A well balanced scheme for the shallow water wave equations in open channels with (discontinuous) varying width and bed

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Abstract

Finite volume methods have proven themselves a powerful tool for finding solutions to the shallow water wave equations. They are based on the conservation laws for the mass and momentum, integrated over discrete finite volumes. These methods tend to do well at the difficult problem of capturing solutions involving shocks. However, one area that causes problems is the approximation of steady or near steady states when there is a sloping bed elevation. The problem arises due to a poor balance between the discretisation of the flux terms across the edge of a finite volume and the pressure terms due to the sloping bed. Methods that overcome these difficulties and reproduce the still lake steady state solution, are called well balanced. In this work we are interested in a well balanced scheme for the one

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dimensional shallow water wave equations but with a modification that allows for varying width in the transverse direction. Here a well balanced method developed by Audusse et al. for the constant width case is extended to the case of varying (possibly discontinuous) width. Numerical validation of this new method is provided.

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1 Introduction

The flow in an open channel of constant width is well modeled by the one dimensional shallow water wave equations, which are given by the system of conservation laws

$$\frac{\partial}{\partial t}\begin{bmatrix} h \\ hu \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix} = \begin{bmatrix} 0 \\ -hgz_x \end{bmatrix}, \tag{1}$$

where $h(x,t)$ is water depth, $u(x,t)$ is depth averaged horizontal velocity, $z(x)$ is the bed elevation, and g is the constant of gravitational acceleration. Another variable, which we use extensively, is $w = h + z$, called the stage, or the vertical position of the water surface (see Figure 1).

Equation (1) has the form
$$\frac{\partial}{\partial t}\mathbf{q} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{q}) = \mathbf{s}(\mathbf{q}), \tag{2}$$
where $\mathbf{q}$ is the vector of conserved quantities, mass and momentum, $\mathbf{f}(\mathbf{q})$ is the flux of the conserved quantities, and $\mathbf{s}(\mathbf{q})$ is the pressure forcing term due to the water pressure generated by the sloping bed. Explicitly
$$\mathbf{q} = \begin{bmatrix} h \\ hu \end{bmatrix}, \quad \mathbf{f}(\mathbf{q}) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix} \quad \text{and} \quad \mathbf{s}(\mathbf{q}) = \begin{bmatrix} 0 \\ -ghz_x \end{bmatrix}.$$

A numerical scheme for the shallow water wave equations is said to be well balanced if the still lake steady state solution (constant stage $w = c$ and zero velocity $u = 0$) is exactly reproduced by the numerical scheme. In particular, when $w = c$ and $u = 0$, the numerical approximation of the flux pressure term $\left(\frac{1}{2}gh^2\right)_x$ needs to balance exactly the numerical approximation of the pressure forcing term $-ghz_x$. This is an important property for accurate solutions of many practical problems such as flows in lakes and estuaries and in tsunami inundation.

Now consider a channel of varying width $b(x)$. The cross sectional area of water at any point along a channel is bh, and the discharge (momentum) across this cross section is bhu. These new conserved variables give a new system of conservative laws:
$$\frac{\partial}{\partial t}\begin{bmatrix} bh \\ bhu \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} bhu \\ bhu^2 + \frac{1}{2}gh^2 b \end{bmatrix} = \begin{bmatrix} 0 \\ -hgz_x b + \frac{1}{2}gh^2 b_x \end{bmatrix}, \tag{3}$$
which are written in the form of Equation (2), but with different expressions for the conserved quantities, flux and forcing term; namely
$$\mathbf{q} = \begin{bmatrix} hb \\ hbu \end{bmatrix}, \quad \mathbf{f}(\mathbf{q}) = \begin{bmatrix} hbu \\ hbu^2 + \frac{1}{2}gbh^2 \end{bmatrix},$$

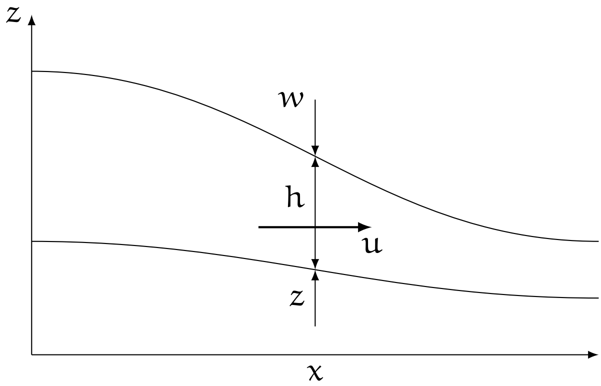

Figure 1: Nomenclature for shallow water wave equations, Equation (1); depth of water, h; elevation of the bed, z; the stage, $w = h + z$; and the depth averaged horizontal velocity, u.

$$\text{and} \quad \mathbf{s}(\mathbf{q}) = \begin{bmatrix} 0 \\ -ghbz_x + \frac{1}{2}gh^2 b_x \end{bmatrix}.$$

In this article we develop a well balanced method to solve the shallow water wave equations for open channel flow with varying width, that is, Equation (3). We consider the situation where both the width b and the bed elevation z of the channel can be discontinuous. Well balancing is important for many practical problems. For instance, for tracking the movement of a surge up a river of greatly varying width and depth over a long distance, it is important that the steady part of the solution is accurately maintained until the surge reaches a point of interest.

