Numerical analysis of an averaged model of turbulent transport near a roughness layer

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Abstract

We formulate and numerically analyse the averaged model of dispersion in turbulent canopy flows. The averaging is carried out across the flow, for example over the river depth. To perform the averaging, we use the general approach suggested by Roberts and co-authors in the late 1980s, which is based on centre manifold theory. We derive an evolution partial differential equation for the depth averaged concentration, involving first, second and higher order derivatives with respect to the downstream coordinate. The coefficients of the equation are expressed in terms of parameters characterising the turbulent flow. Preliminary numerical results are demonstrated. In particular, it is shown that the advection and diffusion coefficients coincide with their values obtained earlier for the flow over a smooth bottom in the limit of large depths.

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Contents

1 Introduction

The centre manifold approach, developed in a series of works originated by Roberts [1, 2] (see also many references therein) is an effective tool enabling one to rigorously derive a low dimensional (one dimensional in simplest case) partial differential equation describing the spreading of contaminants and tracers in environmental and industrial fluid flows,

$$\partial_t C = g_1 \partial_x C + g_2 \partial_x^2 C + g_3 \partial_x^3 C + \cdots , \tag{1}$$

where $C(x, t)$ is the depth averaged concentration of the contaminant. The main idea behind the approach is briefly outlined in the next section. The coefficients g_i, $i = 1, 2, \ldots$, are deduced from the original non-averaged transfer equation as functions of parameters controlling the flow, such as the Reynolds number and the von Karman constant. Equations such as (1) were mostly derived in application to flows near smooth surfaces. However, flows near rough surfaces, for example through agricultural or urban canopies, are not less important; obviously they are extremely complicated.

In the present article we analyse an averaged dispersion model near a canopy based on the three layer structure of the turbulent flow [3]. Note that different structures have also been proposed in the literature, particularly for agricultural canopies [4]. Consider a turbulent inertial layer of constant thickness H and the adjacent canopy layer of thickness h being the size of obstacles. For example, it can be a turbulent flow in an open channel subject to constant shear stress (measured by the friction velocity u_*^2) applied to the fluid surface. Hereafter we refer to channel flow; however, our analysis can be generalised to cases of turbulent boundary layers of varying depth, for instance near a semi-plane. The ensemble averaged concentration of passive substance, $c(x, y, t)$, is subject to the advection-diffusion equation

$$\partial_t c + u(y)\partial_x c = \partial_y[D(y)\partial_y c]\,, \tag{2}$$

where $D(y)$ is the turbulent diffusion coefficient assumed known. The boundary conditions express non-penetration through the surface, $y = H$, and the bottom, $y = 0$,

$$D\partial_y c|_{y=0} = D\partial_y c|_{y=H} = 0\,. \tag{3}$$

2 Centre manifold procedure

Turbulent diffusion tends to spread the substance uniformly across the channel. The velocity shear acts in the opposite way and tends to create vertical nonuniformity. As a result of the competition of these factors, the contaminant not only moves due to the advection but also effectively 'diffuses' along the channel, that is in the x-direction, despite turbulent diffusion in this direction being absent in Equation (2). Following Mercer and Roberts [1] we perform the Fourier transformation of (2) to get

$$\partial_t \hat{c} = L[\hat{c}] - iku(y)\hat{c}\,, \tag{4}$$

where $\hat{c}(y, k, t)$ is the Fourier transform of concentration $c(x, y, t)$ defined by $1/(2\pi)\int_{-\infty}^{\infty} \exp(-ikx)c\,dx$. The linear operator $L[\hat{c}] = \partial_y[D(y)\partial_y\hat{c}]$ expresses the cross flow turbulent diffusion and has a discrete spectrum of eigenvalues. One of the eigenvalues is equal to zero; it corresponds to the neutral eigenmode $\hat{c} =$ constant that is an arbitrary constant level of concentration across the channel. All the other eigenvalues are negative; they correspond to decaying non-uniformities of the concentration across the channel due to the diffusion, provided that there is no flux through the boundaries.

