needs to be specified in some useable form. Whatever form is chosen for events and the influence between them, the state of the system is likely to be dynamic and evolve with time. The strength with which events influence one another may change. It may be useful in some cases to add extra nodes and links.

For instance, suppose that the rugby influence diagram given in Section 2 was created for the 2003 Rugby World Cup year. Then it would need to be adapted for application to a year mid-way between the four yearly Rugby World Cup competition when a losing team in the Bledisloe Cup has longer to recover before contending for the Webb Ellis Rugby World Cup. Perhaps a new diagram should be created with extra events to represent all the Bledisloe Cup matches occurring before the Webb Ellis Rugby World Cup.

4 Transmission of influence

One aspect of influence diagrams identified to be of importance is the concept of the transmission of influence and cumulative influence. Events that are not directly joined by a link, which represents a direct influence, may still be related indirectly through transmission of influence along a path through intermediary events. The identification of the multi-link connections between nodes (events) and calculation of a cumulative measure of influence over time is an important part of interpreting an influence diagram.

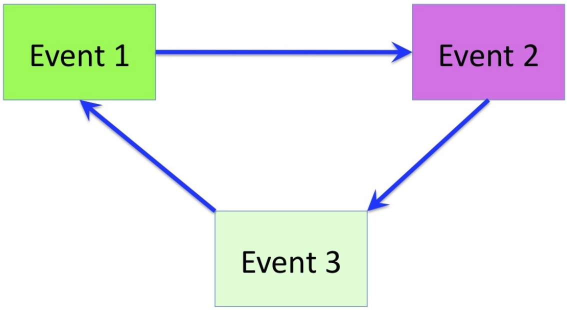

Figure 2: A simple influence diagram connecting three events in a cycle. The links indicate that Event 1 influences Event 2. In turn Event 2 influences Event 3. Finally, Event 3 influences Event 1 and so there is feedback from the original change in Event 1.

In Figure 1 the links between events are coloured to indicate whether the influence is positive (blue) or negative (red). The links in the diagram indicate that Event 1 influences Event 2 positively and Event 2 influences Event 3 negatively. The influence Event 1 has on Event 2 is transmitted to Event 3 through the influence of Event 2 on Event 3. We have both positive and negative influences in the path from Event 1 to Event 3. If Event 2 proved to be of little interest in itself we might consider removing this event and replacing the links with a single link for Event 1 to directly influence Event 3.

Influence diagrams are further complicated by cycles or feedback loops. In the three-event example shown in Figure 2 we assume all the events have associated values on some scale. Suppose that the value associated with Event 1 increases. Then Event 1 has a positive influence on Event 2, and its associated value also increases. Event 2 has a positive influence on Event 3 and so the value associated with Event 3 also grows. Finally Event 3 has a positive influence upon Event 1 and so Event 1's value now grows further and this is an indirect result of its initial growth. The growth will continue through this feedback.

The events in this example could represent confidence in Asian, European and North American Stock Exchanges which influence one another as the working day progresses around the world. The associated event values could be indices for the prices of stocks in each of these markets.

A potential problem with cycles is that an event can affect itself by transmission of influence. For a simple response to transmitted influence there may be unconstrained growth. However, this can be mitigated by renormalisation, as is considered in Section 5. Problems with unconstrained growth can occur because of the simplicity of the model, as in the stock market example above, or perhaps be due to the specific way in which the influence between events is modelled.

Many of the events involved in an influence diagram are likely to be qualitative in nature, as indeed are the overall conclusions to be drawn from the system. However, to perform calculations or implement computational procedures, events need to be assigned quantitative values.

5 Computer simulation and illustration

To further the investigation of influence diagrams a computer program was written to simulate the effects of influence permeating through a system. Homer and Oliva [4] argue for the power of computer simulations. An important part of our investigation is a visualisation scheme for the underlying network. Figure 3 is the visualisation of the simple influence diagram used for this exploration. This model was created to have a greater complexity than the earlier illustrations, which is more realistic. As before, the rectangles represent events (the colours in this diagram have no importance other than for identification). Again, the links between events are coloured blue if the influence passed down the link is positive and red if it is negative. The direction of influence is indicated by a small circle on the link closer to the event which is being influenced.