To develop a well balanced scheme for Equation (3) we extend a well balanced method for Equation (1) developed by Audusse et al. [1]. Audusse's method is particularly simple and flexible while still retaining the desired properties of other numerical schemes.

There are of course other options, such as a simple scheme developed by

Zhou et al. [8] called the surface-gradient method, but this requires z to be defined as a piecewise affine *continuous* function. There are also more complicated methods such as the exact Godunov solver (based on the exact solution to the Riemann problem) which theoretically should solve the problem without giving up other desirable properties but it is computationally expensive [1]. Leveque [5] described a well balanced method which deals with the forcing term and balancing, first by adjusting heights and then applying a numerical method to calculate fluxes.

Audusse's method has no problem with discontinuous z, is computationally efficient (provided an efficient numerical flux algorithm is used), and is obtained as a fairly simple modification of any standard method. So we use this as the basis for our new method. Section 2 provides an introduction to the basic structure of the finite volume method, together with a description of our notation. Then Section 3 overviews the well balancing method developed by Audusse et al. [1].

Audusse's method is not fixed to any specific numerical scheme for computing the fluxes or any particular time stepping algorithm. Therefore these details can be chosen at the time of implementation. Thus Sections 2 and 3 describe our method in the context of a semi-discrete algorithm.

Section 5 presents two sets of numerical experiments, verifying that the new scheme is well balanced for a still lake problem, and that it still works well for a challenging non-stationary problem containing a shock.

2 The standard finite volume method

A domain $[a, b]$ is partitioned into a collection of n intervals $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $i = 1, \ldots, n$. The midpoint of the ith interval is x_i and the mesh size of the ith interval is $\Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$.

We restrict our attention to functions g (here not gravity) on $[a, b]$ which are continuous when restricted to each closed interval I_i (and hence have left and right limits at each of the end points $x_{i+\frac{1}{2}}$). On each interval, we are interested in three discrete values derived from g: the average value g_i of g on that interval, $g_i \Delta x_i = \int_{I_i} g(x)\,dx$; the limiting value of g at right end of that interval, $g_{i,r} = g(x_{i+\frac{1}{2}}-)$; and the limiting value of g at the left end of that interval, $g_{i,l} = g(x_{i-\frac{1}{2}}+)$.

Given a set of discrete integral averages, g_i, associated with each interval in our partition, we consider a procedure to reconstruct a function g such that g is affine when restricted to each interval I_i and maintains the averages, that is, $\int_{I_i} g(x)\,dx = g_i \Delta x_i$. The reconstruction procedure should reconstruct constants, that is, if the g_i's are constant, then the reconstructed g is constant. One such reconstruction procedure is given by the piecewise constant function $\sum_i g_i \chi_{I_i}$. More accurate reconstructions are possible, such as van Leer's classic MUSCL scheme [7]. In these more accurate reconstructions it is necessary to avoid introducing excessive oscillations by ensuring that the total variation of the reconstruction satisfies

$$\mathrm{TV}(g) \leqslant \mathrm{TV}\left(\sum_i g_i \chi_{I_i}\right), \tag{4}$$

where TV is the total variation of a function. This property is usually obtained by 'limiting' the gradient of the reconstructed function so as to satisfy (4).

We describe the finite volume discretization of the shallow water wave equation using the form of the equations given by Equation (2). This allows a simple extension to the varying width equation. The finite volume discretization of Equation (2) is obtained by integrating Equation (2) over each interval I_i and then applying integration by parts to the flux term, to obtain a set of ordinary differential equations for the average of the vector of conserved quantities in each of the intervals. In particular

$$\frac{d}{dt}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{q}(x,t)\,dx + \mathbf{f}(\mathbf{q}(x_{i+\frac{1}{2}},t)) - \mathbf{f}(\mathbf{q}(x_{i+\frac{1}{2}},t)) = \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{s}(\mathbf{q})\,dx\,. \tag{5}$$

Let $\mathbf{q}_i$ be the discrete approximation of the average value of the conserved quantity vector in the interval I_i, that is,

$$\mathbf{q}_i \Delta x_i \approx \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{q}(x, t)\, dx\,.$$

The approximation of the flux at $x_{i+\frac{1}{2}}$ is

$$\mathbf{f}_{i+\frac{1}{2}} = \mathbf{f}_a(\mathbf{q}_{i,r}, \mathbf{q}_{i+1,l}) \approx \mathbf{f}(\mathbf{q}(x_{i+\frac{1}{2}}, t)),$$

where the terms $\mathbf{q}_{i,r}$ and $\mathbf{q}_{i+1,l}$ are the left and right limiting values at $x_{i+\frac{1}{2}}$, of the piecewise affine function reconstructed from the discrete average values $\mathbf{q}_i$. The function $\mathbf{f}_a(\mathbf{q}_-, \mathbf{q}_+)$ is a numerical scheme for approximating the flux generated at the origin by Equation (2) when the initial data consist of a constant state $\mathbf{q}_-$ to the left of the origin, and a constant state $\mathbf{q}_+$ to the right.

