A finite element approximation for the quasi-static Maxwell–Landau–Lifshitz–Gilbert equations

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Abstract

The quasi-static Maxwell–Landau–Lifshitz–Gilbert equations which describe the electromagnetic behaviour of a ferromagnetic material are highly nonlinear. Sophisticated numerical schemes are required to solve the equations, given their nonlinearity and the constraint that the solution stays on a sphere. We propose an implicit finite element solution to the problem. The resulting system of algebraic equations is linear which facilitates the solution process compared to nonlinear methods. We present numerical results to show the efficacy of the proposed method.

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1 Introduction

The Maxwell–Landau–Lifshitz–Gilbert (MLLG) equations describe the electromagnetic behaviour of a ferromagnetic material. For simplicity, we assume that there is a bounded cavity $\tilde{D} \subset \mathbb{R}^3$ (with perfectly conducting outer surface $\partial\tilde{D}$) into which a ferromagnet $D \subset \mathbb{R}^3$ is embeded. We further assume that $\tilde{D}\backslash\bar{D}$ is a vacuum. Over time period $(0, T)$ we let $D_T := (0, T) \times D$ and $\tilde{D}_T := (0, T) \times \tilde{D}$, and let $\mathbb{S}^2$ be the unit sphere. We denote the unit vector of the magnetisation by $\mathbf{m}(t, \mathbf{x}) : D_T \to \mathbb{S}^2$, and the magnetic field by $\mathbf{H}(t, \mathbf{x}) : \tilde{D}_T \to \mathbb{R}^3$ over time t and space $\mathbf{x}$. The quasi-static MLLG system is

$$\mathbf{m}_t = \lambda_1 \mathbf{m} \times \mathbf{H}_{\text{eff}} - \lambda_2 \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}) \quad \text{in } D_T \,, \tag{1a}$$

$$-\mu_0 \tilde{\mathbf{m}}_t = \mu_0 \mathbf{H}_t + \sigma \nabla \times (\nabla \times \mathbf{H}) \quad \text{in } \tilde{D}_T \,, \tag{1b}$$

in which the subscript t indicates a partial derivative with respect to time, $\lambda_1 \neq 0$, $\lambda_2 > 0$, $\sigma \geqslant 0$ and $\mu_0 > 0$ are constants and $\mathbf{H}_{\text{eff}}$ is the effective magnetic field which is dependent on both $\mathbf{H}$ and $\mathbf{m}$. Here $\tilde{\mathbf{m}} : \tilde{D}_T \to \mathbb{R}^3$ is the zero extension of $\mathbf{m}$ onto $\tilde{D}_T$, that is

$$\tilde{\mathbf{m}}(t,\mathbf{x}) = \begin{cases} \mathbf{m}(t,\mathbf{x}) & (t,\mathbf{x}) \in D_T\,, \\ 0 & (t,\mathbf{x}) \in \tilde{D}_T \backslash D_T\,. \end{cases}$$

The system (1a)–(1b) is supplemented with initial conditions

$$\mathbf{m}(0,.) = \mathbf{m}_0 \text{ in } D \quad \text{and} \quad \mathbf{H}(0,.) = \mathbf{H}_0 \text{ in } \tilde{D}\,, \tag{2}$$

and boundary conditions

$$\frac{\partial \mathbf{m}}{\partial \mathbf{n}} = 0 \text{ on } \partial D_T \quad \text{and} \quad (\nabla \times \mathbf{H}) \times \mathbf{n} = 0 \text{ on } \partial \tilde{D}_T\,, \tag{3}$$

where $\mathbf{n}$ is the outward normal vector to the relevant surface.

Equation (1a) is the first dynamical model for the precessional motion of the magnetisation, suggested by Landau and Lifshitz [8] in 1935. In this model, the time derivative of the magnetisation $\mathbf{m}$ is a combination of the precessional movement $\mathbf{m}\times\mathbf{H}_{\text{eff}}$ and the dissipative movement $\mathbf{m}\times(\mathbf{m}\times\mathbf{H}_{\text{eff}})$; see Figure 1.

Cimrák [4] showed existence and uniqueness of a *local* strong solution of (1a)–(3). He also proposed a finite element method to approximate this local solution and provided an error estimation [3]. Baňas, Bartels and Prohl [2] proposed an implicit nonlinear scheme using the finite element method, and proved that the approximate solution converges to a weak *global* solution. Their method required the condition $k = O(h^2)$ on the time step k and space step h for the convergence of the nonlinear system of equations resulting from the discretisation. We propose an implicit *linear* finite element scheme to find a weak *global* solution to (1a)–(3). This approach was initially developed by Alouges and Jaison [1] for the single Landau–Lifshitz equation (1a). We extend their approach to the system (1a)–(1b). The advantage of this approach is that there is no condition imposed on the time step and the space step.