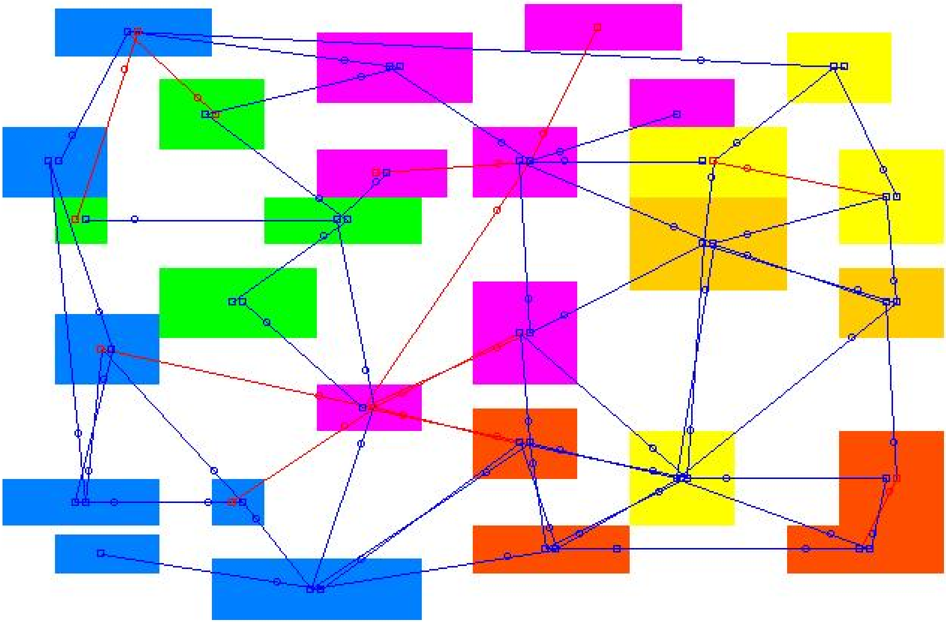

Figure 3: The influence diagram used for the computer simulation. In Figures 4–6 we show ways in which influence is transmitted from the blue rectangle in the top left-hand corner to the red rectangle in the middle of the base of the diagram. These events have no direct connection. The colours of the event boxes have no significance at this stage other than for identification. This diagram was output from the actual computer simulation as are the diagrams in subsequent figures.

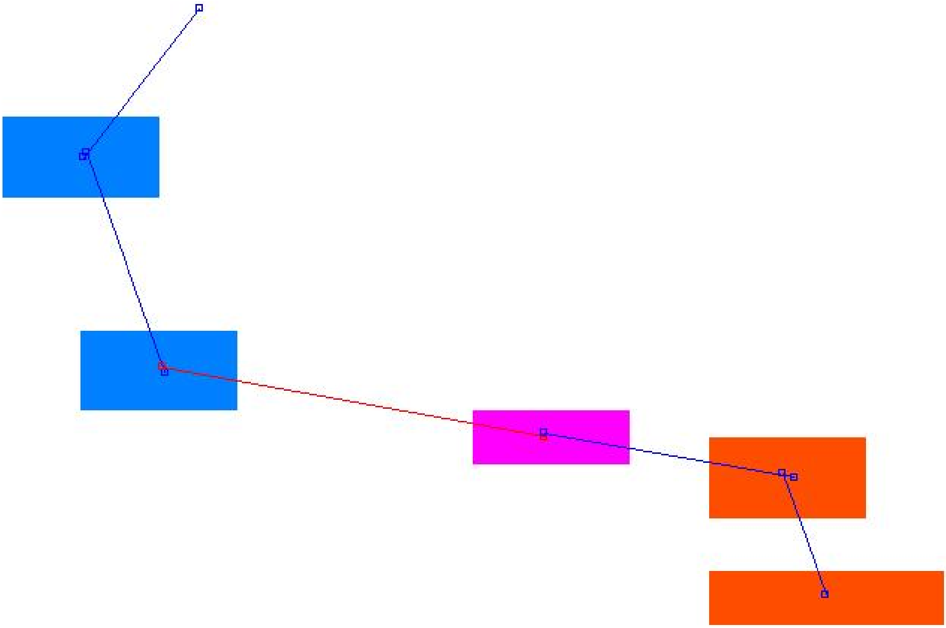

Figure 4: A short path (five links) for transmission of influence between the two events identified in Figure 3. The third link is red, indicating a negative influence.

An early stage in the exploration is to run a depth first search algorithm (DFS) on the influence diagram. This finds the multi-link paths down which influence may be transmitted between events. A variety of connections are found: long and short connections, connections which include cycles, and ones where both positive and negative influences are present. These are illustrated with examples, using the system shown in Figure 3.