The term

$$\mathbf{s}_i \Delta x_i = \begin{bmatrix} 0 \\ -\int_{x_{i-1/2}}^{x_{i+1/2}} g h z_x\, dx \end{bmatrix} \approx \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{s}(\mathbf{q}(x, t))\, dx\,.$$

Since h and z are affine functions when restricted to I_i, the integral is calculated exactly to yield

$$\mathbf{s}_i = \begin{bmatrix} 0 \\ -\frac{1}{2} g (h_{i,r} + h_{i,l}) \frac{(z_{i,r} - z_{i,l})}{\Delta x_i} \end{bmatrix}.$$

This leads to a discrete scheme

$$\frac{d}{dt}\mathbf{q}_i + \frac{1}{\Delta x_i}\left(\mathbf{f}_{i+\frac{1}{2}} - \mathbf{f}_{i-\frac{1}{2}}\right) = \mathbf{s}_i\,. \tag{6}$$

A particular finite volume method is defined in terms of the reconstruction method for $\mathbf{q}_i = [h_i\,, h_i u_i]^T$ and z_i, the approximate flux function $\mathbf{f}_a$, and the particular ordinary differential equation method used to approximate Equation (6).

3 Audusse's well balanced method

Unfortunately the scheme (6) is not guaranteed to reproduce the still lake steady state solution; that is, it is not guaranteed to be well balanced. But Audusse et al. [1] provides a simple method to convert a standard finite volume method into a well balanced scheme.

The Audusse scheme has the form

$$\frac{d}{dt}\mathbf{q}_i + \frac{1}{\Delta x_i}\left(\widehat{\mathbf{f}}_{i+\frac{1}{2}} - \widehat{\mathbf{f}}_{i-\frac{1}{2}}\right) = (\mathbf{s}_i + \mathbf{c}_i)\,, \tag{7}$$

where the flux terms are adjusted to deal with discontinuities in the bed, and an extra forcing term $\mathbf{c}_i$ is added to correct the calculations so that the still lake solution is reproduced.

The adjusted flux calculation is based on a standard flux calculation, but applied to a so-called hydrostatic reconstruction of $\mathbf{q}$. The hydrostatic reconstructions of z and h (see Figure 2) at the point $x_{i+\frac{1}{2}}$ are

$$\begin{aligned} \widehat{z}_{i+\frac{1}{2}} &= \max(z_{i+1,l}, z_{i,r}), \\ \widehat{h}_{i,r} &= \max(0, h_{i,r} + z_{i,r} - \widehat{z}_{i+\frac{1}{2}}), \\ \widehat{h}_{i+1,l} &= \max(0, h_{i+1,l} + z_{i+1,l} - \widehat{z}_{i+\frac{1}{2}}). \end{aligned} \tag{8}$$

This leads to adjusted left and right interface values for the conserved quantities, $\mathbf{q}$ at the point $x_{i+\frac{1}{2}}$:

$$\widehat{\mathbf{q}}_{i,r} = \begin{bmatrix} \widehat{h}_{i,r} \\ \widehat{h}_{i,r}u_{i.r} \end{bmatrix}, \quad \widehat{\mathbf{q}}_{i+1,l} = \begin{bmatrix} \widehat{h}_{i+1,l} \\ \widehat{h}_{i+1,l}u_{i+1,l} \end{bmatrix}.$$

The hydrostatic flux calculation is finally defined as

$$\widehat{\mathbf{f}}_{i+\frac{1}{2}} = \mathbf{f}_a(\widehat{\mathbf{q}}_{i,r}, \widehat{\mathbf{q}}_{i+1,l}).$$

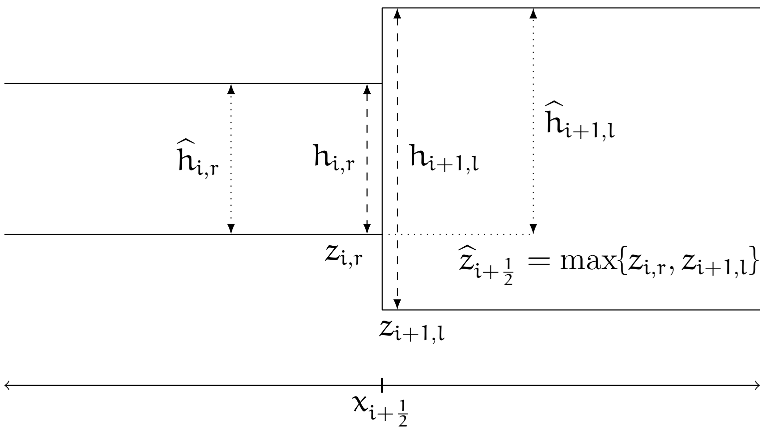

Figure 2: Modified values of height and bed, $\widehat{h}_{i,r}$, $\widehat{h}_{i+1,l}$ and $\widehat{z}_{i+\frac{1}{2}}$, for Audusse's well balanced scheme.

