# Shepherded solitons

I. N. Towers[1] Z. Jovanoski[2]

(Received 10 January 2012; revised 28 June 2012)

**Abstract**

We investigate the onset of vector solitons in a three level, cascade, atomic system. We present an existence curve in the model parameter space for bright vector solitons. Approximate analytical solutions are given and the stability of the solutions discussed. Numerical simulations confirm the analytical predictions.

## Contents



http://journal.austms.org.au/ojs/index.php/ANZIAMJ/article/view/5122 gives this article,  Published July 2, 2012. ISSN 1446-8735. 

# 1 Introduction

Although solitons are ubiquitous in many branches of physics, self-trapped optical spatial solitons possess features which makes them potentially useful for applications such as all-optical switching and routing, interconnections, parallel computing, and optical storage [1]. Among the known mechanisms that support the existence of spatial solitons is self-phase modulation or self-focussing. In this case, due to nonlinearity in the medium, the optical beam modifies the refractive index and induces an effective waveguide, which then self-guides the beam. The spatial soliton can be thought of as the fundamental mode of this waveguide. A wide variety of solitons have been studied and experimentally verified in different media, including Kerr media, liquid crystals, photorefractive and photovoltaic crystals [1], quantum dots [2] and atomic systems [3, 4]. A single soliton described by the nonlinear Schrödinger equation (NLS) propagates at constant speed with a fixed spatial profile. Such solitons are known as scalar solitons. An important problem is the interaction of two such solitons which is governed by a set of coupled NLS equations, the nonlinear coupling between the two fields is governed by cross-phase modulation. A shape preserving solution of such equations is called a vector soliton because of its multicomponent nature [1].

We consider the propagation of two intense optical beams of different frequencies in a Kerr-like medium composed of uniformly distributed, three level, atomic systems in the cascade configuration for which the atoms are initially in the ground level [4, 5, 6, 7]. Two intense optical fields of different frequencies induce dipole allowed transitions from the ground level to an intermediate level, then from the intermediate level to the top level. We assume a closed atomic system where the top level decays to the intermediate level and the intermediate level to the ground level with different decay rates. The direct transition from the top to the ground level is a dipole forbidden transition.

Ansari et al. [5] derived the steady state expressions for the third order susceptibilities experienced by two optical fields propagating through a Kerr-like medium. Using these expressions a version of the coupled NLS equations was derived. We study a normalised version of the system:

$$\begin{aligned} i\frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial x^2} + \sigma_1|v|^2 u &= 0\,, \\ i\alpha\frac{\partial v}{\partial z} + \frac{\partial^2 v}{\partial x^2} + \sigma_2\left(|u|^2 + \eta|v|^2\right) v &= 0\,, \end{aligned} \tag{1}$$

where $\alpha > 0$, $0 < \eta < 1$, and $\sigma_j = \pm 1$. If $\sigma_1 = \sigma_2 = +1$ this indicates the optical medium has a focusing nonlinearity and the system (1) allows so-called bright solitons—solitons with a single central peak in the intensity profile. However, if $\sigma_1 = \sigma_2 = -1$, then the medium has a defocusing nonlinearity and system (1) exhibits dark solitons—solitons with a dip in its otherwise uniform intensity profile. A mixed state of one field as a bright soliton and the other a dark soliton is also possible when the medium is defocusing. Exact bright and dark solutions found by Ansari et al. [5] revealed that the amplitudes of the vector solitons can be equal or vastly different.

The simplest solution of interest of system (1) consists of zero in the first component, $u$, and a soliton in the second component, $v$. This is essentially a scalar soliton. Our interest is to determine the onset of bifurcation from the scalar soliton to the vector soliton. In particular, we study the dynamics in the vicinity of the bifurcation point where the amplitude of $u$ is smaller than $v$, but non-zero. We refer to such solitons as shepherded—the larger amplitude beam acts as a *shepherd* to the smaller one.

## 2 Analytical results

Assuming $\sigma_1 = \sigma_2 = +1$ we look for stationary solutions of the form

$$u(x,z) = \sqrt{\beta_1}e^{i\beta_1 z}u(x), \quad v(x,z) = \sqrt{\beta_1}e^{i\beta_2 z}v(x).$$

After rescaling $x \to x/\sqrt{\beta_1}$ system (1) becomes

$$\frac{d^2u}{dx^2} - u + v^2u = 0\,, \quad \frac{d^2v}{dx^2} - \lambda v + \left(u^2 + \eta v^2\right) v = 0\,, \tag{2}$$

where $\lambda = \alpha\beta_2/\beta_1$ .