We consider two specific events within the influence diagram of Figure 3: that represented by the blue rectangle in the top left-hand corner and that represented by the red rectangle in the middle of the base of the diagram. There is no direct link or influence between these two events; however, there

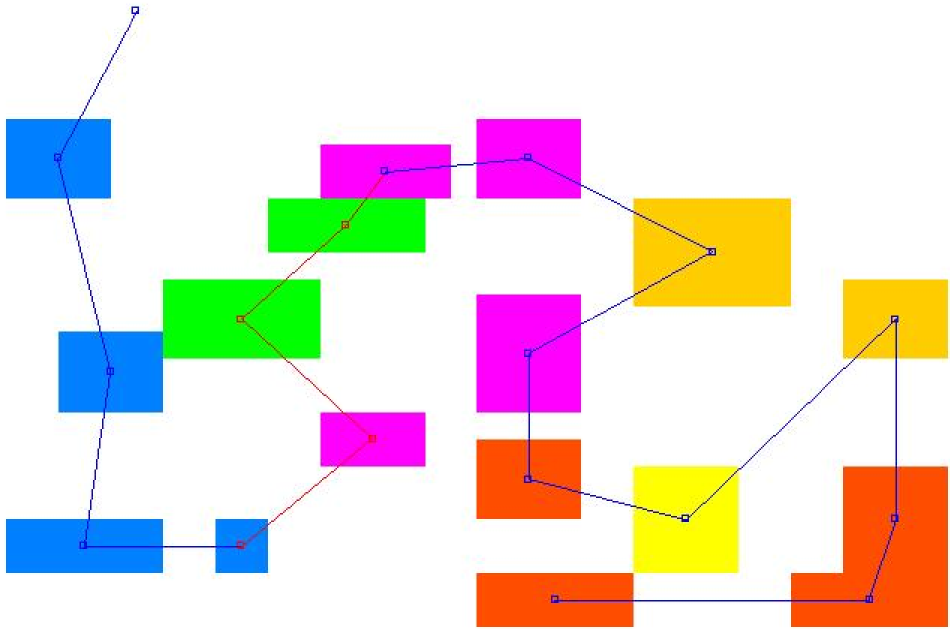

Figure 5: A longer path (seventeen links) for transmission of influence between the same two events as in Figure 4.

are a number of paths through which the blue rectangle indirectly affects the red rectangle through transmission of influence. Figure 4 illustrates one such path for transmission of influence between the two events. This is a comparatively short route only passing down five links. In this case, as the influence is transmitted down the path it passes through links of both positive and negative influence. Figure 5 shows a somewhat longer path.

Figure 6 shows a path containing a cycle. When constructing more complicated influence diagrams, like this one, it is very likely that some sort of cycle with feedback will occur unless this is explicitly avoided in some way. However, for simulations of this kind a simple renormalisation of event values is effective in

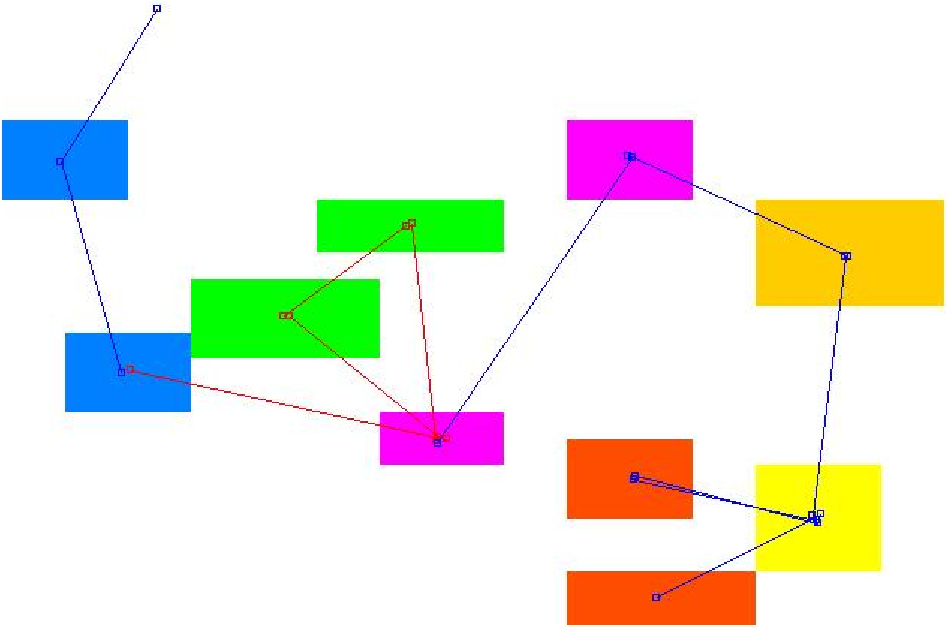

Figure 6: A path for transmission of influence that contains cycles. The path is between the same two events as in Figure 4. The purple rectangle three links down the path is joined with the two green rectangles in a cycle. There is further feedback through links between the yellow rectangle and the upper red rectangle.

keeping parameters manageable. To control feedback in this example, when the values are numerical, all event values are divided by the largest event value and the overall growth in the system values are recorded separately. This is similar to the power method for numerically finding the eigenvalues of a matrix and, indeed, with the simple assumptions made for the present simulation, this is precisely what we are doing! Section 6 gives further details.

