Irrelevance of the fractal dimension term in the modified fractal attrition equation

Lesley Richard Kosowski[1] Adrian Pincombe[2]
Brandon Pincombe[3]

(Received 31 January 2011; revised 16 November 2011)

Abstract

The modified Fractal Attrition Equation (mFAE) models the casualties produced by Map Aware Non-uniform Automata (MANA), an agent based combat modelling distillation, at each MANA time step. The mFAE has three important differences from the simple early twentieth century models of casualties that used only the numbers on each side to make predictions. Firstly, only those agents within range of the enemy may inflict casualties. Secondly, the detection range is assumed to be greater than the range of weapons and a fitting factor assumed to represent the gap between these two ranges, is introduced. This builds in one of the assumptions of Network Centric Warfare, that you will be able to see your enemies before they can shoot at you. Thirdly, and novelly, a fractal dimension is introduced. We postulate that the important part of the information used in calculating the fractal

http://journal.austms.org.au/ojs/index.php/ANZIAMJ/article/view/3963 gives this article, Published December 1, 2011. ISSN 1446-8735.

dimension has already been incorporated into the model through the consideration of range. We test this hypothesis by comparing the outcomes of the mFAE with and without the fractal term on three scenarios: that used by the developers of the mFAE; best practice MANA tactics from the literature; and a rout scenario. When the two models are scaled to fit the MANA casualties there is no significant difference in fit. We conclude that the fractal term in the mFAE is redundant.

Contents

1 Introduction

Within the military establishment, modelling and simulation is commonly used to test various options for force structure and capability improvement. Tactics have an important effect on outcomes and we have a specific interest in the effect of tactics on attrition through casualties.

Differential equations have been used to estimate battle attrition since 1902 [1, 2]. One case, the Lanchester square law, corresponding to a battle in which units use aimed fire [3], involves a pair of coupled first order differential equations,

$$\begin{aligned} \frac{dB}{dt} &= -k_R R(t), \quad B(0) = B_0, \\ \frac{dR}{dt} &= -k_B B(t), \quad R(0) = R_0, \end{aligned} \tag{1}$$

relating the casualty rate on one side (Red or Blue) of the battle with the number of combatants on the other side. Here $B(t)$ and $R(t)$ are the Blue and Red force sizes and k_B and k_R are the killing probabilities for each side.

Numerous deficiencies of these attrition equations have been identified over time including failure to take into account manoeuvre aspects of warfare, and inability to take into account the effects of communication, leadership, command and control [5]. It is also difficult to estimate values for k in advance; values for k are typically obtained retrospectively by fitting these attrition equations to combat data but the fit to actual combat data does not appear to be good [3, 4].

The observation that military conflict is a complex adaptive system led to the development of the Fractal Attrition Equation (FAE) [6, 5] as a replacement for the square law equations of attrition. To investigate the applicability of the FAE, McIntosh and Lauren made use of an agent-based model known as MANA[1] (Map Aware Non-uniform Automata) [7, 8, 5]. The scenario used for this investigation was the Meet scenario [9], which has two forces, Red and Blue, spread out in rectangular regions separated by open space. The interesting feature of the Meet scenario is the formation of a battle front as the two sides approach each other, the onset of instability and the collapse

[1]We used MANA version 4 and thus had access, through the MANA data analyser, to the code used by McIntosh and Lauren to calculate the summary ensemble statistics and fractal dimension used by McIntosh and Lauren [5]. This was the last version of MANA to use the cell-based scheme for agent movement.

of that front as the battle develops in time with the formation of multiple clusters of agents [7]. However, the Meet scenario is only one of multiple possible scenarios so the FAE needs to be tested much more comprehensively.

An interesting collection of tactics using combinations of core skills used by the Army in combat was developed by Shine [10] and modelled using MANA. Some of the tactics investigated by Shine include Blue units backing off on enemy contact then attacking, Blue units using better fire-power, Blue units retreating to friends when shot at by the enemy, Blue units conducting a flanking operation, and Blue units conducting an envelopment operation. Some tactics are easily implemented without modifying the scenario but others such as the flanking operation and also the related envelopment operation would require significant modification to the scenario. The battlefield scenario Shine used to investigate these core skills used in combat had the two sides charge one another, which is an unrealistic battlefield scenario, but the tactics investigated were representative of tactics that the Army would employ in battle. These tactics were also simple and could be inserted into more realistic battle scenarios. One of the best of these tactics, retreat to friends when shot at, was chosen as a candidate for inclusion into the Blue unit's behaviour in the Meet scenario.

We verified the results of McIntosh and Lauren [5], that the FAE and the modified FAE (mFAE) (replacing R with the number of Red agents that have detected at least one Blue agent, and similarly for B) do follow the curves shown in their report as long as both curves are scaled by a prefactor of 0.15. Our graphs are shown in Figure 1. Note that use of the same prefactor for the square law enables us to see that the Lanchester curve is actually closer than the FAE to MANA attrition. We then proceed to modify the Meet scenario (section 3) by changing the force size for each side and introducing a new tactic/behaviour for the Blue side. We show that the results obtained by McIntosh and Lauren [5] for the mFAE could be replicated without the explicit use of a fractal dimension, with the attrition equations being similar to the Lanchester square law with number of shooters within detection range of the enemy in place of remaining force size. This article describes and analyses