After a sufficiently long time, variations of the concentration along the channel become slow; accordingly we suppose that the wave number k is small. Let us add to (4) the trivial equation $\partial_t k = 0$ and formally treat the wave number k as a variable and the term $k\hat{c}$ as 'nonlinear'. As governed by (4) and $\partial_t k = 0$, the dynamics exponentially quickly evolve to a low dimensional state, where each of the fast modes depends on t via the slow neutral mode. As a measure of the 'amplitude' of the neutral mode we choose the depth averaged concentration, $\hat{C}(k, t)$. Accordingly, we have

$$\hat{c} = \hat{c}(\hat{C}, k, y) \quad \text{such that} \quad \partial_t \hat{C} = G(\hat{C}, k)\,. \tag{5}$$

With (5) taken into account, Equation (4) becomes

$$L[\hat{c}] = \frac{\partial \hat{c}}{\partial \hat{C}} G + iku\hat{c}\,. \tag{6}$$

Since the problem is linear, we assume linear asymptotic expansions

$$\hat{c} = \sum_{n=0}^{\infty} c_n(y)(ik)^n \hat{C}\,, \quad G = \sum_{n=1}^{\infty} g_n(ik)^n \hat{C}\,. \tag{7}$$

The definition of $\hat{C}$ as the depth average implies the conditions

$$\frac{1}{h}\int_0^h c_0\,dy = 1\,, \quad \int_0^h c_n\,dy = 0\,, \quad \text{for } n = 1, 2, \ldots\,. \tag{8}$$

Substituting (7) into (6) and collecting similar powers of k we obtain a sequence of equations for the unknown functions $c_n(y)$ and coefficients g_n,

$$L[c_0] = 0\,, \tag{9}$$

$$L[c_n] = \sum_{m=1}^{n} c_{n-m} g_m + u(y) c_{n-1} \,, \quad \text{for } n = 1, 2, \ldots \,. \tag{10}$$

Integrating (10) over the depth we get

$$D \partial_y c|_{y=H} - D \partial_y c|_{y=0} = g_n \overline{c_0} + \overline{u(y) c_{n-1}} = g_n + \overline{u(y) c_{n-1}} \,,$$

where the overline means depth averaging. Once the fluxes through the boundaries are zero, then

$$g_n = -\overline{u(y) c_{n-1}} \quad \text{for } n = 1, 2, \ldots \,. \tag{11}$$

Successively, we can calculate g_n and c_n for any n. Confining our attention to only the three leading terms in the G series in (7), we obtain

$$\partial_t \hat{C} = g_1 (ik) \hat{C} + g_2 (ik)^2 \hat{C} + g_3 (ik)^3 \hat{C} + \cdots \,. \tag{12}$$

Now, applying the inverse Fourier transformation to (12), we obtain the advection-diffusion-dispersion equation for the averaged concentration in the form (1).

3 Analysis of the flow near canopy

The inertial layer is described by the classical semi-logarithmic velocity profile [5],

$$\frac{u(y)}{u_*} = \frac{1}{\kappa} \ln \left(\frac{y - d}{y_0} \right) \,, \tag{13}$$

where y_0 is the roughness height, d is the displacement height and κ is the von Karman constant. The values y_0 and d are usually taken in engineering literature as fractions of the canopy height h.

For the canopy layer, we adopt the model of Macdonald [3]. A parameter that characterises the geometry of obstacles is the frontal area density $\lambda = A_f / A_d$, where A_f is the frontal area of an obstacle exposed to the flow and A_d is the total surface area per obstacle (total area divided by the number of obstacles). Following Cionco [6], Macdonald considered cylindrically shaped obstacles and assumed that within the canopy layer there is a balance between the local shear stress and obstacle drag force. This assumption leads to the exponential velocity profile

$$u(y) = u_h \exp[a(y/h - 1)] \,, \tag{14}$$

where a is the so-called attenuation coefficient and u_h is the velocity at $y = h$. Typically $2 < a < 3$ [7]. Note that the exponential profile (14) does not turn exactly zero at $y = 0$. However, the velocity there is small; we treat it as a virtual zero. The analysis [3] showed that the parameter a linearly depends on the frontal area density,

$$a = m\lambda$$

with $m = 9.6$. For the mixing length ℓ_c in the canopy the following expression in terms of λ, a and h, was obtained,

$$\frac{\ell_c}{h} = \sqrt{\frac{p\lambda\left(1 - e^{-2a}\right)}{4a^3}} \,, \tag{15}$$