If the field $u(x) = 0$, then the system reduces to the canonical NLS which has the well-known fundamental bright soliton solution [8]

$$v(x) = \sqrt{\frac{2\lambda}{\eta}}\,\operatorname{sech}\left(\sqrt{\lambda}x\right).$$

When $\lambda = 1$, an exact solution for a vector soliton (a soliton where both fields have non-zero intensities) exists [5]:

$$u(x) = \sqrt{\frac{2\,(1-\eta)}{\eta}}\,\operatorname{sech}(x)\,, \quad v(x) = \sqrt{\frac{2}{\eta}}\,\operatorname{sech}(x)\,. \tag{3}$$

At some point in the parameter space $(\eta, \lambda)$ of system (2) the vector soliton solutions must bifurcate from the single bright soliton of the standard NLS. In order to investigate where in the parameter space this might occur and what intensity profile the resulting solitons have, we apply a regular perturbation to (2) of the form

$$u(x) = \epsilon\, u_1(x) + \mathcal{O}\left(\epsilon^2\right), \tag{4}$$

$$v(x) = v_0(x) + \epsilon^2 v_2(x) + \mathcal{O}\left(\epsilon^3\right). \tag{5}$$

At zeroth order we reproduce the NLS

$$\frac{d^2v_0}{dx^2} - \lambda\, v_0 + \eta\, v_0^3 = 0\,,$$

and at first order in $\epsilon$ we have

$$\frac{d^2u_1}{dx^2} - u_1 + u_1\, v_0^2 = 0\,. \tag{6}$$

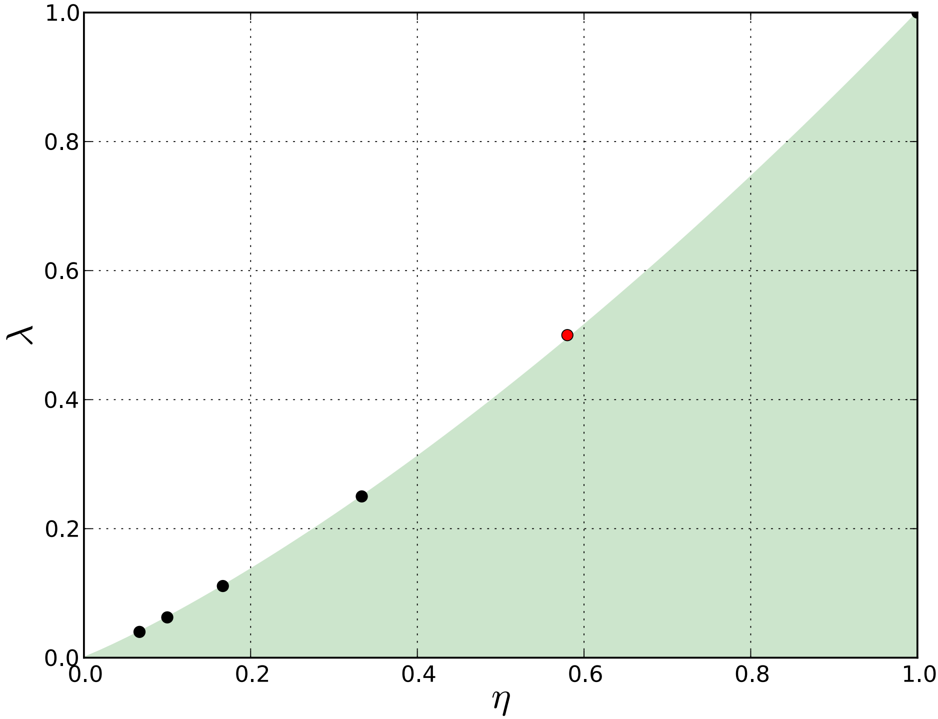


Figure 1: The solid (green) region of the parameter space is where only the NLS soliton may exist. Beyond the solid region one can find bright vector solitons. Close to this boundary the bright solitons will have a small amplitude $\mathfrak{u}$ component. The (red) dot indicates the parameters used in Figures 2, 3 and 4. The black dots are the bifurcation points of the bright vector solitons given by the perturbation analysis.