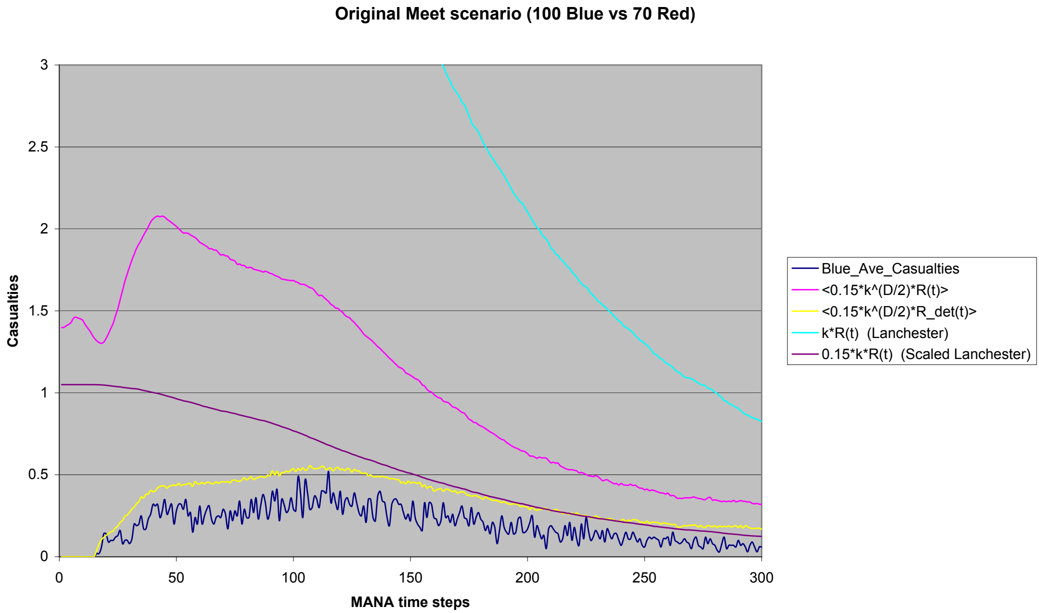

Figure 1: Comparing the FAE with casualty data from the Meet scenario for $k = 0.1$. The upper curve is the Lanchester square law equation. The second curve is the FAE with the prefactor applied. The third curve is Lanchester with the same prefactor applied to it. These two curves show that the Lanchester square law equation gives attrition which is numerically closer to the MANA attrition rate, but the FAE produces a curve that is similar in shape to the MANA attrition.

the results of the simulations generated using the modified Meet scenarios.

There is a question about how to obtain ensemble averages for a non-linear equation: this was not addressed by McIntosh and Lauren [5]. However, we are comparing the FAE and mFAE with an alternative model and we need to ensure that we fairly represent the models of McIntosh and Lauren: this is achieved by using their methodology to generate the ensemble averages. To assist with the processing of the large amount of data generated from the multiple runs, MANA has a data analysis tool which was used to combine the raw data into time-dependent average quantities such as casualty rates, fractal dimensions and units alive: this applies the method used by McIntosh and Lauren [5]. These averaged quantities are used in a form of attrition equation rewritten in terms of ensemble averages[2] from the multiple simulations

$$\begin{aligned}\frac{\langle \Delta B(t)\rangle}{\Delta t} &= -k_R \langle R(t)\rangle\,, \quad B(0) = B_0,\\ \frac{\langle \Delta R(t)\rangle}{\Delta t} &= -k_B \langle B(t)\rangle\,, \quad R(0) = R_0,\end{aligned} \tag{2}$$

where $\langle \Delta B(t)\rangle/\Delta t$ and $\langle \Delta R(t)\rangle/\Delta t$ are the ensemble average Blue and Red casualty rates respectively and $\langle B(t)\rangle$ and $\langle R(t)\rangle$ are the ensemble averages of Blue and Red agents remaining alive at time t. Left and right angled brackets denote ensemble averages.

2 Theory

McIntosh and Lauren [5] noted that both real casualty data and simulated casualty data such as that obtained from MANA can exhibit complex structures in the time series of casualties and the spatial distribution of agents on the

[2] We acknowledge that the ensemble averages on the LHS can be written $\Delta\langle B(t)\rangle$ retaining the same meaning as $\langle \Delta B(t)\rangle$ but we choose to use the latter form which more closely corresponds to the form used by McIntosh and Lauren [5]. Computationally the $\Delta\langle B(t)\rangle$ form appears to be more efficient.

battlefield, particularly the front line separating the two combating forces in the simulation. For simulated battles between cellular automata, a new type of attrition equation was suggested by Lauren et al. [7] that incorporates the force's fractal dimension; they started with the assumption that the spatial distribution of agents at any time t during the simulation are spatial fractals, with dimension $D_B(t)$ and $D_R(t)$, from which they inferred that the time series of casualties, $\Delta B(t)$ and $\Delta R(t)$, would also be fractals. The differential equations using the fractal dimension are

$$\begin{aligned} \frac{\langle \Delta B(t) \rangle}{\Delta t} &= -c_R \left\langle k_R^{D_R(t)/2} \Delta t^{(D_R(t)/2-1)} R(t) \right\rangle, \\ \frac{\langle \Delta R(t) \rangle}{\Delta t} &= -c_B \left\langle k_B^{D_B(t)/2} \Delta t^{(D_B(t)/2-1)} B(t) \right\rangle, \end{aligned} \tag{3}$$

where k_B and k_R are the kill probabilities for Blue and Red respectively, and c_B and c_R are fitting factors used for matching the FAE onto the casualty data [5]. There is an implicit assumption that the terms in the ensemble averages are independent of the factors influencing the values of c_R and c_B. This allows these values to be calculated as scaling factors while the model retains its shape. The method used for calculating the fractal dimension is the correlation dimension [5] attributed to Grassberger and Procaccia [11].

