Sparse inverse and characteristic polynomial of generalized arrow matrix

Murray Dow*

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Abstract

A generalized arrow matrix of order $n$ with $m$ non-zero rows and columns is presented. If a simple condition holds, the inverse of this matrix is also an arrow matrix of the same form. We then derive a simple expression for its characteristic polynomial.

*Supercomputer Facility, Australian National University, Canberra 0200, AUSTRALIA, mailto:m.dow@anu.edu.au

When testing numerical routines, a library of matrices with known inverses or other properties is useful. Most matrices with known inverses have none or at most two parameters that can be varied to provide a family of tests [3, 8]. More complex examples can be generated with the Sherman-Morrison formula or by Schur complements [2], but there appear to be very few simple test matrices of general size with many parameters. Notable exceptions to this are the Vandermonde and some tri-diagonal matrices [1, 3]. The arrow matrix [4, 6, 7] is another example; it is not hard to derive simple expressions for the inverse, determinant and characteristic polynomial of the arrow matrix which has the last row and column non-zero, and a non-zero diagonal [9].

We give a generalization of the arrow matrix, with an arbitrary number of
non-zero columns and rows, whose inverse is also an arrow matrix, and we also
give a simple expression for its characteristic polynomial. Such matrices and
methods are also of interest as pre-conditioners for iterative processes, because
of the sparsity of either the matrix or the inverse, and the freedom in choosing
many of the elements.

2 Inverse of an arrow matrix

Definition 1 Let $M$ be the arrow matrix of order $n$

$$M = \begin{bmatrix} D & e' \\ f & A \end{bmatrix}$$  \hspace{1cm} (1)

where $D$ is a diagonal matrix order $n - m$, $A$ is square of order $m$, $e$ and $f$ are
$m \times (n - m)$ with constant rows, i.e.

$$e = \begin{bmatrix} e_1 & e_1 & \cdots & e_1 \\ e_2 & e_2 & \cdots & e_2 \\ \vdots & \vdots & \ddots & \vdots \\ e_m & e_m & \cdots & e_m \end{bmatrix}, \quad f = \begin{bmatrix} f_1 & f_1 & \cdots & f_1 \\ f_2 & f_2 & \cdots & f_2 \\ \vdots & \vdots & \ddots & \vdots \\ f_m & f_m & \cdots & f_m \end{bmatrix}.$$

In general the inverse of $M$ will be dense, however provided $D$ is diagonal
and a simple condition is satisfied, $M^{-1}$ will be sparse, and indeed will be another
arrow matrix. Let the inverse of $M$ be

$$M^{-1} = \begin{bmatrix} C & p' \\ q & B \end{bmatrix}.$$  

Then using the usual formulae we obtain (see [5]) $C = (D - e' A^{-1} f)^{-1}$. Now for $M^{-1}$ to be an arrow matrix, we require that $C$ be diagonal. This implies that $D$ be diagonal and that $e' A^{-1} f = 0$.

**Theorem 1** Assuming that $A^{-1}$ exists, the inverse of the arrow matrix $M$ is given by

$$M^{-1} = \begin{bmatrix} D^{-1} & p' \\ q & B \end{bmatrix},$$

where

$$q = -A^{-1} f D^{-1},$$

$$p' = -D^{-1} e' A^{-1},$$

$$B = A^{-1} + q D p',$$

provided

$$e' A^{-1} f = 0. \quad (2)$$

Writing the inverse as a matrix of cofactors, this condition can be transformed into

$$\begin{vmatrix} 0 & e' \\ f & A \end{vmatrix} = 0,$$
where $|D|$ denotes a determinant and by $\bar{f}$ we mean the first column of $f$, similarly for $\bar{e}$:

$$
\bar{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix}, \quad \bar{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}.
$$

Matrices $p$ and $q$ have the same size and structure as $e$, that is constant rows. These matrices generate strikingly simple results, and have many pleasing properties, for example the orthogonality conditions hold:

$$
e'(BA-I) = 0, \quad p'(B^{-1} - A) = 0.
$$

**Example 1** Here $|M| = -1$, $D$ unit diagonal:

$$
M^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 2 & -1 \\
0 & 1 & 0 & 0 & 0 & 1 & 2 & -1 \\
0 & 0 & 1 & 0 & 0 & 1 & 2 & -1 \\
0 & 0 & 0 & 1 & 0 & 1 & 2 & -1 \\
0 & 0 & 0 & 0 & 1 & 1 & 2 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 1 & 0 & 0 \\
-2 & -2 & -2 & -2 & -2 & 0 & 2 & 3
\end{bmatrix}^{-1}
$$
Irrespective of the size \( n \) of matrix \( M \), if (2) is satisfied, its inverse will be an arrow matrix. The size of \( D^{-1}, p \) and \( q \) will depend on \( n \) but their elements will not change, and only the elements of \( B \) will depend on \( n \).