Taking the fundamental bright soliton solution of the NLS as the zeroth order solution we substitute for $v_0$ in the equation for $u_1$ to give

$$\frac{d^2u_1}{dx^2} - u_1 + \frac{2\lambda}{\eta}\operatorname{sech}^2\left(\sqrt{\lambda}\,x\right)u_1 = 0\,. \tag{7}$$

Equation (7) is common in quantum mechanics [9] and optics [10]. Using the transformation $y = \tanh\left(\sqrt{\lambda}x\right)$ Equation (7) becomes the associated Legendre equation

$$(1-y^2)\frac{d^2u_1}{dy^2} - 2y\frac{du_1}{dy} + \left(\frac{2}{\eta} - \frac{1/\lambda}{1-y^2}\right)u_1 = 0\,,$$

which has a general solution in terms of associated Legendre functions of the first and second kind [11]:

$$u_1(x) = c_1 P_l^m\left(\tanh\left[\sqrt{\lambda}\,x\right]\right) + c_2 Q_l^m\left(\tanh\left[\sqrt{\lambda}\,x\right]\right).$$

As we desire localised solutions we immediately set $c_2 = 0$. Further we identify that

$$m = \frac{1}{\sqrt{\lambda}}\,, \quad l = \frac{1}{2}\left(-1 + \sqrt{1 + 8/\eta}\right). \tag{8}$$

It follows directly from the requirement of non-zero width beams that $m \neq 0$ and, to ensure localised solutions, $m$ and $l$ must be integer. Thus the first three solutions for $u_1(x)$ are

$$P_1^1(y) = -\operatorname{sech}(x)\,, \quad \lambda = \eta = 1\,, \tag{9}$$

$$P_2^1(y) = -3\tanh(x)\operatorname{sech}(x)\,, \quad \lambda = 1\,, \quad \eta = \frac{1}{3}\,, \tag{10}$$

$$P_2^2(y) = 3\operatorname{sech}^2\left(\frac{x}{2}\right), \quad \lambda = \frac{1}{4}\,, \quad \eta = \frac{1}{3}\,. \tag{11}$$

The correction, $v_2(x)$, to the NLS soliton is the solution of the equation

$$\frac{d^2v_2}{dx^2} - \lambda v_2 + \frac{6\lambda}{\eta}\operatorname{sech}^2\left(\sqrt{\lambda}x\right)v_2 = -\sqrt{\frac{\lambda}{\eta}}\operatorname{sech}\left(\sqrt{\lambda}x\right)u_1\,.$$

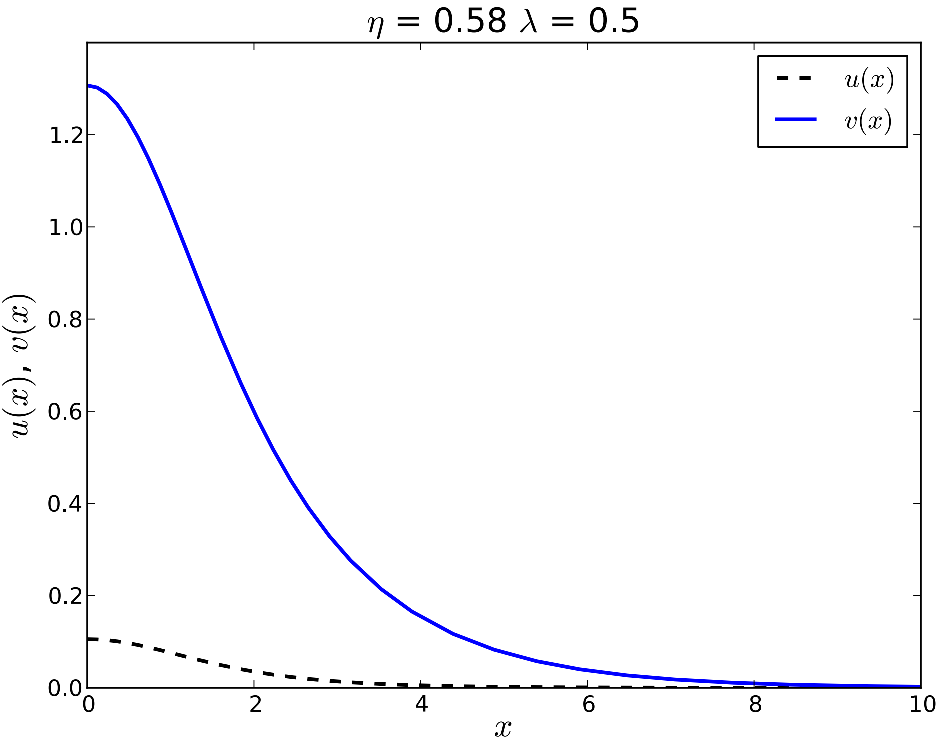


Figure 2: Bright vector soliton profile for $\eta = 0.58$ and $\lambda = 0.5$. The profile is calculated numerically by solving system (2). The boundary for existence is at $\eta = \sqrt{2}/(\sqrt{2}+1) \approx 0.5858$. The ratio of the peak intensities is $|u|^2/(\eta|v|^2) \approx 0.01$.