For simplicity we choose the effective field $\mathbf{H}_{\text{eff}} = \Delta\mathbf{m} + \mathbf{H}$. We focus on implementation issues of the method. In particular, we show how the

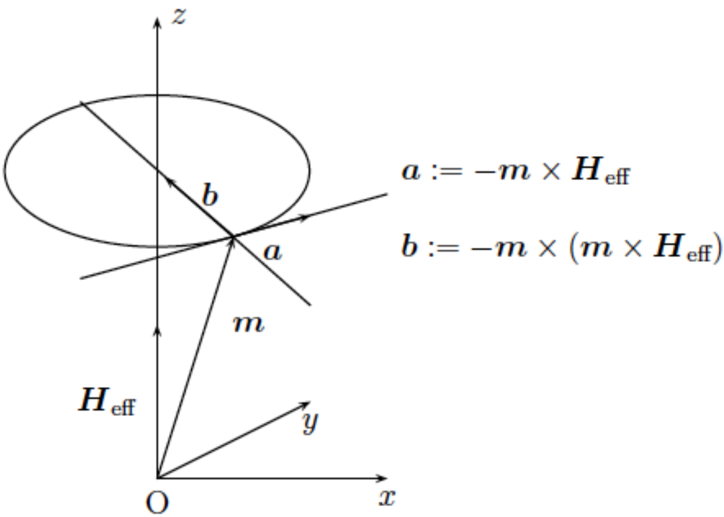

Figure 1: The combination of the precessional movement $\mathbf{m} \times \mathbf{H}_{\text{eff}}$ and the dissipative movement $\mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}})$.

finite element spaces and their bases are constructed. In another article we conducted a convergence analysis [9].

In Section 2 we rewrite (1a) in a form which is more suitable for our approach; see (4). We then introduce a variational formulation of the MLLG system. Section 3 is devoted to the presentation of the implicit linear finite element scheme. Numerical experiments are presented in the last section.

2 A variational formulation of the MLLG equations

Before presenting a variational formulation for the MLLG equations, we show how to rewrite (1a) in the form introduced by Gilbert [6]

$$\lambda_1 \mathbf{m}_t + \lambda_2 \mathbf{m} \times \mathbf{m}_t = \mu \mathbf{m} \times \mathbf{H}_{\text{eff}} \,, \tag{4}$$

in which $\mu = \lambda_1^2 + \lambda_2^2$.

Lemma 1. *Equation (1a) and equation (4) are equivalent.*

Proof: By using the elementary identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \,, \quad \text{for all } \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3 \,,$$

and the property $|\mathbf{m}| = 1$ we obtain

$$\mathbf{m} \times [\mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}})] = -\mathbf{m} \times \mathbf{H}_{\text{eff}} \,. \tag{5}$$

Assume that $\mathbf{m}$ is a solution to (1a). Multiplying both side of (1a) by $\lambda_2 \mathbf{m}$ using the vector product and using (5) we obtain

$$\lambda_2 \mathbf{m} \times \mathbf{m}_t = \lambda_1 \lambda_2 \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}) + \lambda_2^2 \mathbf{m} \times \mathbf{H}_{\text{eff}} \,.$$

Multiplying both sides of (1a) by λ_1 and adding the resulting equation to the above equation, we deduce that $\mathbf{m}$ satisfies (4).

Now assume that $\mathbf{m}$ is a solution to (4). On multiplying both sides of this equation by $\lambda_2 \mathbf{m}$ using the vector product and noting

$$\mathbf{m} \times (\mathbf{m} \times \mathbf{m}_t) = (\mathbf{m} \cdot \mathbf{m}_t)\mathbf{m} - |\mathbf{m}|^2 \mathbf{m}_t = -\mathbf{m}_t \,, \tag{6}$$

we deduce

$$\lambda_1 \lambda_2 \mathbf{m} \times \mathbf{m}_t - \lambda_2^2 \mathbf{m}_t = \lambda_2 \mu \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}) \,.$$

Subtracting this equation from equation (4) times λ_1 and dividing both sides of the resulting equation by μ, we obtain (1a). This proves the lemma. ♠

Before presenting the variational form of this problem, it is necessary to introduce the function spaces

$$\mathbb{H}^1(D,\mathbb{R}^3) = \left\{ \mathbf{u} \in \mathbb{L}^2(D,\mathbb{R}^3) \mid \frac{\partial \mathbf{u}}{\partial x_i} \in \mathbb{L}^2(D,\mathbb{R}^3) \quad \text{for } i = 1,2,3 \right\},$$
$$\mathbb{H}(\text{curl}; \tilde{D}) = \left\{ \mathbf{u} \in \mathbb{L}^2(\tilde{D},\mathbb{R}^3) \mid \nabla \times \mathbf{u} \in \mathbb{L}^2(\tilde{D},\mathbb{R}^3) \right\}.$$

Here $\mathbb{L}^2(\Omega,\mathbb{R}^3)$ is the usual space of Lebesgue integrable functions defined on Ω and taking values in $\mathbb{R}^3$.

