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An adaptive two-level method for hypersingular integral equations in R^3

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Abstract

In this paper an *a posteriori* error estimate for hypersingular integral equations is derived by using hierarchical basis techniques. Based on the properties of a two-level additive Schwarz method easily computable local error indicators are obtained. An algorithm for adaptive error control which allows anisotropic refinements of the boundary elements is formulated and numerical results are included.

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1 A stable two-level subspace decomposition

In recent years adaptive hierarchical basis methods in the finite element method (FEM) [1, 2, 4] have become increasingly popular. This approach has meanwhile been also applied to the boundary element method (BEM) for weakly singular integral equations [8] and to the FEM/BEM coupling [7]. Here we extend it to hypersingular integral equations on surfaces.

We consider the hypersingular integral equation

$$Wv(x) := -\frac{1}{4\pi}\frac{\partial}{\partial n_x}\int_\Gamma v(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}\,d\sigma_y = f(x)\,, \quad x \in \Gamma \tag{1}$$

where Γ is an open plane surface and integration has to be understood in the Hadamard sense. W is a bijective mapping from $\tilde{H}^{1/2}(\Gamma)$ onto $H^{-1/2}(\Gamma)$ and the bilinear form $\langle Wu, v\rangle$ for $u, v \in \tilde{H}^{1/2}(\Gamma)$ is symmetric and positive definite (cf. Costabel [3], Stephan [9]). Hence the unique solution $v_N \in S_N \subset \tilde{H}^{1/2}(\Gamma)$ of the Galerkin scheme

$$\langle Wv_N, \chi\rangle = \langle f, \chi\rangle \quad \forall \chi \in S_N \tag{2}$$

converges quasi-optimally towards the exact solution $v \in \tilde{H}^{1/2}(\Gamma)$ of (1). For standard Sobolev spaces $L^2(\Gamma)$ and $H_0^1(\Gamma)$ (which is the completion of C_0^∞ within $H^1(\Gamma)$) the space $\tilde{H}^{1/2}(\Gamma)$ is an interpolation space between $L^2(\Gamma)$ and $H_0^1(\Gamma)$, and $H^{-1/2}(\Gamma)$ is the dual space of $\tilde{H}^{1/2}(\Gamma)$. The solution u of (1) for given $f \in H^{-1/2}(\Gamma)$ is the jump across Γ of the solution of a Neumann problem for the Laplacian in $\mathrm{R}^3\backslash\bar{\Gamma}$, cf. [9].

In the following we present a two-level method for the h-version of the Galerkin scheme with bilinear continuous elements. For ease of presentation we consider $\Gamma = [-1, 1]^2$.

Let γ_l $(0 \le l \le L)$ be a uniform partition of Γ into squares of side length $h_l = 2^{-l}$ and let

$$\begin{aligned} X_l &= \{u \in \mathcal{C}^0(\Gamma) : u \text{ piecewise bilinear w.r.t. } \gamma_l \text{ and } u_{|\partial\Gamma} = 0\} \ (0 \le l \le L) \\ X_l^0 &= \{u \in X_l : u = 0 \text{ on the nodes of } \gamma_{l-1}\} \quad (1 \le l \le L). \end{aligned}$$

Let $b_{l,i}$ $(0 \le i \le n_l)$ be a piecewise bilinear function with value one at an interior node of γ_l (not belonging to γ_{l-1}) and with zero value in all other nodes of γ_l. Let $X_{L,i}^0 = \operatorname{span}\{b_{L,i}\}$. We have the following two-level decomposition of the space X_L:

$$X_L = X_{L-1} \oplus X^0_{L,1} \oplus \cdots \oplus X^0_{L,n_L} \tag{3}$$

Let

$$P^L_{(2)} = P_{L-1} + \sum_{i=1}^{n_L} P_{L,i} \tag{4}$$

be the two-level additive Schwarz operator belonging to the subspace decomposition (3) and the bilinear form $\langle W\cdot,\cdot\rangle$, i.e.