By setting $\Delta t = 1$ corresponding to one time step in the MANA simulation, the FAEs given by equation (3) simplify to

$$\begin{aligned} -\langle \Delta B(t) \rangle &= c_R \left\langle k_R^{D_R(t)/2} R(t) \right\rangle, \\ -\langle \Delta R(t) \rangle &= c_B \left\langle k_B^{D_B(t)/2} B(t) \right\rangle. \end{aligned} \tag{4}$$

A significant problem with the system (4) is that it predicts casualties for both sides even when they are not actually engaged in battle; this is most evident early in the course of the simulated battles as both sides form a battle front but before they get within sensor range of the enemy. From their investigations into simulated combat scenarios, McIntosh and Lauren [5] stated that rather than using the entire force size of the enemy, the FAE should use the number of enemy agents within sensor/weapons range: creating the mFAE. R_{det} is the number of Red agents within sensor range of Blue agents and are capable of engaging and inflicting casualties onto the Blue agents. B_{det} is the corresponding number of Blue agents. R_{det} and B_{det} come from the detection data within MANA. The modified FAE are

$$\begin{aligned} -\langle \Delta B(t) \rangle &= c_R \left\langle k_R^{D_R(t)/2} R_{det}(t) \right\rangle, \\ -\langle \Delta R(t) \rangle &= c_B \left\langle k_B^{D_B(t)/2} B_{det}(t) \right\rangle. \end{aligned} \tag{5}$$

The mFAE differs from the Lanchester square law in three ways: applying a fractal dimension based power law to the strengths of the forces; considering only those agents in contact with enemy agents when calculating effective force size; and updating the model from the MANA simulation at each time step to eliminate the propagation of errors. While we contend that a model without the fractal term, but with the other two features, would be simpler but no worse than the mFAE, we are grateful to a reviewer for forcing us to consider that applying the power law to the agent totals, $B(t)$ and $R(t)$, may be more appropriate than applying it to the agent strengths, k_B and k_R. If the spatial distribution of agents at any time are, indeed, spatial fractals then the numbers of agents $B_{det}(t)$ and $R_{det}(t)$ within range of each other can be viewed as the region where the boundary of these two fractals intersect. As the 'length' of a fractal's boundary scales as a power of its 'area', $B_{det}(t)$ and $R_{det}(t)$ can be expected to be proportional to the total number of agents $B(t)$ and $R(t)$ raised to some power, that is

$$\begin{aligned} B_{det}(t) &\approx \gamma_B(D_B(t), D_R(t)) B(t)^{\delta_B(t)}, \\ R_{det}(t) &\approx \gamma_R(D_R(t), D_B(t)) R(t)^{\delta_R(t)}, \end{aligned} \tag{6}$$

where, since contact is determined by the intersection of the two fractals, the constants γ_B and γ_R and exponents $\delta_B(t)$ and $\delta_R(t)$ are functions of the dimensions $D_B(t)$ and $D_R(t)$ of the two spatial fractals. Thus it seems more natural to expect the numbers $B_{det}(t)$ and $R_{det}(t)$ of engaged forces in contact scale as power laws determined by the spatial distribution of forces than it does to expect that the strengths k_B and k_R of engaged forces to scale as a power law, and the use of a fractal dimension in this way may produce superior results to the manner in which it is used in the mFAE. However, we compare with the published FAE and mFAE following McIntosh and Lauren and using $k_R = k_B = 0.1$ [5].

At each time step, a considerable amount of calculation is required to estimate the fractal dimensions for each force and to determine the number of agents within range of the sensors. We propose an alternative to the mFAE that eliminates the fractal dimension term. The proposed attrition equations without the fractal dimension are

$$\begin{aligned} -\left\langle \Delta B\left(t\right)\right\rangle &= c_R' k_R \left\langle R_{\mathrm{det}}\left(t\right)\right\rangle , \\ -\left\langle \Delta R\left(t\right)\right\rangle &= c_B' k_B \left\langle B_{\mathrm{det}}\left(t\right)\right\rangle \end{aligned} \tag{7}$$

where c_R' and c_B' are obtained by linear regression through the origin [12] of the observed numbers of agents within sensor range of the enemy versus the observed casualty rates. The regression calculations make use of the ordinary least squares method to calculate the two unknown scaling coefficients c_R' and c_B'. Regression through the origin is necessary as the prediction of casualties is zero when there are zero agents in the force size on the RHS of the equations; it would be unrealistic to expect casualties when there are no forces engaged in combat. We call these equations the Sensor-Range-Limited Attrition Equations (SRLAE)[3]. As for mFAE we set the values of $k_R = k_B = 0.1$. For these experiments we assume that the values of c_R, c_B, c_R' and c_B' are independent of the terms in the ensemble averages. The validity of this assumption needs to be investigated; however, in these experiments we focus on the importance of the fractal dimension and work on the basis that the mFAE and the SRLAE can be simply scaled to fit the MANA casualty data in each scenario.

Our hypothesis is that the important part of the information used in calculating the fractal dimension has already been incorporated into the mFAE

[3]The name for the proposed attrition equations comes from the R_{det} and B_{det} which are numbers of agents within sensor range of their respective enemy agents.

model through consideration of the range. The range referred to is the sensor range for determining the number of agents that are within range to detect the opposing side's agents.

3 Experiment design

Three scenarios were used, a baseline or default scenario based on the Meet scenario, and a variation of the baseline scenario in which the Blue agents retreat to friends when shot at. The latter scenario had two variations, one in which the retreating Blue agents continued to surveil and shoot at the opposing forces when retreating to friends, and one in which they did no surveillance and no shooting when retreating. The retreat to friends action lasted five MANA time steps, after which the Blue forces resumed their default personality. The retreat to friends action was identified from a study of the effectiveness of warfighting tactics [10] as being one of the more effective tactics resulting in reduction of casualties for the side using the tactic and an increase of casualties for the side using only the baseline personality; the ratio of casualties was slightly greater than 2:1 when Blue side used the retreat to friends tactic compared to 1:1 when both sides were using the baseline personality only. The comparison of tactics in the aforementioned report involved equal sized forces but not using the Meet scenario.