Following Lemma 1, instead of solving (1a)–(3) we solve (1b)–(4). A variational form of this problem is as follows. For all $\boldsymbol{\phi} \in \mathbb{C}^\infty(D_T,\mathbb{R}^3)$ and $\boldsymbol{\zeta} \in \mathbb{C}^\infty(\tilde{D}_T,\mathbb{R}^3)$, find $\mathbf{m} \in \mathbb{H}^1(D_T)$ and $\mathbf{H} \in \mathbb{L}^2(\tilde{D}_T)$ such that $\mathbf{H}_t \in \mathbb{L}^2(\tilde{D}_T)$ and $\nabla \times \mathbf{H} \in \mathbb{L}^2(\tilde{D}_T)$ to satisfy the Landau–Lifshitz–Gilbert (LLG) equation

$$\begin{aligned} &\lambda_1 \int_{D_T} \mathbf{m}_t \cdot \boldsymbol{\phi}\, dx\, dt + \lambda_2 \int_{D_T} (\mathbf{m} \times \mathbf{m}_t) \cdot \boldsymbol{\phi}\, dx\, dt \\ &= \mu \int_{D_T} \nabla \mathbf{m} \cdot \nabla(\mathbf{m} \times \boldsymbol{\phi})\, dx\, dt + \mu \int_{D_T} (\mathbf{m} \times \mathbf{H}) \cdot \boldsymbol{\phi}\, dx\, dt, \end{aligned} \tag{7}$$

and Maxwell's equation

$$\mu_0 \int_{\tilde{D}_T} \mathbf{H}_t \cdot \boldsymbol{\zeta}\, dx\, dt + \sigma \int_{\tilde{D}_T} \nabla \times \mathbf{H} \cdot \nabla \times \boldsymbol{\zeta}\, dx\, dt = -\mu_0 \int_{\tilde{D}_T} \tilde{\mathbf{m}}_t \cdot \boldsymbol{\zeta}\, dx\, dt. \tag{8}$$

In the following section we introduce a finite element scheme to approximate the solution $(\mathbf{m}, \mathbf{H})$ of (7)–(8).

3 The finite element scheme

Let $\mathbb{T}_h$ be a regular tetrahedrisation of the domain $\tilde{D}$ into a tetrahedra of maximal mesh size h, and let $\mathbb{T}_h|_D$ be the restriction of $\mathbb{T}_h$ to the domain D.

The set of N vertices is $\mathcal{N}_h := \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ and the set of M edges is $\mathcal{M}_h := \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_M\}$.

To discretise the LLG equation (7) we introduce the finite element space $\mathbb{V}_h$ of all continuous piecewise linear functions on $\mathbb{T}_h|_D$, which is a subspace of $\mathbb{H}^1(D, \mathbb{R}^3)$. A basis for $\mathbb{V}_h$ is chosen to be $(\phi_n)_{1\leqslant n\leqslant N}$, where

$$\phi_n(\boldsymbol{x}_m) = \delta_{n,m},$$

and $\delta_{n,m}$ is the Kronecker delta. The interpolation operator from $\mathbb{C}^0(D, \mathbb{R}^3)$ onto $\mathbb{V}_h$ is

$$I_{\mathbb{V}_h}(\boldsymbol{v}) = \sum_{n=1}^{N} \boldsymbol{v}(\boldsymbol{x}_n)\phi_n \quad \text{for all } \boldsymbol{v} \in \mathbb{C}^0(D, \mathbb{R}^3).$$

To discretise Maxwell's equation (8), we use the space $\mathbb{Y}_h$ of lowest order edge elements of Nedelec's first family [10] which is a subspace of $\mathbb{H}(\text{curl}; \tilde{D})$.

For a function $\boldsymbol{u}$ which is Lebesgue integrable on all edges in $\mathcal{M}_h$, we define [10] the interpolation $I_{\mathbb{Y}_h}$ onto $\mathbb{Y}_h$ as

$$I_{\mathbb{Y}_h}(\boldsymbol{u}) = \sum_{q=1}^{M} u_q \boldsymbol{\psi}_q \quad \text{for all } \boldsymbol{u} \in \mathbb{C}^0(\tilde{D}, \mathbb{R}^3),$$

where

$$u_q = \int_{\boldsymbol{e}_q} \boldsymbol{u} \cdot \boldsymbol{\tau}_q \, ds,$$

in which $\boldsymbol{\tau}_q$ is the unit vector in the direction of edge $\boldsymbol{e}_q$.

Fixing a positive integer J, we choose the time step $k = T/J$, and define $t_j = jk$ for $j = 0, \cdots, J$. The functions $\boldsymbol{m}(t_j, \cdot)$ and $\boldsymbol{H}(t_j, \cdot)$ are approximated by $\boldsymbol{m}_h^{(j)} \in \mathbb{V}_h$ and $\boldsymbol{H}_h^{(j)} \in \mathbb{Y}_h$, respectively, for $j = 1, 2, \ldots, J$. Since

$$\boldsymbol{m}_t(t_j, \cdot) \approx \frac{\boldsymbol{m}(t_{j+1}, \cdot) - \boldsymbol{m}(t_j, \cdot)}{k} \approx \frac{\boldsymbol{m}_h^{(j+1)} - \boldsymbol{m}_h^{(j)}}{k},$$

we evaluate $\mathbf{m}_h^{(j+1)}$ from $\mathbf{m}_h^{(j)}$ using

$$\mathbf{m}_h^{(j+1)} := \mathbf{m}_h^{(j)} + k\boldsymbol{v}_h^{(j)}\,, \tag{9}$$

where $\boldsymbol{v}_h^{(j)}$ is an approximation of $\mathbf{m}_t(t_j,\cdot)$. However, to maintain the condition $|\mathbf{m}_h^{(j+1)}| = 1$, we normalise the right hand side of (9) and therefore define $\mathbf{m}_h^{(j+1)}$ belonging to $\mathbb{V}_h$ by

$$\mathbf{m}_h^{(j+1)} = I_{\mathbb{V}_h}\left(\frac{\mathbf{m}_h^{(j)} + k\boldsymbol{v}_h^{(j)}}{|\mathbf{m}_h^{(j)} + k\boldsymbol{v}_h^{(j)}|}\right) = \sum_{n=1}^{N} \frac{\mathbf{m}_h^{(j)}(x_n) + k\boldsymbol{v}_h^{(j)}(x_n)}{\left|\mathbf{m}_h^{(j)}(x_n) + k\boldsymbol{v}_h^{(j)}(x_n)\right|}\phi_n\,.$$