Using the same transform, as above, $y = \tanh\left(\sqrt{\lambda}x\right)$, localised $v_2(x)$ is found in terms of $P_2^1(y)$ and integrals of $P_2^1(y)$ and $Q_2^1(y)$.

The bright vector solitons should bifurcate from the NLS solitons when the values of $\lambda$ and $\eta$ allow a single peaked intensity profile for $u(x)$. This occurs when $m = l$. The perturbation solution requires that $m, l \in \mathbb{N}$.

# 3 Numerical results

The system (2) was solved numerically using a standard boundary value solver [12]. The boundary values at infinity were approximated as $u(L) = 0$ and $v(L) = 0$ where $L$ was made large enough to allow the amplitudes of the profiles to decay sufficiently so that the approximation was reasonable. Typically $L \approx 20$ was required to ensure the beam was fully localised before the end of the computational window was reached. The other boundary conditions were taken to be $u'(0) = 0$ and $v'(0) = 0$. A typical example of a low amplitude or shepherded soliton is given in Figure 2.

The existence boundary in Figure 1 was found by solving system (2) starting with the known exact vector soliton solution for a given $\eta$. The value of $\lambda$ was then reduced and the system solved again. By repeating the process the stationary vector solitons were found until the integral

$$Q_u = \int_{-\infty}^{\infty} |u|^2 \, dx < 10^{-3},$$

which means that the $u$ field is sufficiently small to be considered a perturbation to the $v$ field. Interestingly, if the restriction that $m$ and $l$ are integer in (8) is lifted, then an approximate existence boundary in parameter space is found for bright vector solitons. The boundary is well described by $\eta \approx 2\lambda/(\sqrt{\lambda} + 1)$ and the difference from the numerically determined boundary shown in Figure 1 is no larger than $10^{-4}$.

The full system of partial differential equations (PDE) (1) was numerically solved using an adaptive step size split–step Fourier method [13]. The split–step Fourier method is a standard PDE solver used in the field of (linear and nonlinear) optics. The basic idea behind the method is that linear and nonlinear operators of the PDE are treated separately. The spatial derivative terms which make up the linear operator are dealt with in Fourier space and a standard ordinary differential equation solver (such as a Runge–Kutta scheme) deals with the nonlinear operator. The approximation to the PDE comes from how one splits these operators and chooses to recombine the results. Sinkin et