In the influence diagram used for the computer simulation each event's state is represented by a numerical value. Initial quantities are assigned to these event values. To investigate the propagation of influence a sequence of steps are made during which influence is simultaneously passed down the links between events.

The values associated with influenced events, that is, those that are in receipt of influence from other events, are modified in a simple algebraic way. For our model we take the simple approach that influence travels the distance of one single link per time step and it travels simultaneously down all the links in the model. Associated with each link is a number representing the strength of the influence over a unit of time. The product of this number and the value associated with the influencing event is added to the event value of the influenced event to obtain the new value for the influenced event at the next time step.

This simple model serves our purpose of illustrating how influence is transmitted, however, one can readily build upon it to implement more complicated transmission rules or different speeds of transmission of influence down the links. Section 6 formalises theoretical details.

A strength of the computer simulation is the visualisation provided of the progression of influence between events in the influence diagram. This is valuable for insight into the process and is a useful tool beyond the strengths of a basic influence diagram. Figures 7, 8, 9 and 10 further illustrate the visualisation.

By providing images at each time step of the system the viewer is able to

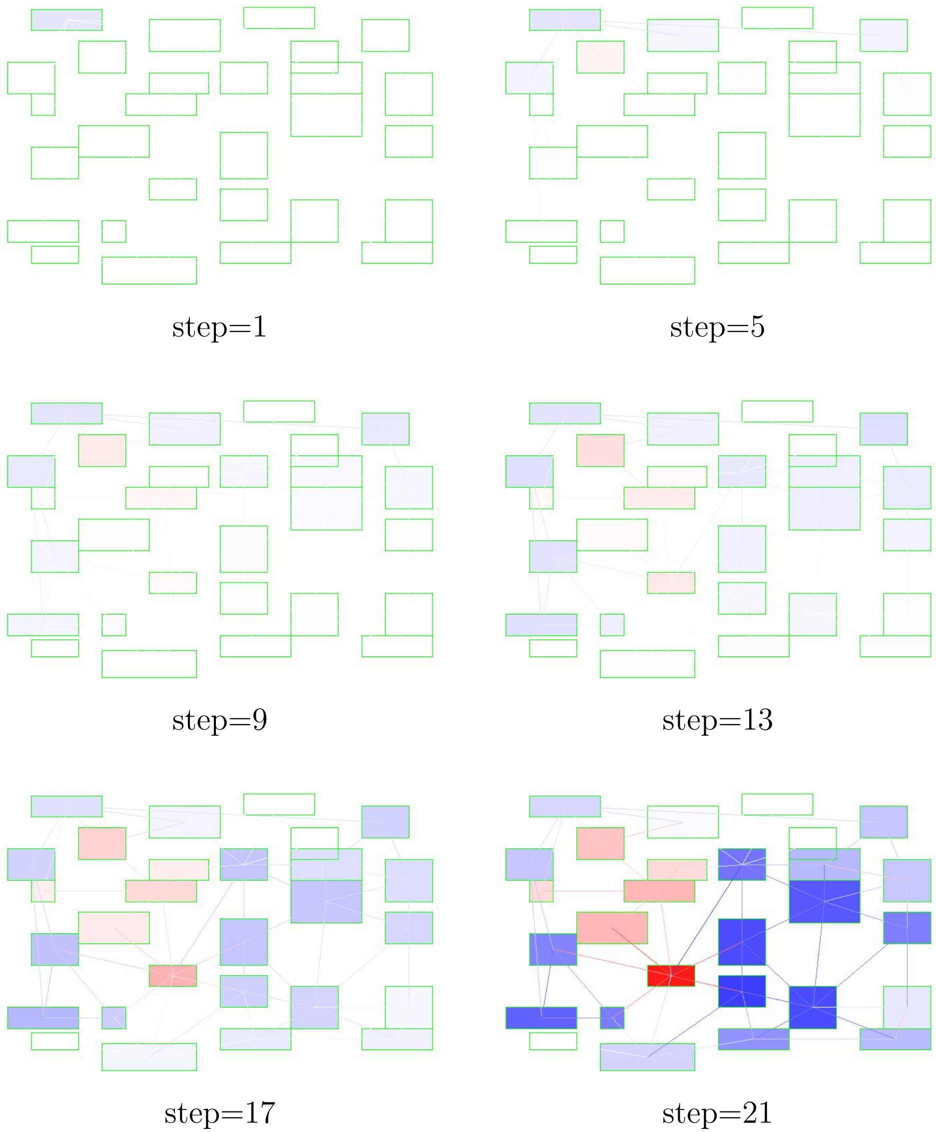

Figure 7: The boxes representing events in Figure 3 are coloured according to their current state, blue for positive and red for negative. The intensity of the colour indicates its relative magnitude. This sequence of pictures shows the evolving state of the influence diagram system during our simulations.

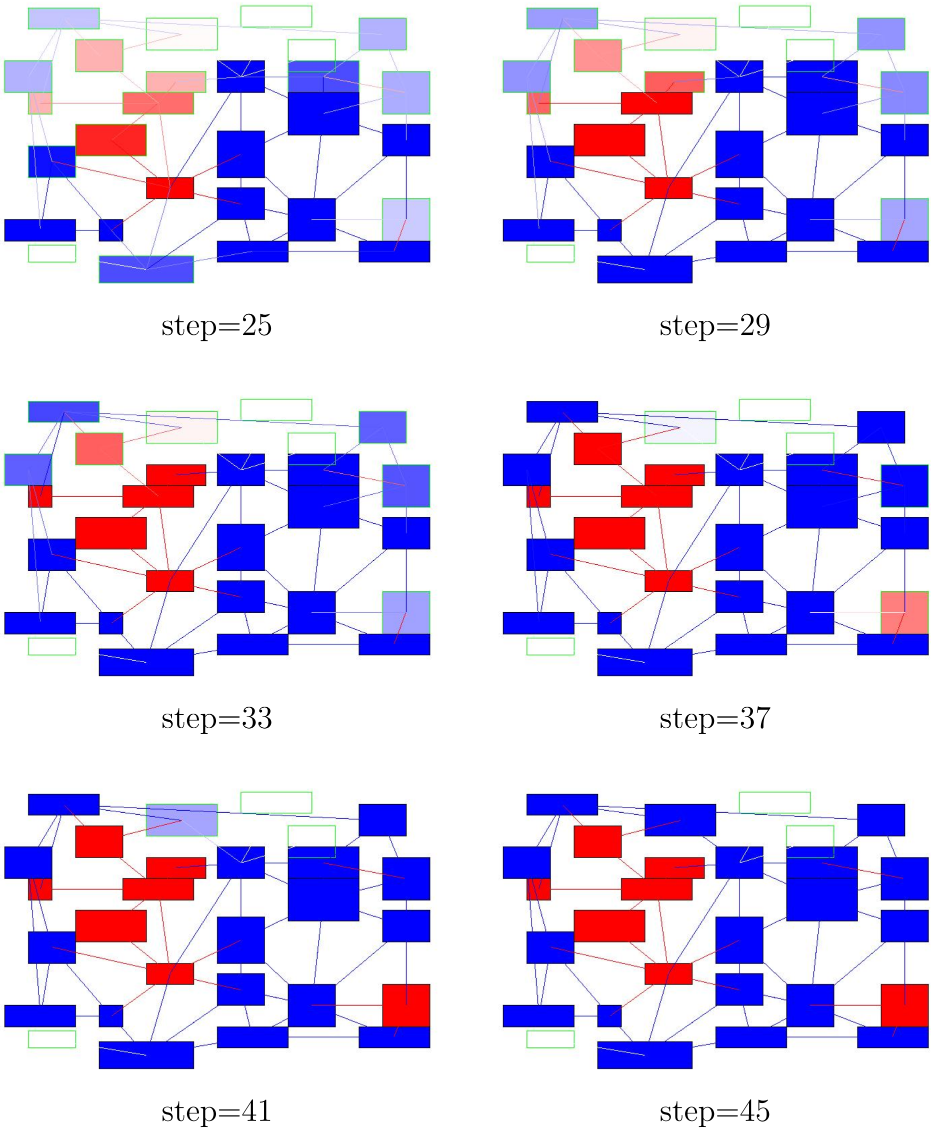

Figure 8: The continuation of the sequence of event states from Figure 7.

see how the event values associated with precursor events gradually change recipient event states. He or she can see the progression of cumulative influence with time. Figures 7 and 8 show one particular computer simulation using the influence diagram of Figure 3. Each individual picture shows a state of the system after a different number of time steps. Events are coloured blue if they have a positive state (event value) and red if they have a negative state, the intensity of the colour indicates relative magnitude: deep red for very negative, pale red for marginally negative. During the sequence of pictures some event values (colour/colour intensity) change relatively smoothly whereas other event values progress through a mixture of positive and negative states.