Hence it suffices to propose a scheme to compute $\boldsymbol{v}_h^{(j)}$.

Motivated by $\mathbf{m}_t \cdot \mathbf{m} = 0$, we find $\boldsymbol{v}_h^{(j)}$ in the space $\mathbb{W}_h^{(j)}$ defined by

$$\mathbb{W}_h^{(j)} := \left\{\boldsymbol{w} \in \mathbb{V}_h \mid \boldsymbol{w}(x_n)\cdot\mathbf{m}_h^{j}(x_n) = 0,\ n = 1,\dots,N\right\}. \tag{10}$$

Given $\mathbf{m}_h^{(j)} \in \mathbb{V}_h$, if (7) is used to compute $\boldsymbol{v}_h^{(j)}$, an approximation to $\mathbf{m}_t(t_j,\cdot)$, then a different test space from $\mathbb{V}_h$ is required because the test function $\boldsymbol{\phi}$ in (7) is not perpendicular to $\mathbf{m}$, unlike $\mathbf{m}_t$. To circumvent this difficulty, we use (6) to rewrite (7) as

$$\begin{aligned}\lambda_2\int_{D_T}\mathbf{m}_t\cdot\boldsymbol{w}\,dx\,dt - \lambda_1\int_{D_T}(\mathbf{m}\times\mathbf{m}_t)\cdot\boldsymbol{w}\,dx\,dt \\ = -\mu\int_{D_T}\nabla\mathbf{m}\cdot\nabla\boldsymbol{w}\,dx\,dt + \mu\int_{D_T}\mathbf{H}\cdot\boldsymbol{w}\,dx\,dt\end{aligned} \tag{11}$$

where $\boldsymbol{w} = \mathbf{m}\times\boldsymbol{\phi}$. Now both $\mathbf{m}_t$ and $\boldsymbol{w}$ are perpendicular to $\mathbf{m}$ for all $(t,\boldsymbol{x}) \in D_T$. Hence, given $\mathbf{m}_h^{(j)} \in \mathbb{V}_h$ and $\mathbf{H}_h^{(j)} \in \mathbb{Y}_h$, we compute the approximations $\boldsymbol{v}_h^{(j)}$ and $\mathbf{H}_h^{(j+1)}$ of $\mathbf{m}_t(t_j,\cdot)$ and $\mathbf{H}(t_{j+1},\cdot)$, respectively, as follows. Find $\boldsymbol{v}_h^{(j)} \in \mathbb{W}_h^{(j)}$ and $\mathbf{H}_h^{(j+1)} \in \mathbb{Y}_h$ satisfying, for all $\boldsymbol{w}_h^{(j)} \in \mathbb{W}_h^{(j)}$ and

$\zeta_h \in \mathbb{Y}_h$,

$$\lambda_2 \int_D \boldsymbol{v}_h^{(j)} \cdot \boldsymbol{w}_h^{(j)}\,dx - \lambda_1 \int_D (\mathbf{m}_h^{(j)} \times \boldsymbol{v}_h^{(j)}) \cdot \boldsymbol{w}_h^{(j)}\,dx$$
$$= -\mu \int_D \nabla\left(\mathbf{m}_h^{(j)} + \theta k \boldsymbol{v}_h^{(j)}\right) \cdot \nabla \boldsymbol{w}_h^{(j)}\,dx + \mu \int_D \mathbf{H}_h^{(j+1/2)} \cdot \boldsymbol{w}_h^{(j)}\,dx\,dt\,, \quad (12)$$

and

$$\mu_0 \int_{\tilde{D}} d_t \mathbf{H}_h^{(j+1)} \cdot \xi_h\,dx\,dt + \sigma \int_{\tilde{D}} \nabla \times \mathbf{H}_h^{(j+1/2)} \cdot \nabla \times \xi_h\,dx\,dt$$
$$= -\mu_0 \int_{\tilde{D}} \boldsymbol{v}_h^{(j)} \cdot \xi_h\,dx\,dt\,. \quad (13)$$

Here,

$$\mathbf{H}_h^{(j+1/2)} := \frac{\mathbf{H}_h^{(j+1)} + \mathbf{H}_h^{(j)}}{2} \quad \text{and} \quad d_t \mathbf{H}_h^{(j+1)} := k^{-1}(\mathbf{H}_h^{(j+1)} - \mathbf{H}_h^{(j)})\,.$$

The parameter θ is arbitrarily chosen to be in $[0, 1]$. The method is explicit when $\theta = 0$ and fully implicit when $\theta = 1$.