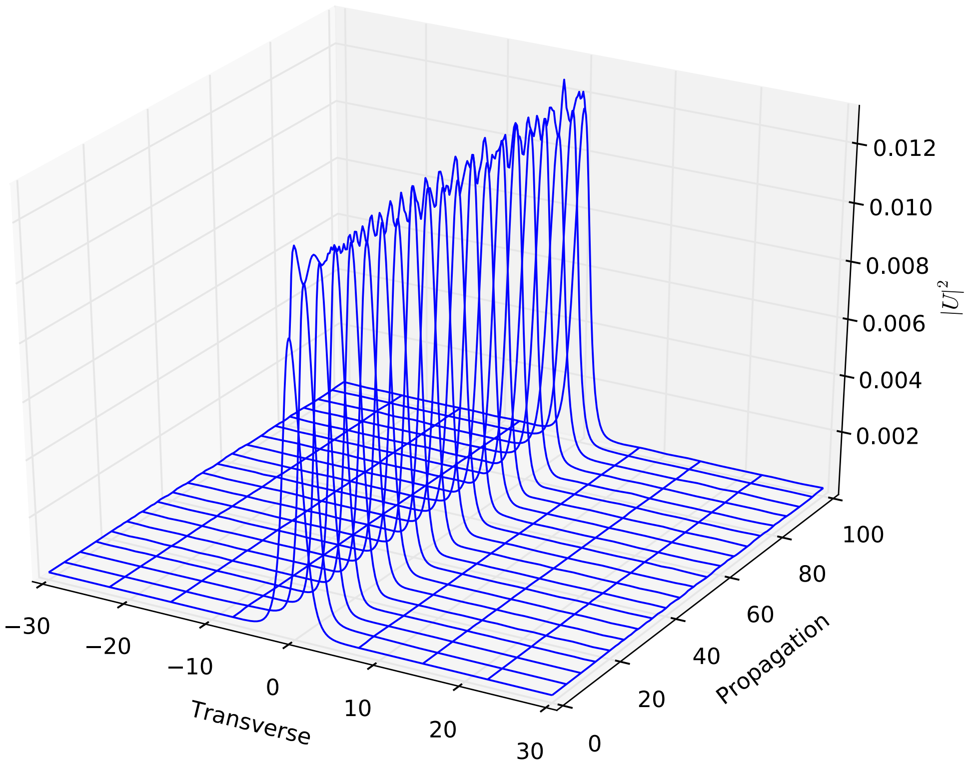


Figure 3: The intensity of the $u(x, z)$ resulting from a simulation of system (1) with $\alpha = 0.5$, $\eta = 0.58$, $\sigma_1 = \sigma_2 = 1$ and an initial profile $u(x, 0) = 0.1 \operatorname{sech}\left(\sqrt{\alpha}\, x\right)$. The initial profile is not a soliton solution and so the beam profile is not stationary during propagation.

al. [13] detailed a version of the method which maintains global third–order accuracy by adapting the step size in the propagation direction—in the case of (1) $z$ is the propagation direction—and it is this version of the method we utilise here.

Figures 3 and 4 show the propagation of the $u(x, z)$ and $v(x, z)$ fields respectively. Initially the fields are set to a profile which is close to but not exactly

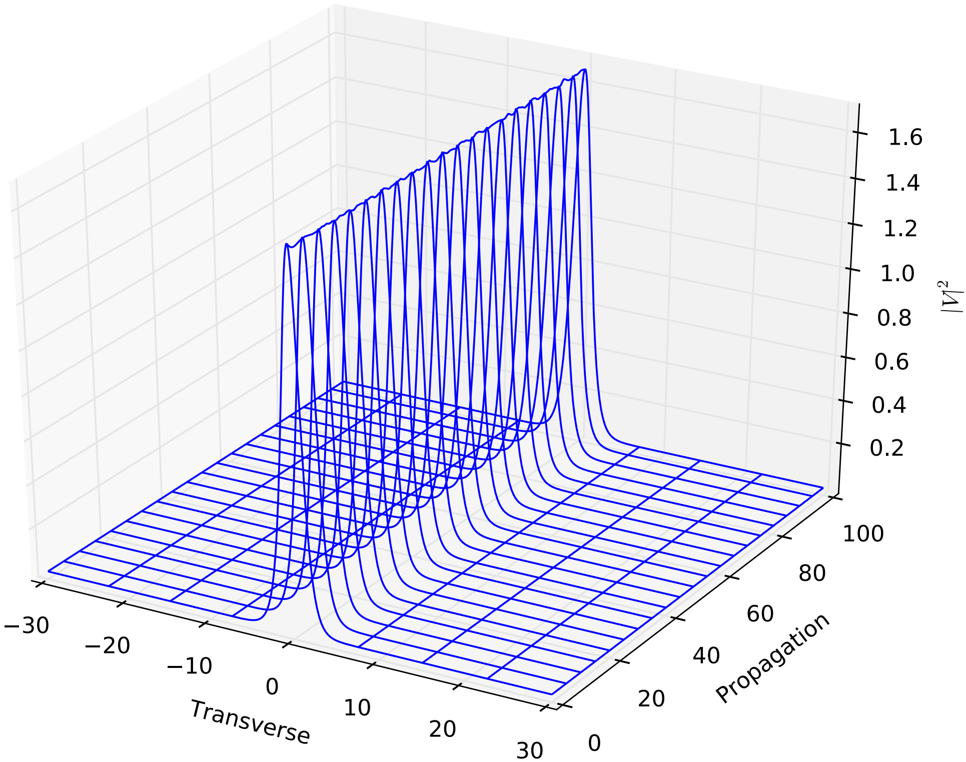


Figure 4: The intensity of the $v(x, z)$ under the same conditions as Figure 3 and initial condition $u(x, 0) = 1.3 \operatorname{sech}\left(\sqrt{\alpha}\, x\right)$.

the shape of the stationary solution shown in Figure 2. The profiles are then propagated numerically for a distance of $100$ units. Because the initial profiles are not stationary solutions, small oscillations in the beam amplitude are visible in Figure 3. The scale of the intensity axis in Figure 4 masks the oscillations. A linear stability analysis of the solutions was not performed but the modulational instability studies by Ansari et al. [5] and long propagation distance shown here suggests that the bright solitons are, at least, observable, if not actually stable. The fact the beam does not collapse or self-focus to a singularity indicates the soliton solution about which the beam oscillates is stable at least to small perturbations.