The extra forcing term, $\mathbf{c}_i$, in Equation (7) has two components, associated with the flux pressure terms on the left and right of the ith interval,

$$\mathbf{c}_i = \mathbf{c}_{i,r} + \mathbf{c}_{i,l}\,,$$

where

$$\begin{aligned} \mathbf{c}_{i,r} &= \frac{1}{\Delta x_i}\left[\begin{array}{c} 0 \\ \frac{1}{2}g\,\widehat{h}^2_{i,r} - \frac{1}{2}g\,h^2_{i,r} \end{array}\right], \\ \mathbf{c}_{i,l} &= \frac{1}{\Delta x_i}\left[\begin{array}{c} 0 \\ \frac{1}{2}g\,h^2_{i,l} - \frac{1}{2}g\,\widehat{h}^2_{i,l} \end{array}\right]. \end{aligned}$$

Audusse et al. [1] provided a proof that this method is well balanced, consistent with the shallow water wave equations and is formally second order accurate.

If the stage values are constant ($w_i = c$) and the velocity values zero ($u_i = 0$), then the interface values of stage and velocity will be c and 0 respectively. In this case

$$\widehat{\mathbf{f}}_{i+\frac{1}{2}} = \mathbf{f}_a(\widehat{\mathbf{q}}_{i+1,l}, \widehat{\mathbf{q}}_{i,r}) = \left[\begin{array}{c} 0 \\ \frac{1}{2}g\,\widehat{h}^2_{i,r} \end{array}\right],$$

and the correction term $\mathbf{c}_{i,r}$ cancels with the numerical flux to give

$$\frac{1}{\Delta x_i}\widehat{\mathbf{f}}_{i+\frac{1}{2}} - \mathbf{c}_{i,r} = \frac{1}{\Delta x_i}\begin{bmatrix} 0 \\ \frac{1}{2}g\, h_{i,r}^2 \end{bmatrix}.$$

Similarly for the left interface,

$$-\frac{1}{\Delta x_i}\widehat{\mathbf{f}}_{i-\frac{1}{2}} - \mathbf{c}_{i,l} = -\frac{1}{\Delta x_i}\begin{bmatrix} 0 \\ \frac{1}{2}g\, h_{i,l}^2 \end{bmatrix}.$$

Thus the numerical fluxes and the forcing terms satisfy

$$\begin{aligned}
&\frac{1}{\Delta x_i}\left(\widehat{\mathbf{f}}_{i+\frac{1}{2}} - \widehat{\mathbf{f}}_{i-\frac{1}{2}}\right) - (\mathbf{c}_{i,r} + \mathbf{c}_{i,l} + \mathbf{s}_i) \\
&\quad = \frac{1}{\Delta x_i}\begin{bmatrix} 0 \\ \frac{1}{2}g\, h_{i,r}^2 - \frac{1}{2}g\, h_{i,l}^2 - \frac{1}{2}g\ (h_{i,r} + h_{i,l})\,(z_{i+1,l} - z_{i,r}) \end{bmatrix} \\
&\quad = \frac{1}{\Delta x_i}\begin{bmatrix} 0 \\ \frac{1}{2}g\ (h_{i,r} + h_{i,l})\,(h_{i+1,l} + z_{i+1,l} - h_{i,r} - z_{i,r}) \end{bmatrix} \\
&\quad = 0\,,
\end{aligned}$$

since the reconstructed stage is constant, $c = h_{i+1,l} + z_{i+1,l} = h_{i,r} - z_{i,r}$. So Audusse's method reproduces a still lake solution.

4 Modified method for variable width equations

Equation (3) models flow in a channel of varying width. A well balanced scheme for this problem must balance the flux pressure term $(\frac{1}{2}gbh^2)_x$ with the pressure forcing term $-ghbz_x + \frac{1}{2}gh^2b_x$. The problem of developing a well balanced scheme can be significantly simplified if the width is restricted to be a piecewise affine continuous function or a piecewise constant (discontinuous) function. In these cases simple methods give the well balanced property. We are interested in the more difficult situation when both the width and the bed elevation are piecewise affine discontinuous functions. To tackle this problem we set up a similar situation as in the constant width case, where in the steady state case we have known exact flux pressure terms at the edges.

First we have the problem of defining b at the interval edges (as we previously had to define a hydrostatic approximation for z). Given piecewise affine reconstructions for b, z and $\mathbf{q}$, at time t we let

$$\widehat{b}_{i+\frac{1}{2}} = \frac{1}{2}(b_{i+1,l} + b_{i,r})$$

and define adjusted conserved quantities as

$$\widehat{\mathbf{q}}_{i,r} = \begin{bmatrix} \widehat{h}_{i,r}\widehat{b}_{i+\frac{1}{2}} \\ \widehat{h}_{i,r}u_{i.r}\widehat{b}_{i+\frac{1}{2}} \end{bmatrix}, \quad \widehat{\mathbf{q}}_{i+1,l} = \begin{bmatrix} \widehat{h}_{i+1,l}\widehat{b}_{i+\frac{1}{2}} \\ \widehat{h}_{i+1,l}u_{i+1,l}\widehat{b}_{i+\frac{1}{2}} \end{bmatrix}.$$