The algorithm for the numerical approximation of the MLLG system is summerised in Algorithm 1. By the Lax–Milgram theorem, for each $j > 0$ there exists a unique solution $(\boldsymbol{v}_h^{(j)}, \mathbf{H}_h^{j+1}) \in \mathbb{W}_h^{(j)} \times \mathbb{Y}_h$ of equations (12)–(13). Since

$$\left|\mathbf{m}_h^{(0)}(x_n)\right| = 1 \quad \text{and} \quad \boldsymbol{v}_h^{(j)}(x_n) \cdot \mathbf{m}_h^{(j)}(x_n) = 0 \quad \text{for all } n = 1, \ldots, N\,,$$

and $j = 0, \ldots, J$, by induction,

$$\left|\mathbf{m}_h^{(j)}(x_n) + k\boldsymbol{v}_h^{(j)}(x_n)\right| \geqslant 1 \quad \text{and} \quad \left|\mathbf{m}_h^{(j)}(x_n)\right| = 1\,.$$

Therefore, the algorithm is well defined.

We now comment on the construction of basis functions for $\mathbb{W}_h^{(j)}$ and $\mathbb{Y}_h$ which are necessary in solving (12)–(13) in Step 5 of Algorithm 1.

Algorithm 1: Numerical approximation of the MLLG system

begin
  Set $j = 0$;
  Choose $\mathbf{m}_h^0 = I_{\mathbb{V}_h}\mathbf{m}_0$ and $\mathbf{H}_h^0 = I_{\mathbb{Y}_h}\mathbf{H}_0$;
  for $j = 0, 1, 2, \ldots, J-1$ **do**
    Solve (12) and (13) to obtain $(\boldsymbol{v}_h^{(j)}, \mathbf{H}_h^{(j+1)}) \in \mathbb{W}_h^{(j)} \times \mathbb{Y}_h$;
    Define

$$\mathbf{m}_h^{(j+1)}(\boldsymbol{x}) := \sum_{n=1}^{N} \frac{\mathbf{m}_h^{(j)}(\boldsymbol{x}_n) + k\boldsymbol{v}_h^{(j)}(\boldsymbol{x}_n)}{\left|\mathbf{m}_h^{(j)}(\boldsymbol{x}_n) + k\boldsymbol{v}_h^{(j)}(\boldsymbol{x}_n)\right|}\phi_n(\boldsymbol{x});$$

  end
end

3.1 A basis for $\mathbb{W}_h^{(j)}$

From (10), the basis functions of $\mathbb{W}_h^{(j)}$ at each iteration depend on the solution $\mathbf{m}_h^{(j)}$ computed in the previous iteration. Therefore they must be computed again for each iteration of j in Algorithm 1. For each $\boldsymbol{w} \in \mathbb{W}_h^{(j)}$, let

$$\boldsymbol{\alpha}_n = \boldsymbol{w}(\boldsymbol{x}_n) \quad n = 1, \cdots, N,$$

and let $\boldsymbol{\alpha}_n^{(1)}$ and $\boldsymbol{\alpha}_n^{(2)}$ be two basis vectors of the plane tangential to the vector $\mathbf{m}_h^{(j)}(\boldsymbol{x}_n) = (u_1, u_2, u_3)^T \in \mathbb{R}^3$. It follows from (10) that

$$\boldsymbol{\alpha}_n = \beta_n \boldsymbol{\alpha}_n^{(1)} + \gamma_n \boldsymbol{\alpha}_n^{(2)},$$

for some real numbers β_n and γ_n. In our computation we take

$$\boldsymbol{\alpha}_n^{(1)} = \mathbf{A}\mathbf{m}_h^{(j)}(\boldsymbol{x}_n) \quad \text{and} \quad \boldsymbol{\alpha}_n^{(2)} = \mathbf{m}_h^{(j)}(\boldsymbol{x}_n) \times \boldsymbol{\alpha}_n^{(1)},$$

where

$$\mathbf{A} = \begin{pmatrix} u_2u_3 & 1 & 1 \\ -1 & u_2u_3 & 1 \\ -1 & -2 & u_2u_3 \end{pmatrix}.$$

A basis for $\mathbb{W}_h^{(j)}$ is defined from bases of the 2D planes which are perpendicular to $\mathbf{m}_h^{(j)}(\mathbf{x}_n)$ for all $n = 1, \ldots, N$. Therefore $\dim(\mathbb{W}_h^{(j)}) = 2N$ and $\boldsymbol{w}$ is expressed in terms of $\boldsymbol{\alpha}_n^{(1)}$ and $\boldsymbol{\alpha}_n^{(2)}$ by

$$\boldsymbol{w}(\mathbf{x}) = \sum_{n=1}^{N} \left(\beta_n \boldsymbol{\alpha}_n^{(1)} + \gamma_n \boldsymbol{\alpha}_n^{(2)}\right) \phi_n(\mathbf{x}). \tag{14}$$

It can be shown that $\left\{(\boldsymbol{\alpha}_n^{(1)}\phi_n, \boldsymbol{\alpha}_n^{(2)}\phi_n)\right\}_{1\leqslant n\leqslant N}$ is a basis for the vector space $\mathbb{W}_h^{(j)}$.