We used a three factor experimental design consisting of the following factors

- two levels of Red force size (50 and 80),
- four levels of Blue force size (25, 40, 50 and 80),
- three levels of Tactics category (Baseline, Baseline + Retreat to friends, no surveillance and no shooting, Baseline + Retreat to friends with surveillance and shooting).

McIntosh and Lauren [5] used one hundred replications for their simulations. We used 200 replications for each scenario assigned to the three-tuples (Red

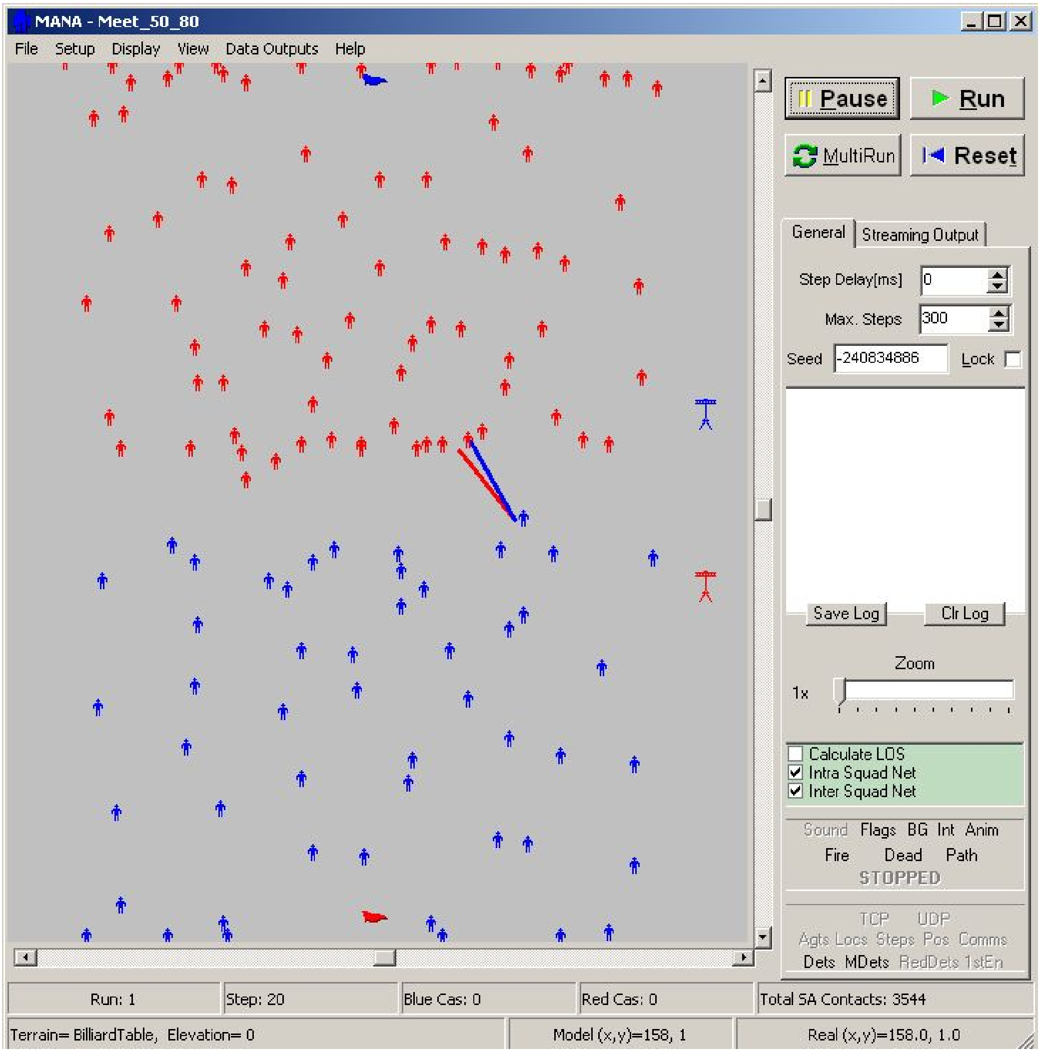

Figure 2: An image of the MANA window showing the Meet scenario in which Blue is using the default tactic and Red is using the default tactic + retreat to friends tactic. The Blue side had an initial force size of 50 and Red had an initial force size of 80. The image shows the two forces shortly after coming into contact and the formation of a battle front. There is an instance of one agent from each side shooting at their enemy.

force size, Blue force size, Tactics category). The number of replications was a compromise value chosen to minimise the time required for conducting the MANA simulations and replaying the data with MANA Data Analysis Tool while ensuring summary statistics that are valid according to the method of McIntosh and Lauren [5].

As with the original meet scenario the maximum duration of the scenario was set to 300 time steps. In some cases one side had all its agents destroyed well before this time limit was reached and in other cases both sides still had agents alive and engaging the opposing forces at the final time step.

After the ensemble averages for each time step of a scenario are known, we scale the calculated casualties to best match the simulated casualties. We use a simple linear regression approach to calculate the scaling factor c'.

4 Results

The analysis of our experiments had three parts. First, it had been unclear whether the values of c were expected to fit all scenarios, so we tested the $c = 0.15$ in our scenarios. Second, we estimated the values of c' for each scenario. Finally, we estimated values of c'' for the mFAE in all scenarios. We discuss one scenario where the fractal dimensions on visual inspection did not appear to vary substantially; using the definition in Appendix A we see there is a substantial variation in the range of D_B.

4.1 Comparison of attrition equations

In the theory section above we introduced an alternative form of attrition equation, the SRLAE. We have a posteriori estimated the scaling coefficients for the SRLAE.