where $p = 1.2$. At the top of the canopy the shear stress must be continuous. Substituting the velocity (14) into the Prandtl expression for the shear stress, $\ell_c^2(\partial u/\partial y)^2$, and equating it to the shear stress in the inertial layer u_*^2, gives

$$\frac{u_*}{u_h} = \frac{a\ell_c}{h} \,.$$

The two velocity profiles—the canopy one expressed by (14) and semi-logarithmic one expressed by (13)—must match each other. Macdonald [3] achieved this by assuming that the mixing length ℓ varies linearly against y when moving from the top of the canopy to some yet-to-be-determined contact point with the semi-logarithmic layer, $y = y_w$. At that point, the length ℓ must coincide with the mixing length on the lower edge of the semi-logarithmic layer,

$$\ell = \kappa(y - d) \tag{16}$$

(this is precisely the expression that eventually produces (13)). At the point $y = y_w$ formula (16) becomes $\ell_w = \kappa(y_w - d)$. Macdonald adopted the following linear interpolation of the length connecting the point $\ell = \ell_c$ at $y = h$ and the point $\ell = \ell_w$ at $y = y_w$,

$$\ell(y) = \ell_c + \left(\frac{y-h}{y_w - h}\right)[\kappa(y_w - d) - \ell_c] = A + By\,, \tag{17}$$

where the coefficients A and B are defined by the left-hand side of (17),

$$A = \ell_c - \frac{h}{y_w - h}[\kappa(y_w - d) - \ell_c]\,, \quad B = \frac{1}{y_w - h}[\kappa(y_w - d) - \ell_c]\,.$$

Further, Macdonald assumed that within the layer connecting the canopy and inertial layer, the friction velocity is constant, giving

$$\frac{\partial u}{\partial y} = \frac{u_*}{\ell}\,, \tag{18}$$

where ℓ is represented by (17). Integrating (18) gives the velocity in the connecting layer of the form

$$u(y) = \frac{u_*}{B}\ln\left(\frac{A+By}{A+Bh}\right) + u_h\,. \tag{19}$$

Here the constant of integration was chosen to meet the boundary condition $u = u_h$ at $y = h$. At the matching point $y = y_w$ formula (19) should give the same value as the semi-logarithmic law (13), that is

$$\frac{u_*/u_h}{B}\ln\left(\frac{A+By_w}{A+Bh}\right) + 1 = \frac{u_*/u_h}{\kappa}\ln\left(\frac{y_w - d}{y_0}\right)\,. \tag{20}$$

Relation (20) presents an implicit equation with respect to the coordinate y_w, which needs to be found. Calculations [3] based on available experimental data showed that roughly $y_w/h \approx 2$.

The above analysis identifies the velocity profiles in the three layers: canopy layer (14), connecting layer (19), and inertial (semi-logarithmic) layer (13). Using the profiles, it is easy to get expressions for the turbulent diffusion coefficient for momentum in each layer. Rearranging the Prandtl formula for the stress, $\ell^2(\partial u/\partial y)^2$ as $(\ell^2\partial u/\partial y)\partial u/\partial y = D_{\text{mom}}\partial u/\partial y$, we have

$$D_{\text{mom}} = \ell^2 \frac{\partial u}{\partial y} \,.$$

Finally, we assume that the diffusion coefficient for the passive substance in each layer is proportional to D_{mom} [8],

$$D = K\,D_{\text{mom}} \,. \tag{21}$$

The diffusion coefficient is continuous through the layers.

The transfer equations (2), boundary conditions (3) and diffusion coefficient expressed by (21) form a self-consistent model involving the following independent dimensional parameters: the friction velocity u_*, thickness of the inertial layer H, height of the canopy h, and frontal area density of the canopy λ. As for the roughness height, y_0, and displacement height, d, we assume these to be fractions of the canopy height, h, and, in the final analysis, to be functions of λ. Choosing the canopy height, h, as the length scale and friction velocity, u_*, as the velocity scale, we arrive at just two nondimensional independent parameters: H/h and λ. Note that implicitly the model also depends on the shape of obstacles—they can be square or staggered.

Figure 1 shows some velocity profiles plotted using values of the parameters from Macdonald [3] for square cube arrays at various packing densities. Below the level $y/h = 1$ lies the canopy layer, above $y = y_w \approx 2$ lies the semi-logarithmic layer and between them is the matching layer.