3.2 A basis for $\mathbb{Y}_h$

A basis $\{\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_M\}$ of $\mathbb{Y}_h$ is defined as follows [10, Section 5.5.1]. Consider an edge $\boldsymbol{e}_q$, $q = 1, \ldots, M$, and let K be the tetrahedron having $\boldsymbol{e}_q$ as an edge. Let $\lambda_q^{(1)}$ and $\lambda_q^{(2)}$ be the barycentric coordinate functions corresponding to the endpoints of $\boldsymbol{e}_q$. We define

$$\boldsymbol{\psi}_q|_K := \lambda_q^{(1)}\nabla\lambda_q^{(2)} - \lambda_q^{(2)}\nabla\lambda_q^{(1)}.$$

The relation

$$\nabla \times \left(\boldsymbol{\psi}_q|_K\right) = 2\nabla\lambda_q^{(1)} \times \nabla\lambda_q^{(2)},$$

is useful in the computation of $\nabla \times \left(\boldsymbol{\psi}_q|_K\right)$.

4 Numerical experiments

In order to carry out physically relevant experiments, the initial conditions of the MLLG equations should be chosen to satisfy the divergence-free constraint [7]

$$\operatorname{div}(\mathbf{H}_0 + \chi_D \mathbf{m}_0) = 0 \quad \text{in} \quad \tilde{D},$$

where χ_D is the characteristic function of D. This is achieved by taking

$$\mathbf{H}_0 = \mathbf{H}_0^* - \chi_D \mathbf{m}_0\,, \tag{15}$$

where $\mathbf{H}_0^*$ is some function defined on $\tilde{D}$ satisfying $\operatorname{div} \mathbf{H}_0^* = 0$. In our experiment, for simplicity, we choose $\mathbf{H}_0^*$ to be a constant.

We solve the standard problem #4 proposed by the Micromagnetic Modeling Activity Group at the National Institute of Standards and Technology [5]. In this model, the initial conditions $\mathbf{m}_0$ and $\mathbf{H}_0$, and the effective field $\mathbf{H}_{\text{eff}}$ are

$$\mathbf{m}_0 = (1,0,0) \text{ in } D\,, \quad \mathbf{H}_0^* = (0.01, 0.01, 0.01) \text{ in } \tilde{D}\,, \quad \mathbf{H}_{\text{eff}} = \frac{2A}{\mu_0^* M_s}\Delta\mathbf{m} + \mathbf{H}\,.$$

The parameters

$$\lambda_1 = -\frac{\gamma}{1+\alpha^2} \quad \text{and} \quad \lambda_2 = \frac{\gamma\alpha}{1+\alpha^2}\,,$$

where the positive physical constants are the damping parameter α, the gyromagnetic ratio γ, the vacuum permeability μ_0^*, the exchange constant A, and the magnitude of magnetisation M_s. The values of the physical constants are

$$\begin{aligned}
&\alpha = 1\,, \quad \sigma = 10^{-13}\,\mathrm{S^{-1}m\,s^{-1}}\,, \quad \mu_0 = 2.211739\times 10^{9}\,\mathrm{H\,m^{-1}s^{-1}}\,,\\
&A = 1.3\times 10^{-11}\,\mathrm{J\,m^{-1}}\,, \quad \mu_0^* = 1.25667\times 10^{-6}\,\mathrm{H\,m^{-1}}\,,\\
&\gamma = 2.2\times 10^{9}\,\mathrm{m\,A^{-1}s^{-1}}\,, \quad M_s = 8\times 10^{5}\,\mathrm{A\,m^{-1}}\,.
\end{aligned}$$