The graphical results are interpreted qualitatively in an attempt to identify any obvious patterns in the results. The data of relevance to testing our hypothesis are the graphs of casualties generated by the MANA simulation, casualties predicted by the mFAE, and casualties predicted by the SRLAE. Unlike McIntosh and Lauren, we looked at the results for both sides (Red and Blue) of the coupled mFAE and SRLAE. These graphs show how well the predictor equations track the casualty data. Originally [5] the fitting factor for the mFAE was claimed to be a function of the difference between the sensor and weapons ranges; these ranges were unchanged for our experiments.

The mFAE was developed using a single scenario, except that several values were tried for the detection range [5]. The fitting factor c was thus shown to be a function of the difference between detection and shooting ranges [5] but it was unclear if c also depends on other factors. We first compared the SRLAE to the mFAE with the predetermined value of c. This gave a poor fit for the mFAE that varied with tactics and with force numbers (see Figures 3 and 4 for examples). For subsequent comparisons, the mFAE was fitted to the casualty data, introducing a new fitting factor c''.

4.2 Analysis of empirical data for one scenario

Typically the mFAE of equation (5) using a fixed value of c either over-estimates or under-estimates the casualty rate depending on the scenario. The sample scenario shown in the following graphs is of 40 Blue agents using the Baseline + Retreat to friends tactic versus 50 Red agents using the Baseline Meet tactic. Looking at the graphs of casualties for this sample scenario we see that the mFAE over-estimates both Blue casualties and Red casualties. The SRLAE (7), for this scenario, calculates an estimate of the casualty rate that is consistently closer to the actual ensemble mean value of casualties generated with the MANA simulations. This can be seen in the selection of graphs (Figures 3 to 4); the blue curve is the casualties, the pink curve is the mFAE, and the yellow curve is the SRLAE.

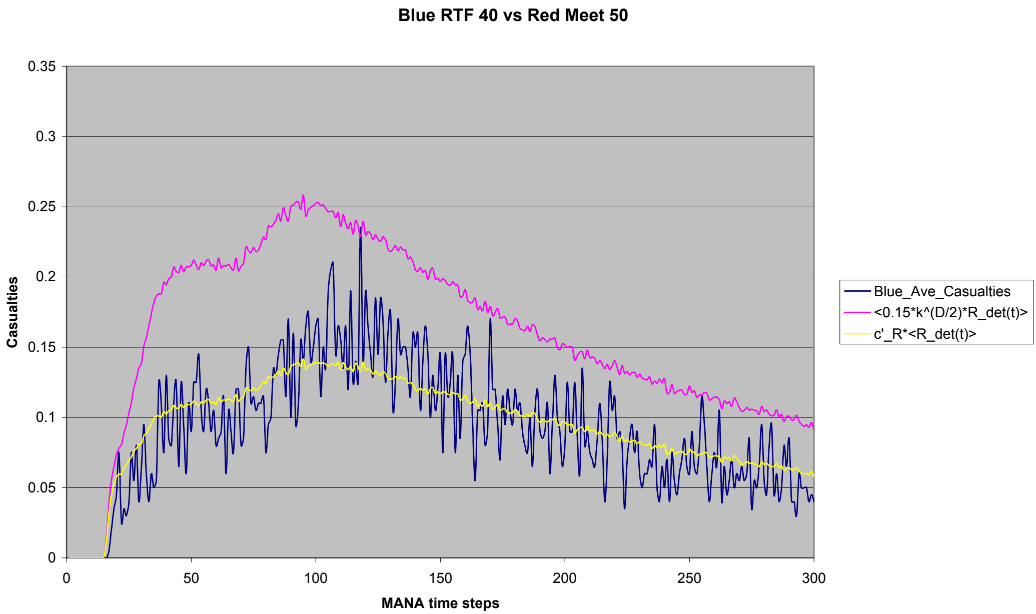

Figure 3: Comparing the casualties of the Blue force generated by the MANA simulations and the predicted casualties from the mFAE (pink) and the SRLAE (yellow) for the Baseline + Retreat to friends with surveillance and shooting scenario

We also applied the same ordinary least squares fitting technique to find a fitting factor a posteriori for the mFAE. This could be considered controversial as the RHS of the mFAE is non-linear, but if we ignore the apparent non-linearity of the RHS we view the data as a linear model[4] of the form $y = mx$.

[4]For the scenario we have shown here, of 40 Blue agents using the default + retreat to friends tactic versus 50 Red agents using the default tactic, the fractal dimensions for both sides, on initial visual inspection, does not appear to vary substantially over the simulation (see Figure 5). Using the definition in Appendix A we see that there is substantial variation in D_B. As there is little difference in the coefficient of determination for both the Red and Blue cases of the SRLAE and fitted mFAE we therefore suggest that the linear model of attrition fitting the mFAE is sensible and can be compared against the SRLAE.