We now define a 'correction' term $\mathbf{c}_i = \mathbf{c}_{i,l} + \mathbf{c}_{i,r}$ so that in the case of a steady lake we get the same cancellation as for Audusse's method, leaving us with some form of 'exact' flux pressure term. The correction terms are

$$\mathbf{c}_{i,r} = \frac{1}{\Delta x_i}\begin{bmatrix} 0 \\ \frac{1}{2}g\,\widehat{h}^2_{i,r}\widehat{b}_{i+\frac{1}{2}} - \frac{1}{2}g\,h^2_{i,r}b_{i,r} \end{bmatrix},$$

$$\mathbf{c}_{i,l} = \frac{1}{\Delta x_i}\begin{bmatrix} 0 \\ \frac{1}{2}g\,h^2_{i,l}b_{i,l} - \frac{1}{2}g\,\widehat{h}^2_{i,l}\widehat{b}_{i+\frac{1}{2}} \end{bmatrix}.$$

If the stage is constant and the velocity zero, then $\widehat{\mathbf{q}}_{i,r} = \widehat{\mathbf{q}}_{i+1,l}$ and

$$\widehat{\mathbf{f}}_{i+\frac{1}{2}} = \mathbf{f}_a(\widehat{\mathbf{q}}_{i+1,l}, \widehat{\mathbf{q}}_{i,r}) = \begin{bmatrix} 0 \\ \frac{1}{2}g\,\widehat{h}^2_{i,r}\widehat{b}_{i+\frac{1}{2}} \end{bmatrix},$$

so that

$$\frac{1}{\Delta x_i}\widehat{\mathbf{f}}_{i+\frac{1}{2}} - \mathbf{c}_{i,r} = \frac{1}{\Delta x_i}\begin{bmatrix} 0 \\ \frac{1}{2}g\,h^2_{i,r}b_{i,r} \end{bmatrix}.$$

Similarly at the left edge,

$$-\frac{1}{\Delta x_i}\widehat{\mathbf{f}}_{i-\frac{1}{2}} - \mathbf{c}_{i,l} = -\frac{1}{\Delta x_i}\begin{bmatrix} 0 \\ \frac{1}{2} g\, h_{i,l}^2 b_{i,l} \end{bmatrix}.$$

In the still lake case, $w = c$ and $u = 0$, we have

$$\begin{aligned} \frac{1}{\Delta x_i}\left(\widehat{\mathbf{f}}_{i+\frac{1}{2}} - \widehat{\mathbf{f}}_{i-\frac{1}{2}}\right) - (\mathbf{c}_{i,l} + \mathbf{c}_{i,r}) &= \frac{1}{\Delta x_i}\begin{bmatrix} 0 \\ \frac{1}{2} g\, h_{i,r}^2 b_{i,r} - \frac{1}{2} g\, h_{i,l}^2 b_{i,l} \end{bmatrix} \\ &= \frac{1}{\Delta x_i}\begin{bmatrix} 0 \\ \int_{x_{i-1/2}}^{x_{i+1/2}} \left(\frac{1}{2} g h^2 b\right)_x \, dx \end{bmatrix}. \end{aligned}$$

Now we need to approximate $\mathbf{s}_i$ to ensure well balancing. This is quite simple, we just set

$$\mathbf{s}_i = \frac{1}{\Delta x_i}\begin{bmatrix} 0 \\ \int_{x_{i-1/2}}^{x_{i+1/2}} \left(-g h b z_x + \frac{1}{2} g h^2 b_x\right) \, dx \end{bmatrix}.$$

The integral is evaluated exactly. The pressure forcing term is quadratic in nature. Both z and b are affine functions when restricted to I_i and so z_x and b_x are constants. The first part of the forcing term has a bhz_x component which is the product of two affine functions and a constant, and the second term contains $h^2 b_x$, the product of the square of an affine function and a constant, so together the forcing term is quadratic. Hence we use Simpson's rule to calculate the exact value of the integral

$$p_i = \int_{x_{i-1/2}}^{x_{i+1/2}} \left(-g h b z_x + \frac{1}{2} g h^2 b_x\right) \, dx\,, \tag{9}$$

to obtain

$$\begin{aligned} p_i = &-\frac{g}{6}(b_{i,l} h_{i,l} + 4 b_i h_i + b_{i,r} h_{i,r})(z_{i,r} - z_{i,l}) \\ &+ \frac{g}{12}(h_{i,l}^2 + 4 h_i^2 + h_{i,r}^2)(b_{i,r} - b_{i,l}). \end{aligned}$$

So

$$\mathbf{s}_i = \frac{1}{\Delta x_i} \begin{bmatrix} 0 \\ p_i \end{bmatrix}.$$

Note that if $w = c$ then

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \left(\frac{1}{2} g h^2 b\right)_x \, dx = \int_{x_{i-1/2}}^{x_{i+1/2}} \left(-g h b z_x + \frac{1}{2} g h^2 b_x\right) \, dx\,.$$

Hence if we have a still lake condition, $w = c$ and $u = 0$, then

$$\frac{1}{\Delta x_i} \left(\widehat{\mathbf{f}}_{i+\frac{1}{2}} - \widehat{\mathbf{f}}_{i-\frac{1}{2}}\right) - (\mathbf{c}_{i,l} + \mathbf{c}_{i,r} + \mathbf{s}_i) = \mathbf{0}\,,$$

and so the method is well balanced.