The domains D and $\tilde{D}$ are chosen to be

$$D = (0, 0.5)\times(0, 0.125)\times(0, 0.003)\,,$$

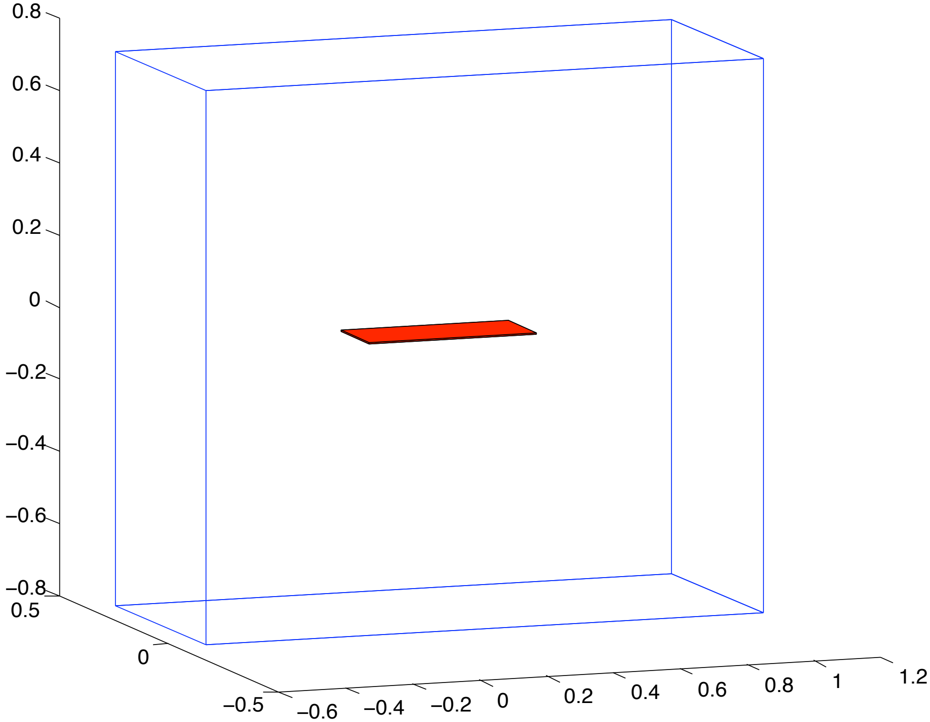

Figure 2: The magnetisation domain D (in red) is a thin film. The magnetic domain $\tilde{D}$ is in blue.

and

$$\tilde{D} = (-0.583, 1.083) \times (-0.146, 0.271) \times (-0.767, 0.770),$$

with distances measured in μm; see Figure 2. The domain D is uniformly partitioned into cubes of dimensions $0.042 \times 0.010 \times 0.003\,\mu\text{m}^3$, where each cube consists of six tetrahedra. We generate a nonuniform mesh for the magnetic domain $\tilde{D}$ in such a way that it is identical to the mesh for D in the region near D, and the mesh size gradually increases away from D. A cross section of the mesh at $x_3 = 0$ is displayed in Figure 3.

At each iteration of Algorithm 1, the system to be solved is linear and of

Table 1: A comparision between Baňas, Bartels and Prohl's (BBP) method and our method.

	BBP	Our method
Discrete system	nonlinear (uses fixed-point iteration)	linear (solved directly)
Degrees of freedom	$3N + M$	$2N + M$
Required condition	$k = O(h^2)$	None
Basis functions of solution space	same for all iterations	different in each iteration

size $2N + M$. A comparision of our method and the method proposed by Baňas, Bartels and Prohl [2] is presented in Table 1.

For $j = 0, 1, 2, \dots$, let $\mathcal{E}_T^{(j)}$ be the total energy at time $t_j = jk$ defined by

$$\begin{aligned}\mathcal{E}_T^{(j)} &:= \frac{2\sigma A}{M_s}\int_D |\nabla \mathbf{m}_h^{(j)}|^2\,dx + 2\sigma\mu_0\int_{\tilde{D}} |\mathbf{H}_h^{(j)}|^2\,dx + \frac{\lambda_2}{\mu}\int_{\tilde{D}} |\nabla\times\mathbf{H}_h^{(j)}|^2\,dx \\ &:= \mathcal{E}_{ex}^{(j)} + \mathcal{E}_H^{(j)} + \mathcal{E}_E^{(j)}\,,\end{aligned}$$

where $\mathcal{E}_{ex}^{(j)}$, $\mathcal{E}_H^{(j)}$ and $\mathcal{E}_E^{(j)}$ are the exchange energy, magnetic field energy and electric field energy, respectively. Our computation shows that the total energy decreases, that is

$$\mathcal{E}_T^{(j+1)} \leqslant \mathcal{E}_T^{(j)} \quad \text{for all } j \geqslant 0\,; \tag{16}$$

see Figure 4. The decrease of the discrete energy $\mathcal{E}_T^{(j)}$, $j = 1, \dots, J$, suggests the gradient stability of the MLLG solutions. In Figure 4 we plot different scaled versions of the different energies versus $\log t$. The change in $\mathcal{E}_T^{(j)}$ is dominated by the change in $\mathcal{E}_H^{(j)}$.

Figure 4 also shows that the sequence $\{\nabla \mathbf{m}_h^{(j)}\}_{j\geqslant 0}$ is bounded in $\mathbb{L}^2(D)$, and the sequences $\{\mathbf{H}_h^{(j)}\}_{j\geqslant 0}$ and $\{\nabla\times\mathbf{H}_h^{(j)}\}_{j\geqslant 0}$ are bounded in $\mathbb{L}^2(\tilde{D})$. In a forthcoming paper, we will show that our numerical solution converges to

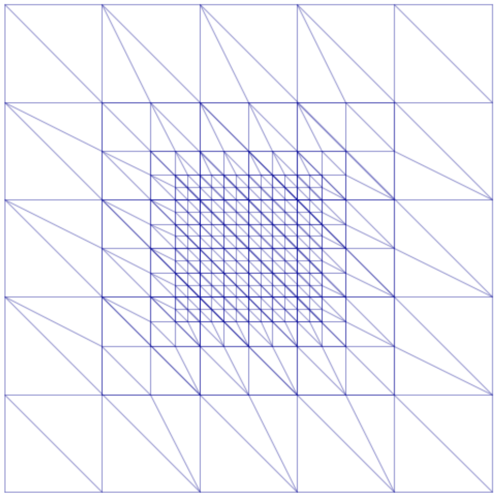

Figure 3: Mesh for the domain $\tilde{D}$ at $x_3 = 0$.

a weak solution of the problem (1b)–(4) by proving that the inequality (16) holds.