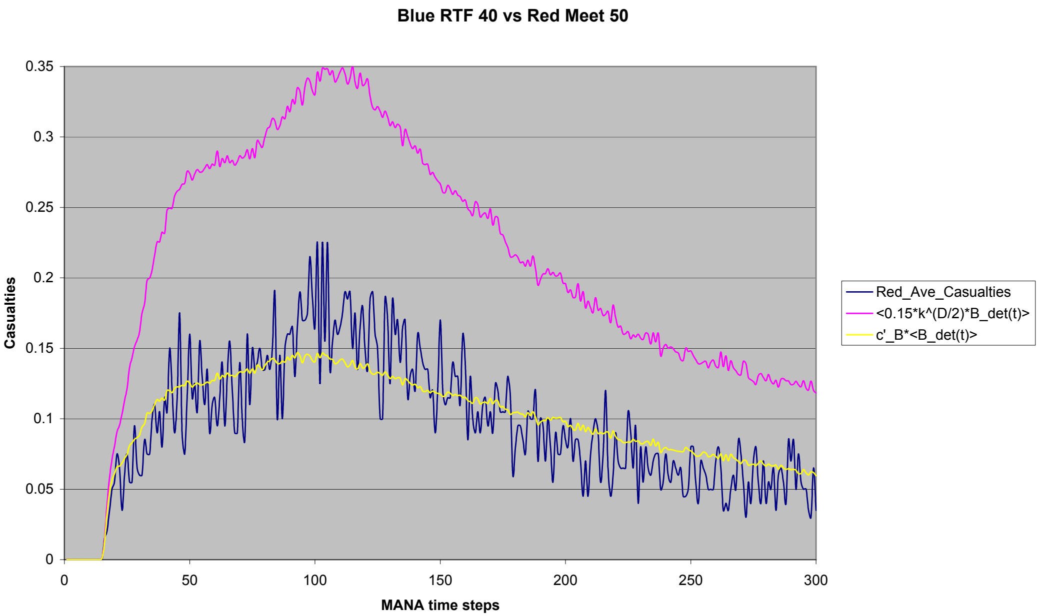

Figure 4: Comparing the casualties of the Red force generated by the MANA simulations and the predicted casualties from the mFAE (pink) and the SRLAE (yellow) for the Baseline + Retreat to friends with surveillance and shooting scenario.

The suggested model has the form

$$-\langle \Delta B(t) \rangle = c''_R \left\langle k_R^{D_R(t)/2} R_{\text{det}}(t) \right\rangle , \qquad (8)$$
$$-\langle \Delta R(t) \rangle = c''_B \left\langle k_B^{D_B(t)/2} B_{\text{det}}(t) \right\rangle .$$

When the RHS of the mFAE is fitted to the casualty data, the fitted mFAE tends to be a much better fitting graph than the original mFAE using the a priori determined fitting factor. The shape of the fitted mFAE is now very similar to the SRLAE.[5] These results are shown in the selection of graphs

[5]The naming of these equations may be confusing but we have tried to be consistent.

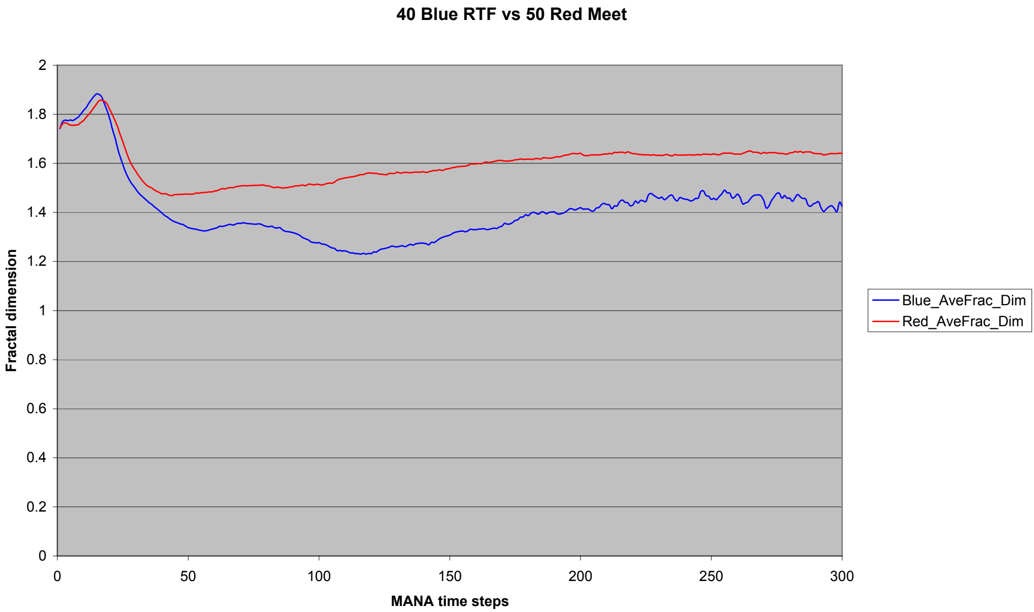

Figure 5: Here we plot the fractal dimension for each side for the duration of the simulation for the Baseline + Retreat to friends with surveillance and shooting scenario. The fractal dimension for this scenario is contained in a narrow range for much of the simulation. The early spike in the fractal dimension coincides with the two sides manoeuvring into position before the front forms; during this time the values of R_{det} and B_{det} are zero as are the corresponding casualty figures.