5 Numerical results

In the following numerical tests we use the central-upwind numerical flux function described by Kurganov et al. [4]. The time stepping algorithm used is the second order strong stability-preserving Runge–Kutta algorithm [3]. The limiter used is the standard van Leer limiter [6].

Two test cases will be presented. A still lake test case is used to verify that the scheme is well balanced, and a dam break problem is used to verify that the well balanced scheme can reproduce the results of the original scheme in the case of non-steady flow with a shock. The original scheme has already been validated against analytical solutions by Zoppou and Roberts [9].

5.1 Steady lake tests

Our first test considers a situation with varying width and bed elevation, with initially constant stage and zero velocity (still lake steady state). The numerical results confirm that the new numerical method maintains the steady state solution.

The numerical domain corresponds to a channel of length 1000 m and is partitioned into 100 intervals. Initial velocity (u) is set to zero. Initial stage is set to 10 m. The width and bed elevation are both defined randomly for each interval. Each vertex value for $z(x)$ and $b(x)$ (treating the vertex on each side of an interface separately) is assigned a random value drawn from a normal distribution with mean 3 m and standard deviation 1 m. This makes them both very discontinuous and often very steep within each cell. The simulation is run to 100 s.

The results of this test using the method without well balancing is shown in Figure 3 and the results for the well balanced scheme are shown in Figure 4. The results for the non well balanced scheme are poor. Obviously the numerical solution is not steady. Clearly this is not a satisfactory solution. In contrast the well balanced method seems to reproduce the steady state exactly. There are errors in the velocity on the order of 10^{-6} but these are negligible.

This kind of width and bed elevation is perhaps not particularly realistic, especially in terms of the width (one can perhaps imagine some sort of very jagged rock formation). So problems in the non well balanced method are greatly exaggerated here. Still it is comforting that even in this extreme case our well balanced method copes well.

5.2 Radial dam break test

This test is used as a validation of the performance of the new method when faced with a varying width problem in a more realistic situation. In this test the solution contains a shock, so we gauge the scheme's ability to capture this shock and obtain the correct shock speed.

This test is the same as used by Birman and Falcovitz [2]. Here we use our one dimensional scheme to solve a two dimensional problem by exploiting its

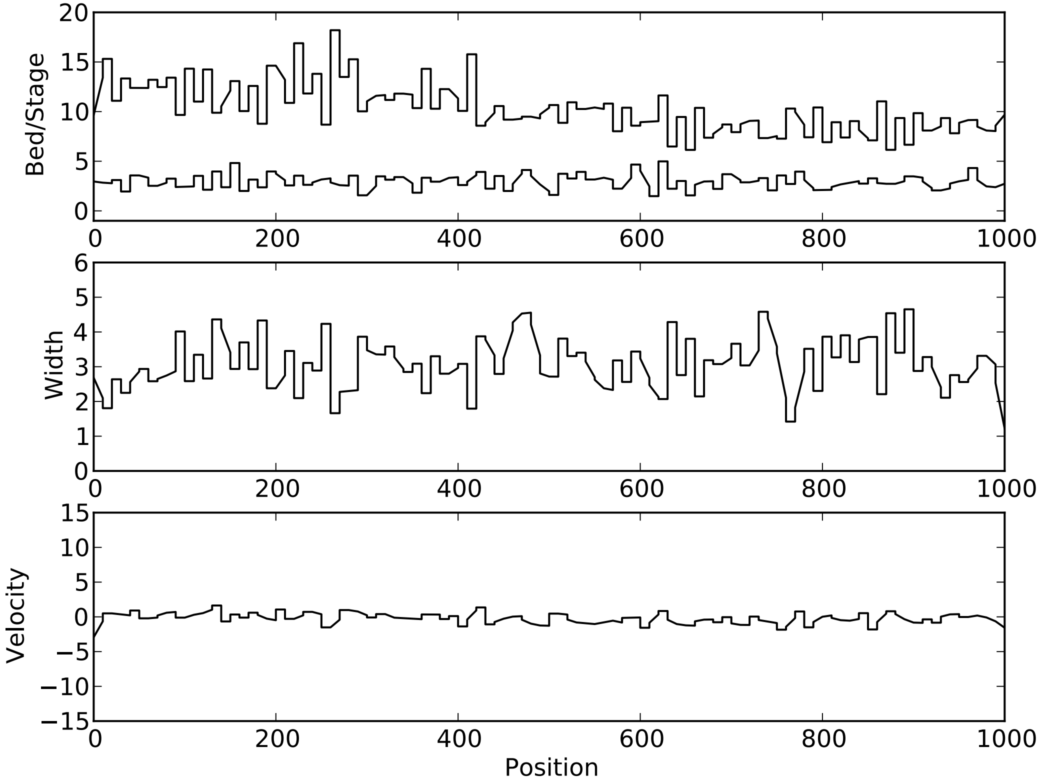

Figure 3: Numerical results of unmodified scheme (not well balanced), at time 100 s, with random width and bed elevation, and initial stage equal to 10 m. Note the unsteady stage and velocity.