In Figure 9, the boxes representing events are split into two halves, blue for positive and red for negative. The brightness of the colours are adjusted according to how much positive and how much negative influence has been transmitted to the event.

Figure 10 shows the dominant eigenstate of our system. This is the state that our system evolved towards during the simulations, illustrated in Figures 7 and 8. As before, events are coloured blue if they have a positive state (event value) and red if they have a negative state, the intensity of the colour indicating magnitude. The calculation of such states and associated theoretical development is described in Section 6.

6 The underlying dynamical system

In order to develop a simulation, some fixed model of influence and its transmission is required. There is likely to be uncertainty about the precise form that the system should take. This uncertainty might be moderated by conducting a series of simulations across a grid of suitable realisations. It might also be possible to track uncertainties through the model. These possibilities are readily illustrated within the simulation framework of Section 5, perhaps with event boxes graduated in colour to show the range of uncertainty.

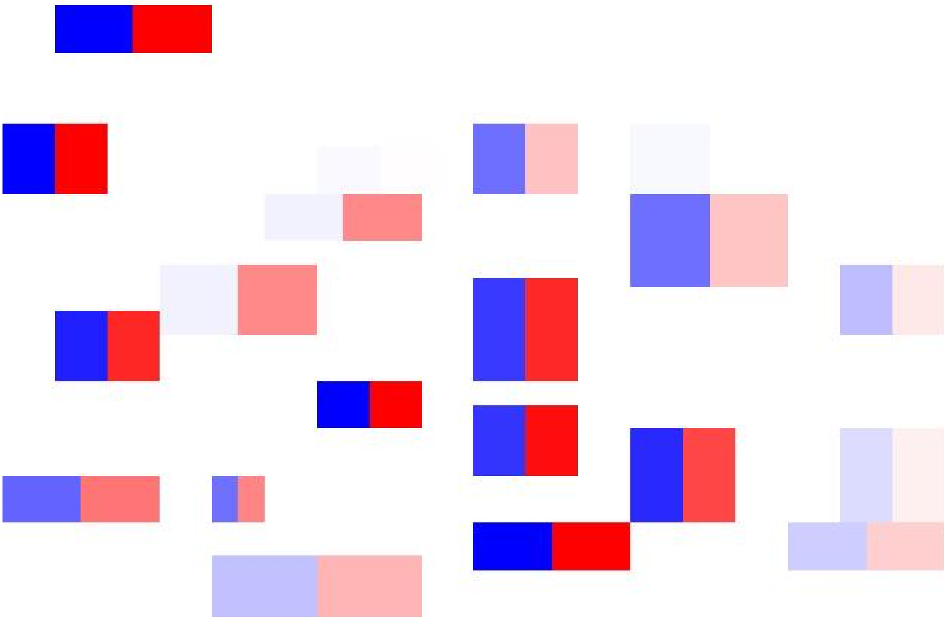

Figure 9: The boxes representing events in Figure 3 have been split in two halves, one is blue for positive influence and one is red for negative influence. After a number of time steps the brightness of the colours are adjusted according to how much positive and how much negative influence has been received at the event.