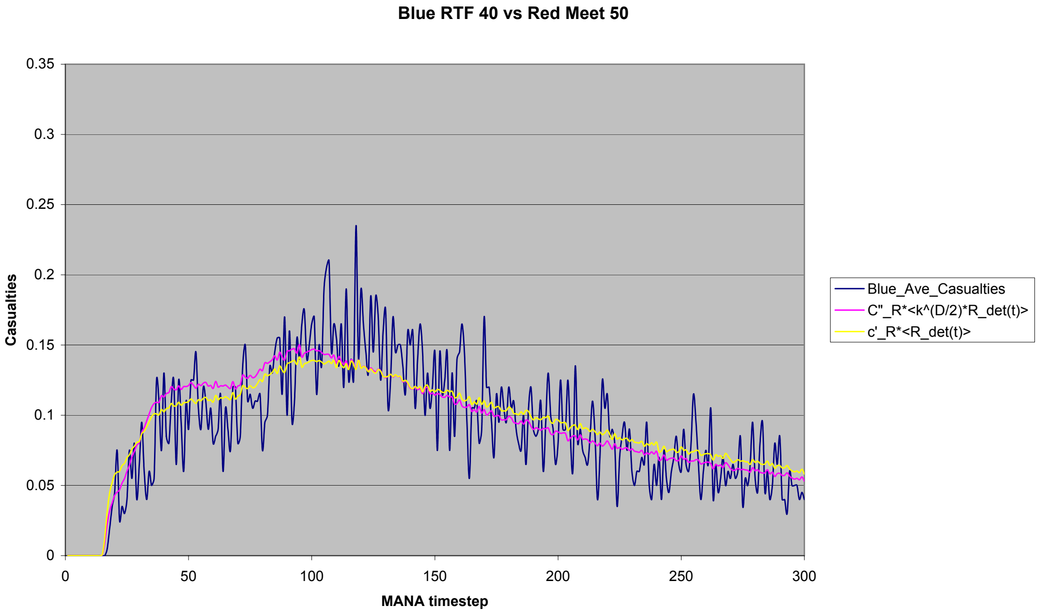

Figure 6: Comparing the casualties of the Blue force generated by the MANA simulations and the predicted casualties from the fitted mFAE (pink) and the SRLAE (yellow) for the Baseline + Retreat to friends with surveillance and shooting scenario.

(Figures 6 to 7); the blue curve is the casualties, the pink curve is the fitted mFAE, and the yellow curve is the SRLAE.

The FAE acronym was used by McIntosh and Lauren for two different equations, namely the original FAE and the FAE using the number of agents that have detected at least one enemy agent (that is, $\mathsf{R}_{\mathrm{det}}$ and $\mathsf{B}_{\mathrm{det}}$); this latter form of equation we named the mFAE. The SRLAE is the equation we developed by fitting the casualty data versus the sensor range limited force size data. Incidentally the mFAE also makes use of the same sensor range limited force size data as well as the fractal dimension, so it could be called a sensor range limited fractal attrition equation. At this point we have introduced another equation (8), the fitted mFAE which does not use the a priori fitting factor but uses a fitting coefficient calculated by fitting the casualty data to the RHS of the mFAE.

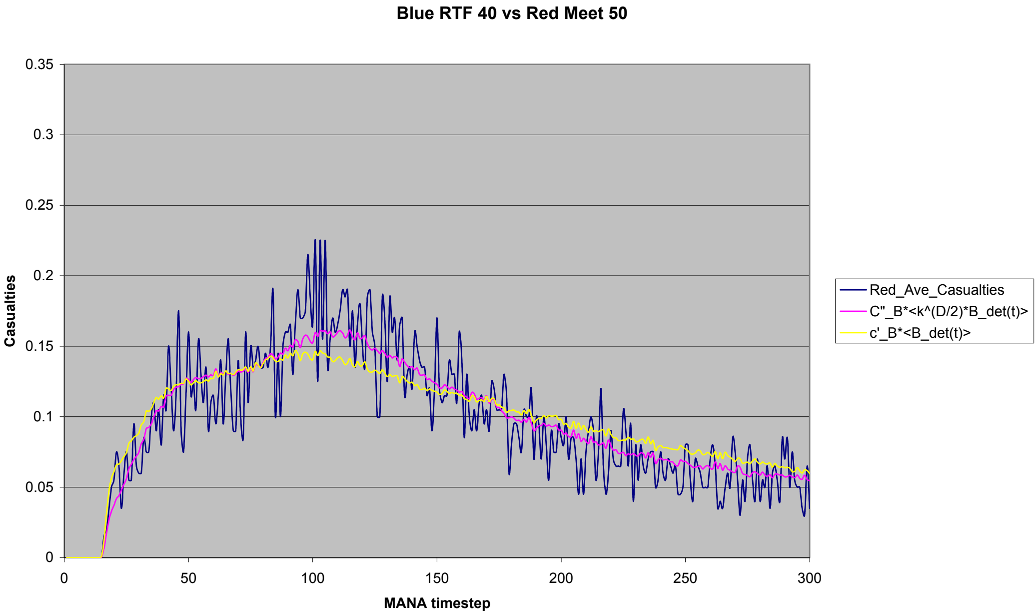

Figure 7: Comparing the casualties of the Red force generated by the MANA simulations and the predicted casualties from the fitted mFAE (pink) and the SRLAE (yellow) for the Baseline + Retreat to friends with surveillance and shooting scenario.

5 Discussion

By fitting the R_{det} and B_{det} to the corresponding Blue casualty rate and Red casualty rate we introduced corresponding fitting/scaling factors c'_B and c'_R and showed that the SRLAE is a good alternative to the mFAE. Additionally, the SRLAE does not require the fractal dimension terms; this reduces the computational complexity in estimating the casualty rates that was introduced into the modified FAE. Each battle is of course different, particularly when tactics or initial force sizes are changed, as well as the time distribution of R_{det} and B_{det} so the corresponding casualty rates will require a different value for the c' for each of the coupled DEs.

The fitted mFAE has also been shown to be as good a fit to the casualty data as the SRLAE. The difference between the two models is the lack of the fractal dimension term in the SRLAE. As there is little difference between the two models when viewing their graphs and also when comparing the coefficients of determination, and this being true whether or not there is a substantial variation in the fractal dimension, we apply Occam's Razor and conclude that the SRLAE, being simpler, is the preferred model.

The over-estimates and under-estimates of casualties by the mFAE for the different combinations of Red and Blue force sizes and Blue tactics shows that the fitting factor c, introduced by McIntosh and Lauren [5] in order to re-scale the mFAE to the casualty data, cannot be merely a function of the disparity between the sensor range and weapons range but must also depend on tactics and scenario.

The different values of c' for both Red and Blue sides and for different battle scenarios are an expected result for the SRLAE as are the different values of k for the Lanchester equations. The c' in the SRLAE depends on other parameters that describe the scenario and tactics, both Red and Blue.