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References

[1] F. Alouges. A new finite element scheme for Landau–Lifchitz equations. *Discrete Contin. Dyn. Syst. Ser. S*, **1**:187–196 (2008). doi:10.3934/dcdss.2008.1.187 C683

[2] L. Baňas, S. Bartels, and A. Prohl. A convergent implicit finite element discretization of the Maxwell–Landau–Lifshitz–Gilbert equation. *SIAM*

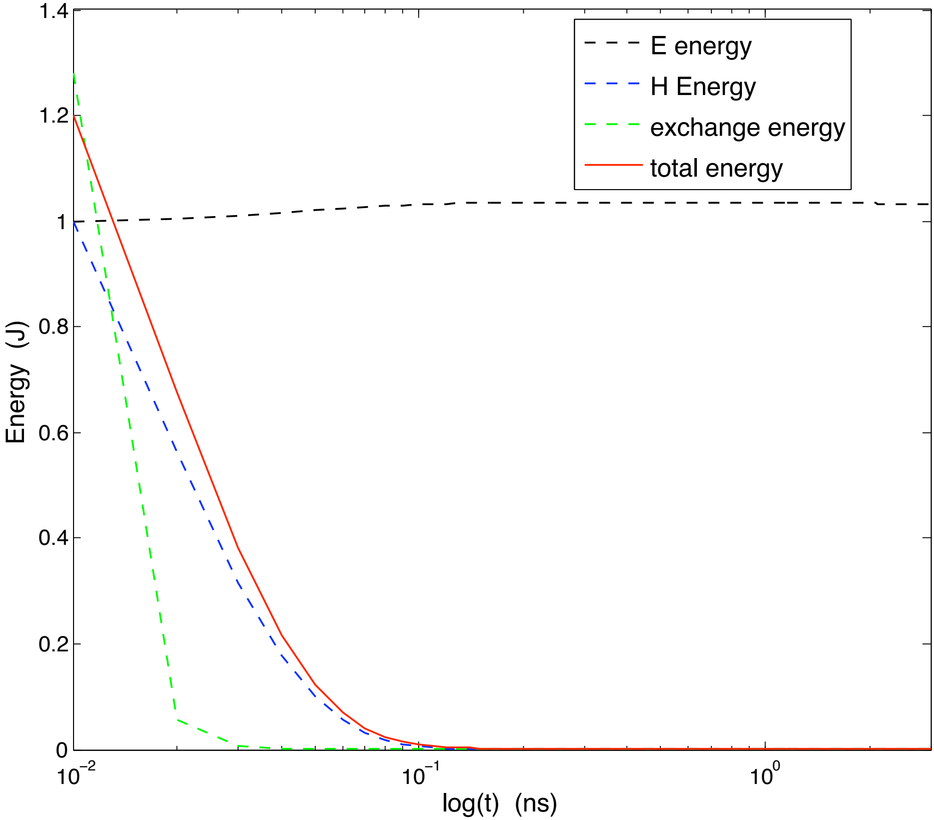

Figure 4: Evolution of the energies, $\log(t) \mapsto \mathcal{E}_E(t)/600$, $\mathcal{E}_H(t)/10^6$, $10^4\mathcal{E}_{ex}(t)$, $\mathcal{E}_T(t)/10^6$.

J. Numer. Anal., **46**:1399–1422 (2008). doi:10.1137/070683064 C683, C694

[3] I. Cimrák. Error analysis of a numerical scheme for 3D Maxwell–Landau–Lifshitz system. *Math. Methods Appl. Sci.*, **30**:1667–1683 (2008). doi:10.1002/mma.863 C683

[4] I. Cimrák. Existence, regularity and local uniqueness of the solutions to the Maxwell–Landau–Lifshitz system in three dimensions. *J. Math. Anal. Appl.*, **329**:1080–1093 (2007). doi:10.1016/j.jmaa.2006.06.080 C683

[5] Micromagnetic Modeling Activity Group, Center for Theoretical and Computational Materials Science, National Institute of Standards and Technology, USA. `http://www.ctcms.nist.gov/~rdm/mumag.org.html` C692

[6] T. L. Gilbert. A Lagrangian formulation of the gyromagnetic equation of the magnetic field. *Phys Rev*, **100**:1243–1255 (1955). Abstract only; full report, Armor Research Foundation Project No. A059, Supplementary Report, May 1, 1956 (unpublished). C685

[7] B. Guo and S. Ding. *Landau–Lifshitz Equations*, volume 1 of *Frontiers of Research with the Chinese Academy of Sciences*. World Scientific, Hackensack NJ (2008). doi:10.1142/9789812778765 C692

[8] L. Landau and E. Lifschitz. On the theory of the dispersion of magnetic permeability in ferromagnetic bodies. *Phys Z Sowjetunion*, **8**:153–168 (1935); *Perspectives in Theoretical Physics*, pp. 51–65, Pergamon, Amsterdam (1992). doi:10.1016/B978-0-08-036364-6.50008-9 C683

[9] K.-N. Le and T. Tran. A convergent finite element approximation for the quasi-static Maxwell–Landau–Lifshitz–Gilbert equations. Research Report, arXiv:1212.3369, 2012, submitted to *Computers & Mathematics with Applications*. `http://arxiv.org/abs/1212.3369` C684

[10] P. Monk. *Finite Element Methods for Maxwell's equations*. Numerical Mathematics and Scientific Computation. Oxford University Press, New York (2003). C687, C691

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