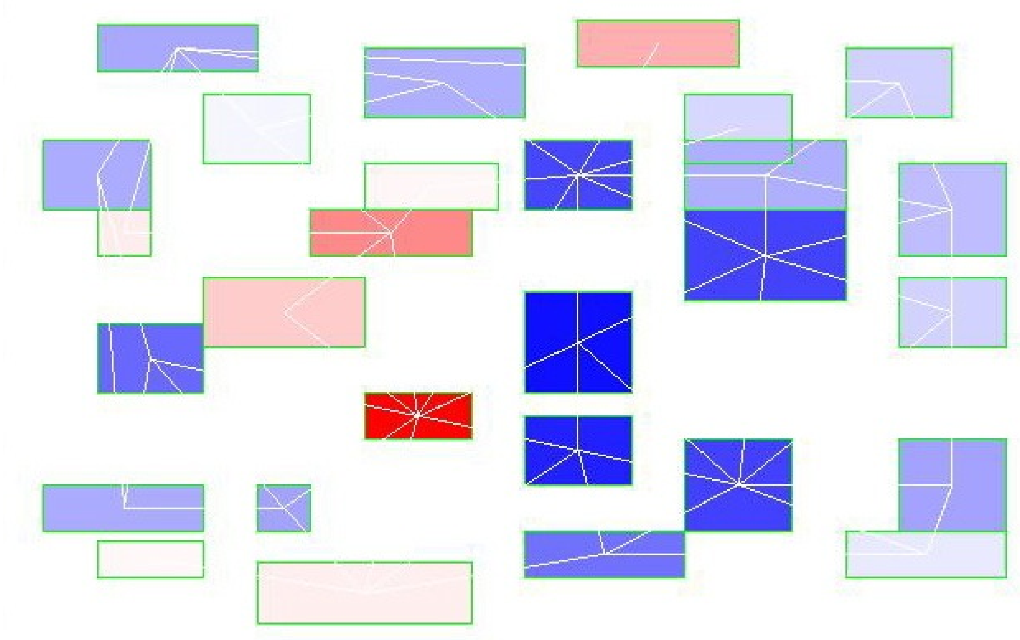

Figure 10: The boxes representing events in Figure 3 are coloured according to their current state, blue for positive and red for negative. As before, the intensity of the colours indicate the relative magnitude. This particular picture shows the dominant eigenstate for this particular system. The simulations illustrated in Figures 7 and 8 eventually tended towards this state.

The theory underpinning the above influence diagram simulations is described in this section. The end states of event values relate to eigenfunctions of the system and these depend purely on the network of influences. If one of the associated eigenvalues is real and dominant then, within this idealised model, any time limited initial intervention is likely to result in the same end-state. In any case, the eigenvalues of the linear system give an indication of the relative growth or decline of the system states. To avoid a stable dominant end-state in this model requires continual intervention or a change in the network of influences and incorporated events.

We begin by formalising some ideas within the context of our specific model.

6.1 Formalisation

We consider an influence diagram D, containing n distinct event nodes labelled x_i, where $i = 1, \ldots, n$. Each node has an associated value w_i to represent the event state. We let $X = \{x_i\}$ be the set of nodes and $W = \{w_i\}$ be the set of event values. The vector of event values is written $\mathbf{w}$.

Definition 1 (Adjacency)**.** *For all $i, j \in \{1, \ldots, n\}$ a directed adjacency indicator a_{ij} is defined:*

- $a_{ij} = 1$ *if there is a directed path or link from the ith event node to the jth event node; and*
- $a_{ij} = 0$ *otherwise.*

This gives the adjacency matrix for the event nodes $A = (a_{ij})$ in the usual manner. As A shows the connections between nodes over one link and $A_k = A^k$ shows the links of length k, the matrix

$$A^{\{k\}} = \sum_{j=1}^{k} A_j \tag{1}$$

shows the total number of paths (through possibly multiple event nodes) of k or less time steps. The links between events convey influence.

Definition 2 (Influence)**.** *For all $i, j \in \{1, \ldots, n\}$ we define γ_{ij} where*

- $\gamma_{ij} \in R$ *if there is a directed path conveying influence from the ith event node to the jth event node in a single step, and*
- $\gamma_{ij} = 0$ *otherwise.*

The parameter γ_{ij} is a multiplicative measure of how the event value w_i of event node x_i directly influences event value w_j at node x_j over one unit of time. As with adjacency, we define the influence matrix $\Gamma = (\gamma_{ij})$. This matrix details the specific ("per-capita") influence each of the events has on each other event over one time step. Computationally, A can be easily constructed from Γ.

All relevant information about the influence diagram's structure is encapsulated in Γ, X and W, giving us $D = D(\Gamma, X, W)$.

6.2 Progression of influence

As described in Section 5, in each time step we take the new event value to be its present value plus a sum of terms from events that have a direct influence upon it. Within the present notation this can be written

$$w_i^{\text{new}} = w_i^{\text{initial}} + \sum_j \gamma_{ji} w_j^{\text{initial}} \Delta \,, \tag{2}$$

where Δ is a constant based on the time step we are considering. (In the simulation we took $\Delta = 1$ for simplicity.) The whole progression of influence can be written in matrix form:

$$\mathbf{w}^{\text{new}} = (I + \Gamma^T \Delta)\mathbf{w}^{\text{initial}}, \tag{3}$$

where I is the identity matrix. To get the state of the system (event values) after two steps we just multiply $\mathbf{w}^{\text{new}}$ above by $(I + \Gamma^T \Delta)$. After m time steps the state of the system is given by the event value vector