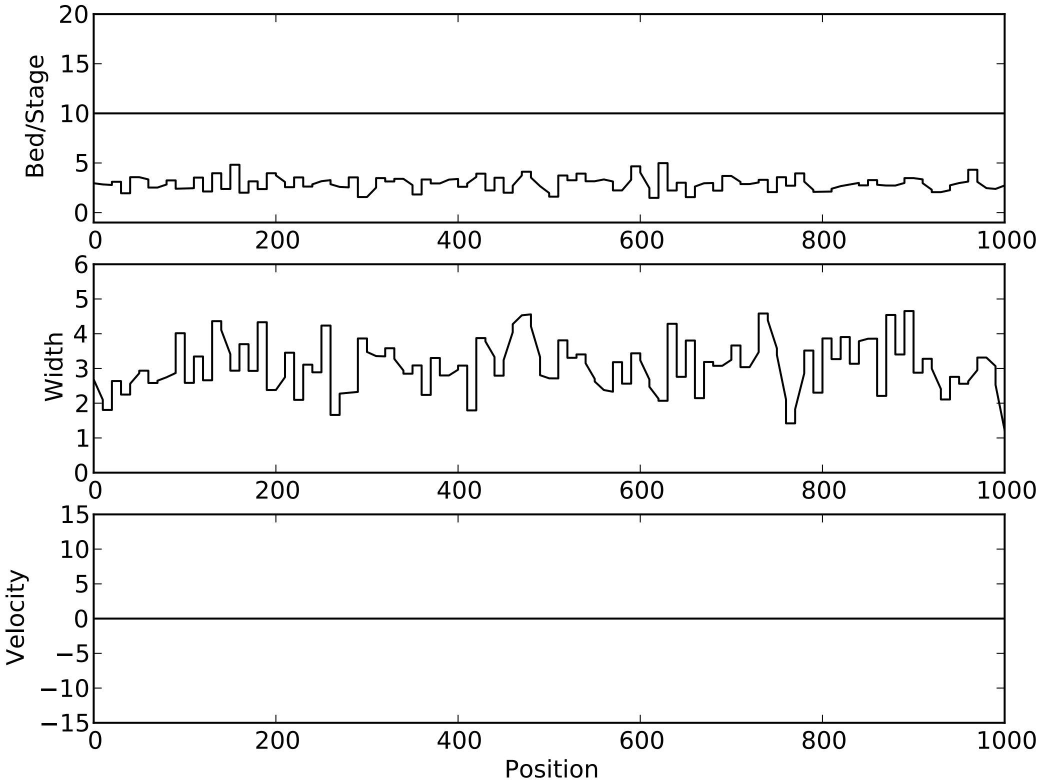

Figure 4: Numerical results of modified scheme (well balanced), at time 100 s, with random width and bed elevation, and initial stage equal to 10 m. Note the steady stage and zero velocity.

symmetry. The two dimensional problem is a cylindrical dam break. We get this situation by setting width to be $b(x) = 2\pi x$. Hence, as water flows from the left, the expanding channel mimics the two dimensional situation.

The length of the channel is set at 100 m, we partition the domain into 100 intervals. As mentioned before $b(x) = 2\pi x$. Stage is set to be 10 m from $x = 0$ m to $x = 50$ m and elsewhere the stage is set to 2 m. The problem is run until $t = 2$ s.

As noted by Birman and Falcovitz [2] this problem has no analytic solution and so we use the solution on a fine grid (of 1000 intervals) as an accurate approximation to the exact solution. In Figure 5, the results of the original non well balanced scheme are presented. In particular the results of using a fine grid of 1000 intervals is compared with the solution obtained using 100 intervals. Similar results, for the new well balanced scheme are presented in Figure 6. In this non-steady state test case, the results for the well balanced scheme are essentially identical to the non well balanced scheme.

6 Conclusion

The well balanced scheme presented here is very flexible in that one can use any numerical flux function satisfying some very common properties. Here we have used a Central-Upwind type numerical flux along with a Runge–Kutta time stepping algorithm and a van Leer limiter to provide numerical examples. These numerical examples show that the scheme is well balanced. Additionally they show that the addition of the well balancing terms does not degrade the quality of the solution for non steady-state problems, in particular the shock speeds remain the same.