From the formulated model we derived expressions for the advection coefficient, g_1, and diffusion coefficient, g_2. Due to space limitations we do not present them in this article; they are very large and we plan to report them elsewhere. However, we analyse some of their features here. It is interesting

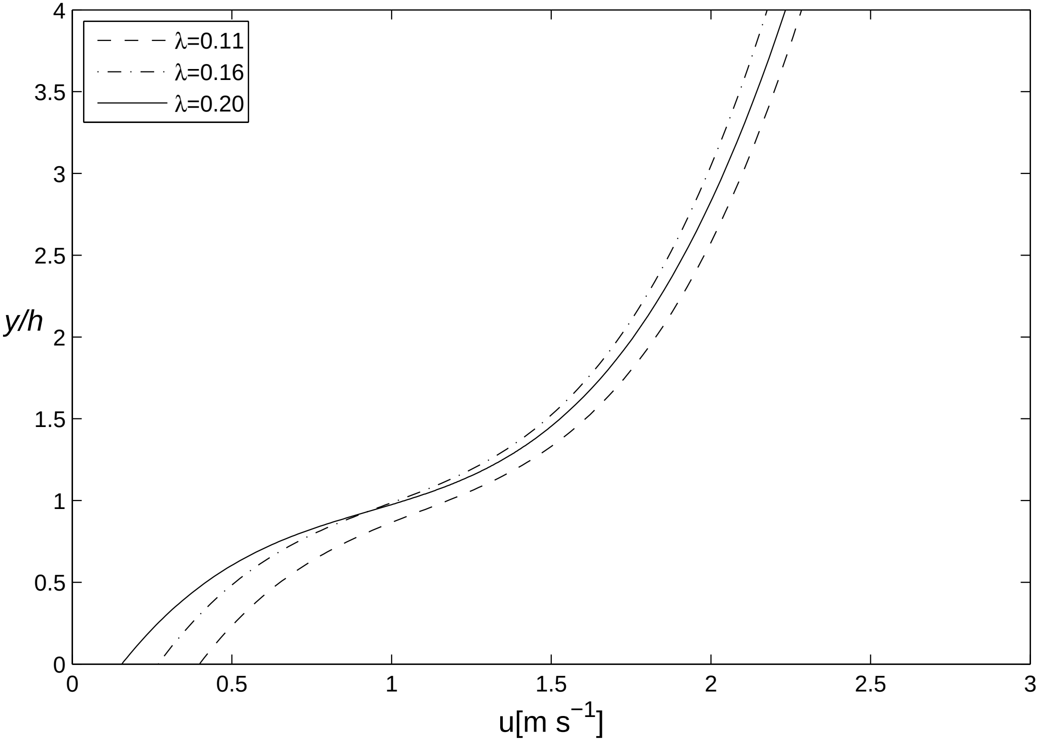

Figure 1: Mean velocity profiles for different λ for square cube arrays.

to compare the results for g_1 and g_2 with those for the smooth channel flow analysed by Strunin [9]. In the limit of very large Reynolds numbers he obtained

$$g_1 \to -\frac{1}{\kappa}\ln R = -\frac{1}{\kappa}\ln\left(\frac{u_* H}{\nu}\right),$$

where the Reynolds number is based on the total depth of the channel, that is $R = u_* H/\nu$ with ν standing for the kinematic viscosity. Taking the limit in our resulting formula for g_1 as $H \to \infty$, we get

$$g_1 \to -\frac{h}{H}\left\{\frac{H-d}{h\kappa}\ln\left(\frac{H-d}{y_0}\right) - \frac{H}{h\kappa}\right\} \to -\frac{1}{\kappa}\ln\frac{H}{y_0} = -\frac{1}{\kappa}\ln\left(\frac{u_* H}{\nu}\cdot\frac{\nu}{u_* y_0}\right)$$