$$(I + \Gamma^T \Delta)^m \mathbf{w}^{\text{initial}}. \tag{4}$$

From Equation (3) it is apparent that the simulated model system has a number of steady event-value-vector directions, namely when the constituent values correspond to eigenvectors of the matrix $(I + \Gamma^T \Delta)$. By multiplying a vector of initial event values multiple times by the matrix, we are exactly replicating the process by which the power method is used to find an eigenvalue of a matrix. So we know that if the matrix $(I + \Gamma^T \Delta)$ satisfies some simple conditions (completeness, for example) and has a unique dominant eigenvalue, then the event value vector will approach a multiple of the corresponding dominant eigenvector from almost any set of initial event values. This is what happened in the simulation of Section 5: regardless of any initial intervention in event states the system always approached a multiple of the eigenstate shown in Figure 10. Just as for the power method, the rate of approach to this end-state depends upon the relative sizes of the two most dominant eigenvalues.

6.3 Continuous formulation

We take the model described by Equation (3) and consider the limit as the time step Δ tends to zero and obtain, rescaling time if necessary, the differential equation

$$\frac{d\mathbf{w}}{dt} = \Gamma^T \mathbf{w} \,. \tag{5}$$

This equation has solutions

$$\mathbf{w}(t) = \sum \exp(\lambda t)\mathbf{v} \,, \tag{6}$$

where λ and $\mathbf{v}$ are eigenvalues and eigenvectors of Γ^T, respectively (the eigenvectors are not normalised). This formulation allows us to study the long term behaviour of the solutions of the system.

We consider the evolution of the system from some initial state. If an event x_j has no influences acting upon it then its event value will not change. If we include a constant input or output b_i, Equation (2) becomes

$$w_i^{\text{new}} = w_i^{\text{initial}} + \left(\sum_j \gamma_{ji} w_j^{\text{initial}} + b_i \right) \Delta \,, \tag{7}$$

and Equation (5) is correspondingly

$$\frac{d\mathbf{w}}{dt} = \Gamma^{\mathsf{T}}\mathbf{w} + \mathbf{b}\,. \tag{8}$$

This is a non-homogeneous version of the system, which has a particular solution

$$\mathbf{w} = -(\Gamma^{\mathsf{T}})^{-1}\mathbf{b}$$

which is added to the right-hand side of Equation (6). Thus again we solve the system analytically and in a standard way calculate the asymptotic state.

7 Discussion and conclusions

We considered influence diagrams in general and then outlined a specific dynamical approach to the generic model. This approach has been used for simulations and examined analytically. It is not the only way a model can be constructed. As is clear, we approached this project naively, as necessitated by the project outline. Our solutions and models are simple and effective but there are a variety of very specific and in-depth studies on the solving, measuring and quantifying of influence diagrams for specific cases [2, 6, 5, e.g.]. Elsewhere, there is discussion of perturbation methods for the underlying matrices in the context of population models [1] and network measures when considering the influence diagram as a complex network [3]; however, further consideration of these is beyond the scope of this work.

We used an example influence diagram for illustration. In practice, constructing such diagrams can, in themselves, be quite a challenge and may need to be combined with a process of interpretation. There is a need to examine the robustness of the network using standard evaluation techniques (NET). This may embody procedures using Monte Carlo simulation, combinatorial optimisation, dynamical systems (as we have used) and data mining. Special attention needs to be given to investigating the sensitivity of the diagram to either:

1. adding or subtracting a node;
2. altering the event values on any nodes and/or the influence values between them.

As far as our model went, a reassuring agreement was found between the computer simulations of Section 5 and the underpinning algebra of Section 6. In particular, for the simulation of the influence diagram illustrated in Figure 3, we did indeed find that the system tended towards the dominant eigenstate (Figure 10) as identified by theory. However, the tendency of the event values to grow (or shrink) because of feedback and hence the requirement for renormalisation is unfortunate. The process is probably non-linear and the approach of our conceptual model may be better reformulated as

$$\frac{d\mathbf{w}}{dt} = \left[\Gamma^T + F(\mathbf{w})\right]\mathbf{w} + \mathbf{b}\,, \tag{9}$$

where $F(\mathbf{w})$ is a real matrix, and the function F is itself linear. This may make it possible to constrain the growth of event values.

Theoretical analysis and computer simulation of an influence diagram provides a means for interrogating the nature of the underlying system and predicting outcomes. The approach can be applied to a variety of situations: financial markets and sports are mentioned in the present article but other areas of application, such as ecology, exist. With sufficient detail known, the nature of the whole system (typically tens of state variables) can be evaluated. For conveying ideas and gaining a rapid insight into what was happening, the visual display approach used here appeared appropriate and valuable. It nicely illustrated the power of using influence diagrams to convey ideas.

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