The fractal dimension does appear to be of value as "a sensitive probe of different events occurring on the battlefield" [5] identifying "precise times at which the battle changed its character or key events occurred", but we disagree with McIntosh and Lauren that the apparent correlation between the fractal dimension and casualty rates provides justification for using it in an attrition equation. The simpler method of linear regression through the origin to obtain the SRLAE shows that there is sufficient information in the sensor-range-limited force sizes to accurately estimate the casualty rates without using the fractal dimension. The mFAE does make use of the sensor-range-limited force sizes but it was made unnecessarily complicated with the fractal dimension term. When we performed a linear regression on the mFAE, subject to the fractal dimension terms not varying substantially, we see that it can be made to fit the casualty data as well as the SRLAE; this tells us that the SRLAE is potentially a good linear model of attrition and is simpler than the mFAE.

Further work is needed to see if any relationships can be defined for c' as well as to ensure that the factors underlying it are independent of the SRLAE.

A Statistical results

The SRLAE (equation (7)) and the fitted mFAE (equation (8)) were both fitted using an ordinary least squares method with the intercept passing through the origin. The typical method of evaluating the fit of these models is to calculate the coefficient of determination R^2. The R^2 for the three models are presented in Tables 1, 2 and 3. The variation in the fractal dimension over the duration of the battle can be very complicated. It may remain range bound for the entire battle for one side or both sides. For one side it may be range bound for a short time then rapidly decline to zero as that side's force size is rapidly annihilated. A slow decline in the fractal dimension for one side for the scenarios investigated in this article occurs with a slow decline in the force size for that side. A visual inspection of the fractal dimension graphs may be sufficient to decide whether or not the fractal dimension is varying substantially; however, it is not a precise indicator of the variability of the fractal dimension and the corresponding variability of the $k^{D/2}$ term in the mFAE. An objective means of identifying a substantial variation is needed and we suggest the following definition for deciding what is a substantial variation of D.

- If the range of D is greater than $\pm 20\%$ of the mean value of D, then this is a substantial variation of D,
- otherwise it is not a substantial variation.

An increase in the value of the fractal dimension, D, will have less effect on the value of $k^{D/2}$ than will a decrease in D. Additionally these effects are

Table 1: R^2 values for each of the default scenarios.

Scenario	Red SRLAE	Red fitted MFAE		Blue SRLAE	Blue fitted MFAE	
25 Blue 50 Red	0.946	0.950		0.932	0.885	†
25 Blue 80 Red	0.874	0.921	†	0.893	0.889	†
40 Blue 50 Red	0.953	0.953		0.948	0.955	
40 Blue 80 Red	0.947	0.955		0.956	0.942	†
50 Blue 50 Red	0.956	0.966		0.949	0.959	
50 Blue 80 Red	0.961	0.960		0.952	0.958	†
80 Blue 50 Red	0.954	0.956	†	0.954	0.957	
80 Blue 80 Red	0.972	0.971		0.967	0.966	

amplified for smaller values of k.

The tables show those scenarios where there is a substantial variation in the fractal dimension (indicated with a †). The reader is reminded that $k = 0.1$ for all these scenarios. Scenarios where the value of k is greater than or less than 0.1 may need alternative values for the range of D values to delineate the substantial variation from the insubstantial variation. The Red columns refer to the RHS of the equation containing the R_{det} and the Blue columns refer to the RHS of the equation containing the B_{det}.

The tables of coefficients of determination show that there is little difference between the SRLAE and the fitted MFAE and this is true for those scenarios where there is a substantial variation in the fractal dimension as well as for those scenarios where there is no substantial variation.

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Table 2: R^2 values for each of the scenarios where Blue used Default + "Retreat to friends".

Scenario	Red			Blue		
	SRLAE	fitted mFAE		SRLAE	fitted mFAE	
25 Blue 50 Red	0.927	0.936	†	0.925	0.936	†
25 Blue 80 Red	0.844	0.896	†	0.848	0.930	†
40 Blue 50 Red	0.942	0.941		0.949	0.963	†
40 Blue 80 Red	0.925	0.945		0.931	0.957	†
50 Blue 50 Red	0.954	0.954		0.953	0.955	†
50 Blue 80 Red	0.933	0.945		0.935	0.939	†
80 Blue 50 Red	0.964	0.962	†	0.959	0.949	
80 Blue 80 Red	0.970	0.964		0.973	0.974	†

Table 3: R^2 values for each of the scenarios where Blue used Default + "Retreat to friends" (no surveillance and no shooting when retreating).

Scenario	Red			Blue		
	SRLAE	fitted mFAE		SRLAE	fitted mFAE	
25 Blue 50 Red	0.926	0.931		0.908	0.872	†
25 Blue 80 Red	0.876	0.906		0.910	0.896	†
40 Blue 50 Red	0.974	0.966		0.908	0.859	†
40 Blue 80 Red	0.928	0.930		0.930	0.915	†
50 Blue 50 Red	0.972	0.970		0.930	0.893	†
50 Blue 80 Red	0.949	0.936		0.942	0.918	†
80 Blue 50 Red	0.974	0.966	†	0.940	0.942	†
80 Blue 80 Red	0.974	0.954		0.947	0.939	†

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Author addresses

1. **Lesley Richard Kosowski**, Land Operations Division, Defence Science and Technology Organisation, PO Box 1500, Edinburgh, South Australia 5111, Australia.
   mailto:les.kosowski@dsto.defence.gov.au
2. **Adrian Pincombe**, Land Operations Division, Defence Science and Technology Organisation, PO Box 1500, Edinburgh, South Australia 5111, Australia.
   mailto:adrian.pincombe@dsto.defence.gov.au
3. **Brandon Pincombe**, Land Operations Division, Defence Science and Technology Organisation, PO Box 1500, Edinburgh, South Australia 5111, Australia.
   mailto:brandon.pincombe@dsto.defence.gov.au