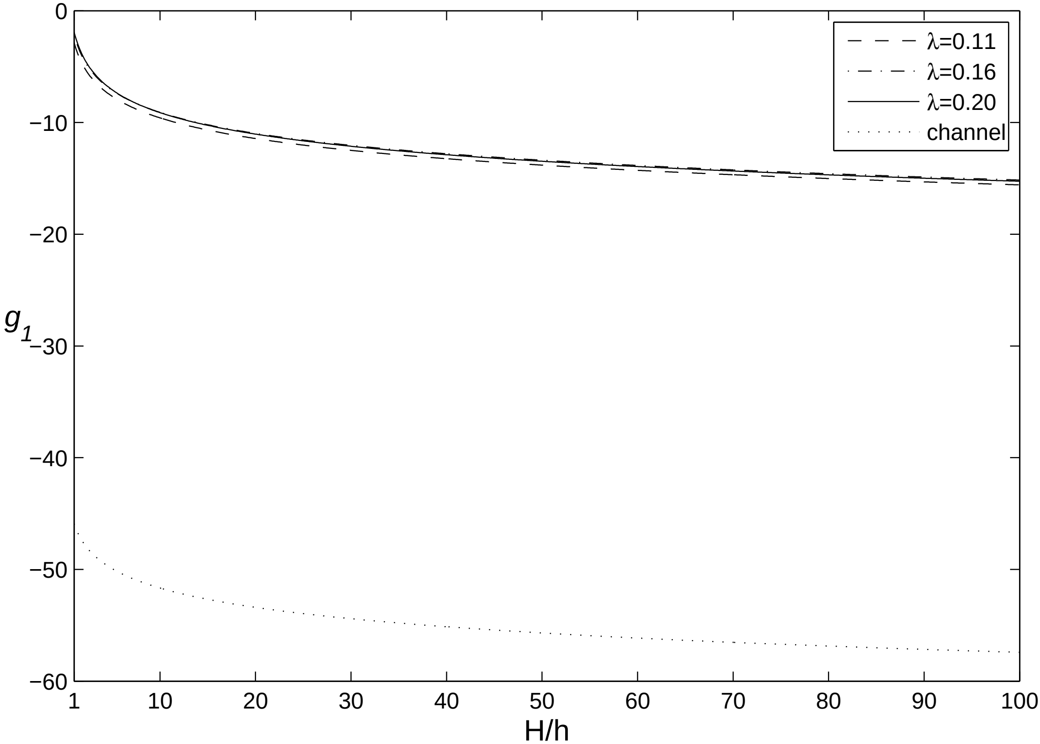

Figure 2: The advection coefficient versus total depth for different λ.

$$= -\frac{1}{\kappa}\left[\ln\left(\frac{u_* H}{\nu}\right) + \ln\left(\frac{\nu}{u_* y_0}\right)\right] \to -\frac{1}{\kappa}\ln\left(\frac{u_* H}{\nu}\right), \qquad (22)$$

which is exactly the earlier result given above. From (22) we see that the advection coefficient for the canopy flow is smaller in absolute value than that for the smooth channel flow. The reason for this is the negative second term in the square brackets. It represents the inverse Reynolds number based on the roughness height, y_0; although y_0 is small, the velocity is large so that the Reynolds number is large enough. From a physical viewpoint, the canopy just slows down the flow relative to the smooth flow with the same shear stress. The advection coefficient g_1 is the depth average velocity; accordingly,