### 3 Characteristic Polynomial

First, to find the determinant of \( M \), we need a lemma. Note that in the remainder of this paper it is not necessary that (2) holds.
Lemma 1 Define the matrices $Q_{n,m}$ and $Q$ as

$$Q_{n-m,m} = \begin{bmatrix} 0 & e' \\ d_2 & \vdots \\ \vdots & \ddots \\ d_{n-m} & A \\ f & \ldots & d_{n-m} & A \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & \tilde{e}' \\ \frac{f}{i} & A \end{bmatrix},$$

and put $Q_{m+1,m} = Q$. Then

$$|Q_{n,m}| = \prod_{i=2}^{n-m} d_i |Q|, \quad n > m.$$

Proof. Expand $|Q_{m+i,m}|$ along the row $i$.

$$|Q_{m+i,m}| = d_i \begin{vmatrix} 0 & e' \\ d_2 & \vdots \\ \vdots & \ddots \\ d_{i-1} & A \\ f & \ldots & A \end{vmatrix} - e_1 \begin{vmatrix} 0 & d_2 & \vdots & \vdots \\ 0 & \ldots & 0 & \tilde{A} \\ f & \ldots & d_{i-1} & 0 \end{vmatrix} +$$
where $\tilde{A}$ indicates one column has been deleted from a matrix $A$.

Because $f$ has constant rows, all the determinants are zero except the first, giving $|Q_{m+i,m}| = d_i |Q_{m+i-1,m}|$, and the lemma follows.

**Theorem 2**  The determinant of the arrow matrix (1) is given by

$$|M| = |A| \prod_{j=1}^{n-m} d_j + |Q| \sum_{i=1}^{n-m} \prod_{j=1, j\neq i}^{n-m} d_j, \quad n \geq m.$$  

If $D$ has a constant diagonal $d$, this simplifies to

$$|M| = |A|d^{n-m} + |Q|(n - m)d^{m-m-1},$$  

with $Q$ as above.
Proof. Expand determinant along top row:

\[
|M| = d_1 \begin{vmatrix} 
    d_2 & e' \\
    d_3 & \ddots \\
    \vdots & \ddots & \ddots \\
    d_{n-m} & \ddots & \vdots \\
    f & \cdots & A \\
\end{vmatrix} + \\
\begin{vmatrix} 
    d_2 & e' \\
    \vdots & \ddots \\
    d_2 & \ddots & \ddots \\
    d_{n-m} & \ddots & \ddots \\
    f & \cdots & A \\
\end{vmatrix}.
\]

Using the notation \( F(n, m, d_1 : d_{n-m}) = |M| \), then

\[
|M| = F(n, m, d_1 : d_{n-m}) = d_1 F(n - 1, m, d_2 : d_{n-m}) + Q_{n,m} \\
= d_1 (d_2 F(n - 2, m, d_3 : d_{n-m}) + Q_{n-1,m}) + Q_{n,m} \\
= \prod_{j=1}^{n-m} d_j F(m, m, 0) + \prod_{j=1}^{n-m-1} d_j Q_{m+1,m} + \cdots + d_1 Q_{n-1,m} + Q_{n,m}.
\]

Applying the above lemma and the identity \( F(m, m, 0) = |A| \) the result follows.

We can now write out the characteristic polynomial of \( M \).

**Theorem 3** The characteristic polynomial of the matrix \( M \) defined as above is given by

\[
|A - \lambda I| \prod_{j=1}^{n-m} (d_j - \lambda) + |Q_\lambda| \sum_{i=1}^{n-m} \prod_{j=1, j \neq i}^{n-m} (d_j - \lambda),
\]
where

\[ Q_\lambda = \begin{bmatrix} 0 & \bar{c}' \\ \bar{f} & A - \lambda I \end{bmatrix}. \]

If \( D \) has a constant diagonal \( d \), the characteristic polynomial is

\[ |A - \lambda I| (d - \lambda)^{n-m} + |Q_\lambda| (n - m) (d - \lambda)^{n-m-1}, \ n \geq m. \]

For example, the eigenvalues of \( M \) in Example 1 (for any \( n \)) are the roots of

\[
(1 - \lambda)^{n-4} \left( (1 - \lambda) (-1 - 2 \lambda + 4 \lambda^2 - \lambda^3) + (n - 3) (23 \lambda - 7 \lambda^2) \right) = 0
\]

with \( n = 8 \) in this case. Numerically the roots are \(-5.09395, 0.00879162, 1, 1, 1, 1, 3.2824, 6.80275\).

References