# 4 Conclusion

We analysed the case in which one field is so weak compared with the other that it does not affect the propagation of the much larger amplitude and therefore more intense field. However, the intense field still affects the weak field through cross-phase modulation induced coupling. Under appropriate conditions, the soliton can trap the weak field completely through soliton-induced waveguiding. Physically, the intense field propagating as a soliton changes the refractive index of the nonlinear medium and creates an effective waveguide. A weak field is trapped by this waveguide and propagates as a guided mode. This phenomenon may be referred to as beam shepherding. We have shown that the mechanism behind beam shepherding leading to the formation of vector solitons is associated with bifurcations of a scalar soliton. We found an approximate analytic expression for the boundary that partitions the parameter space into scalar solitons and vector solitons. In the vicinity of the boundary where shepherding occurs we obtained approximate analytic solutions for the vector solitons. Numerical simulations indicate that these vector solitons are stable.

# References

[1] M. Segev. Optical spatial solitons. *Optical and Quantum Electronics*, **30**, 1998, 503–533. C313

[2] J. Li, J. Liu and X. Yang. Superluminal optical soliton via resonant tunneling in coupled quantum dots. *Physica E*, **40**, 2008, 2916–2920. doi:10.1016/j.physe.2008.02.004 C313

[3] G. Huang, L. Deng and M. G. Payne. Dynamics of ultraslow optical solitons in a cold three-state atomic system. *Phys. Rev. E*, **72**, 2005, 016617. doi:10.1103/PhysRevE.72.016617 C313

[4] N. A. Ansari, Z. Jovanoski, H. S. Sidhu and I. N. Towers. Non-linear interaction of two intense fields with a three-level atomic system. *J. Nonlin. Opt. Phys. Mat.*, **15**, 2006, 401–413. doi:10.1142/S0218863506003402 C313

[5] N. A. Ansari, I. N. Towers, Z. Jovanoski and H. S. Sidhu. A semi-classical approach to two-frequency solitons in a three-level cascade atomic system. *Opt. Commun.*, **274**, 2007, 66–73. doi:10.1016/j.optcom.2007.02.019 C313, C314, C315, C321

[6] C. R. Menyuk, *IEEE J. Quantum Electron.*, **QE-23**, 174 (1987). C313

[7] Y. S. Kivshar and G. P. Agrawal. *Optical Solitons: From fibers to photonic crystals*, Chapter 9, San Diego: Academic Press, 2003. C313

[8] N. N. Akhmediev and A. Ankiewicz. *Solitons: Nonlinear pulses and beams*, p.27, London : Chapman & Hall, 1997. C315

[9] S. Flugge. *Practical quantum mechanics*, Vol. 1, pg. 94–100, Berlin; New York : Springer-Verlag, 1971. C317

[10] A. W. Snyder and J. D. Love. *Optical Waveguide Theory*, pp. 264–8, London; New York : Chapman and Hall, 1983. C317

[11] F. W. J. Olver. *Introduction to Asymptotics and Special Functions*, p.169, New York; London: Academic Press, 1974. C317

[12] L. F. Shampine, P. H. Muir and H. Xu. A User-Friendly BVP solver. *JNAIAM*, **1**, 2006, 201–17. `http://cs.stmarys.ca/~muir/JNAIAM_Shampine_Muir_Xu2006.pdf` C319

[13] O. V. Sinkin, R. Holzlöhner, J. Zweck and C. R. Menyuk. Optimization of the Split-Step Fourier Method in Modeling Optical-Fiber

Communications Systems. *J. Lightw. Technol.*, **21**, 2003, 61–8. doi:10.1109/JLT.2003.808628 C319, C320

## Author addresses

1. **I. N. Towers**, School of Physical, Environmental and Mathematical Sciences, University of New South Wales, Canberra 2600, Australia.
mailto:i.towers@adfa.edu.au
2. **Z. Jovanoski**, School of Physical, Environmental and Mathematical Sciences, University of New South Wales, Canberra 2600, Australia.
mailto:z.jovanoski@adfa.edu.au