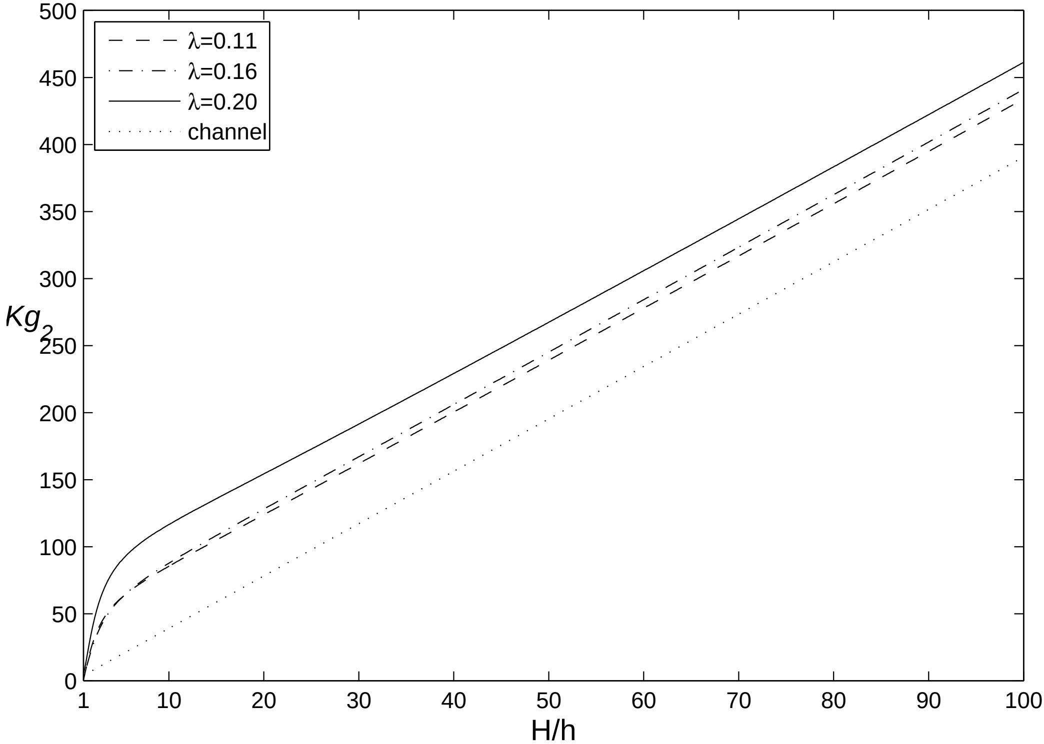

Figure 3: The diffusion coefficient versus total depth for different λ.

for the canopy flow, it is smaller in absolute value. Analysing the resulting formula for the diffusion coefficient g_2 (also omitted in this short article) in the limit $H \to \infty$, we find

$$\lim_{H\to\infty} g_2 = \frac{1}{4K\kappa^3}\frac{H}{h}\,. \tag{23}$$

This conclusion also agrees with the result of earlier work [9]. Figures 2 and 3 present numerical results for our canopy model for finite depths.

4 Conclusion

We constructed an averaged model of shear dispersion in the turbulent flow near a canopy. The core of the model is the standard advection-diffusion equations (2) with no-flux boundary conditions (3). The model contains as independent dimensional parameters the friction velocity u_*, total thickness of the flow H, height of the canopy h, and frontal area density of the canopy λ. The model is reduced to the averaged form (1) by the centre manifold procedure. The advection and diffusion coefficients, governing the transfer of substances along the flow, are found in terms of the independent parameters.

References

[1] Mercer, G. N. and Roberts, A. J., A centre manifold description of contaminant dispersion in channels with varying flow properties, *SIAM J. Appl. Math.*, **50**, 1990, 1547–1565. doi:10.1137/0150091 C143, C144

[2] Roberts, A. J. and Strunin, D. V., Two-zone model of shear dispersion in a channel using centre manifolds, *Quarterly J. Mech. Appl. Math.*, **57**, 2004, 363–378. doi:10.1.1.102.854 C143

[3] Macdonald, R. W., Modelling the mean velocity profile in the urban canopy layer, *Boundary-Layer Meteorol.*, **97**, 2000, 25–45. doi:10.1023/A:1002785830512 C143, C146, C147, C148, C149

[4] Harman, I. N. and Finnigan, J. J., A simple unified theory for flow in the canopy and roughness sublayer, *Boundary-Layer Meteorol.*, **123**, 2007, 339–363. doi:10.1007/s10546-006-9145-6 C144

[5] H. Tennekes and J. L. Lumley. *A First Course in Turbulence.* MIT Press, Cambridge, MA, 1972. C146

[6] Cionco, R. M, Mathematical model for air flow in a vegetative canopy, *J. Appl. Meteorol.*, **4**, 1965, 517–522.
doi:10.1175/1520-0450(1965)004¡0517:AMMFAF¿2.0.CO:2 C147

[7] Cionco, R. M., A wind-profile index for canopy flow, *Boundary-Layer Meteorol.*, **3**, 1972, 255–263. doi:10.1007/BF0203923 C147

[8] Barenblatt, G. I., Transfer of a passive additive in a turbulent boundary layer at very large Reynolds numbers, *Proc. Nat. Acad. Sci.*, **100**, 2003, 1481–1483. doi:10.1073/pnas.0337426100 C149

[9] Strunin, D. V., Universality of turbulent dispersion in a steady flow in an open channel, *Quarterly J. Mech. Appl. Math.*, **64**, 2011, 197–214.
doi:10.1093/qjmam/hbr002 C150, C152

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