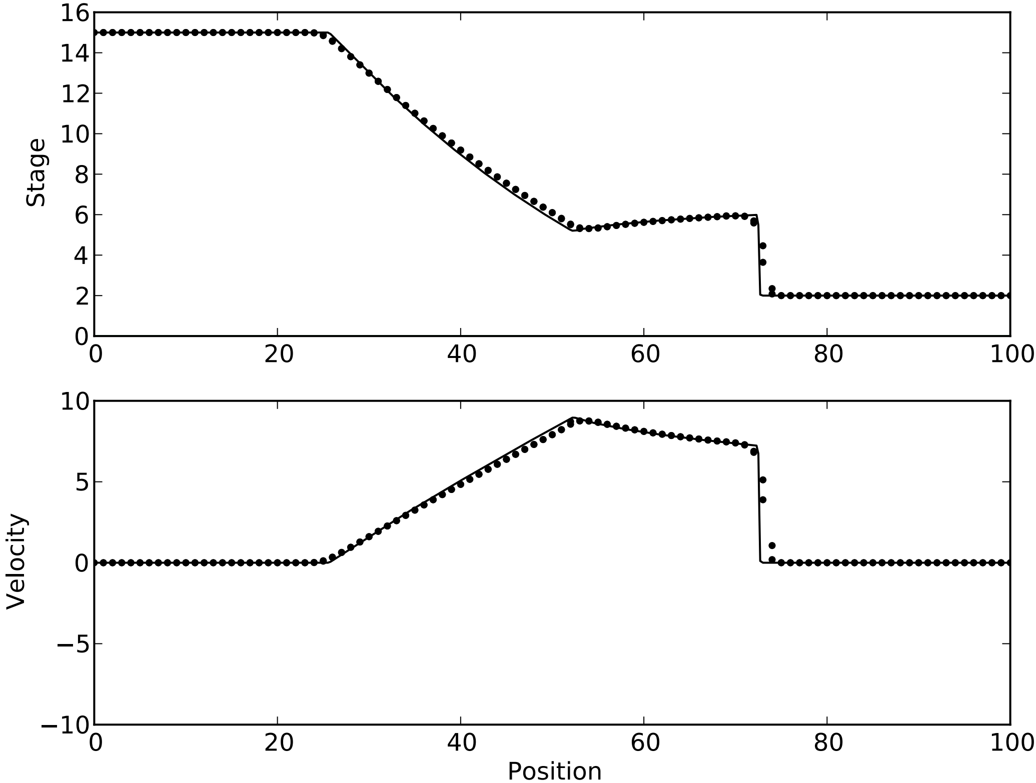

Figure 5: Radial dam test case of varying width non steady flow using original (non well balanced) scheme. Solution using 100 intervals is given by the black dots, and solution using 1000 intervals is given by the unbroken line.

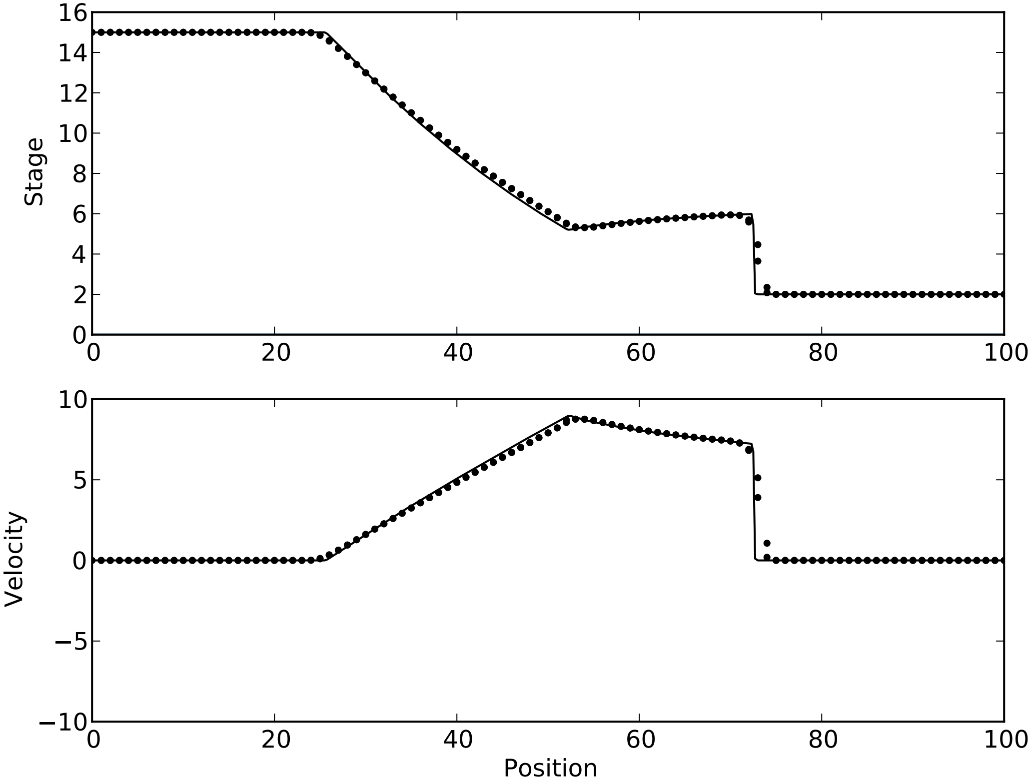

Figure 6: Radial dam test case of varying width non steady flow using the well balanced scheme. Solution using 100 intervals is given by the black dots, and solution with 1000 intervals is given by the unbroken line.

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