$$\begin{aligned} \langle WP_{L-1}\phi,\psi\rangle &= \langle W\phi,\psi\rangle \quad \forall\psi\in X_{L-1}, \phi\in X_L, \\ \langle WP_{L,i}\phi,\psi\rangle &= \langle W\phi,\psi\rangle \quad \forall\psi\in X^0_{L,i}, \phi\in X_L. \end{aligned}$$

Then there holds

Theorem 1 *There exist constants $c_1, c_2 > 0$, independent of L, such that*

$$c_1\langle Wu,u\rangle \le \langle WP^L_{(2)}u,u\rangle \le c_2\langle Wu,u\rangle \quad \forall u\in X_L\,. \tag{5}$$

The proof of Theorem 1 is based on the following lemmas where always $u \in X_L$ arbitrary with

$$u = u_{L-1} + \sum_{i=1}^{n_L} u_{L,i}$$

where $u_{L-1} \in X_{L-1}$ and $u_{L,i} \in X^0_{L,i}$. Let $I_{L-1}u \in X_{L-1}$ be the bilinear interpolant of u at the nodes of γ_{L-1}.

Lemma 2 *[6, Lemma 2.5, Lemma 3.10] There exists a constant c, independent of L and u, such that*

$$\|u - I_{L-1}u\|_{L^2(\Gamma)} \leq ch_L\|u\|_{H^1(\Gamma)}. \tag{6}$$

Furthermore for any $s \in [0,1]$ there exists a constant $c = c(s) > 0$ such that

$$\|I_{L-1}u\|_{\tilde{H}^s(\Gamma)} \leq c\|u\|_{\tilde{H}^s(\Gamma)} \quad \textit{for any } u \in X_L. \tag{7}$$

Lemma 3 *[6, Lemma 3.12] Let $\Gamma_i^L \in \gamma_L$ be a square with vertices x_k $(1 \leq k \leq 4)$. Let v, w be bilinear functions on Γ_i^L with $v(x_1) = w(x_1)$ and $w(x_2) = w(x_3) = w(x_4) = 0$. Then there holds*

$$\|w\|_{L^2(\Gamma_i^L)} \leq \frac{4}{3}\|v\|_{L^2(\Gamma_i^L)} \quad .$$

Lemma 4 *[5] Let $\{\Gamma_i, i = 1, \dots, N\}$ be a finite covering of Γ with rectangles Γ_i and covering constant $\sigma \in \mathrm{N}$, i.e. we can colour $\{\Gamma_i, i = 1, \dots, N\}$ by at most σ different colours such that subdomains with same colour are disjoint. Let $\phi = \sum_{i=1}^N \phi_i \in \tilde{H}^s(\Gamma)$ for $s \in \mathrm{R}$ with $\phi_i \in \tilde{H}^s(\Gamma_i)$. Then there holds*

$$\|\phi\|^2_{\tilde{H}^s(\Gamma)} \leq \sigma \sum_{i=1}^N \|\phi_i\|^2_{\tilde{H}^s(\Gamma_i)} \quad .$$

Proof: (of Theorem 1) We show that there exist constants $c_1, c_2 > 0$ such that

$$\frac{1}{c_2}\|u\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq \|u_{L-1}\|^2_{\tilde{H}^{1/2}(\Gamma)} + \sum_{i=1}^{n_L} \|u_{L,i}\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq \frac{1}{c_1}\|u\|^2_{\tilde{H}^{1/2}(\Gamma)}. \tag{8}$$

The left inequality follows directly from Lemma 4 with $c_2 = 9$ since to any $x \in \Gamma$ there belong at most 8 different $b_{L,i}$'s with $x \in \mathrm{supp} b_{L,i}$ $(1 \leq i \leq n_L)$ and since

$$\|u_{L,i}\|_{\tilde{H}^{1/2}(\Gamma)} = \|u_{L,i}\|_{\tilde{H}^{1/2}(\mathrm{supp} b_{L,i})} \quad .$$

It remains to show the right inequality in (8). Since $u_{L-1} = I_{L-1}u$ Lemma 2 implies

$$\|u_{L-1}\|_{\tilde{H}^{1/2}(\Gamma)} \leq c\|u\|_{\tilde{H}^{1/2}(\Gamma)} \quad . \tag{9}$$

Let $\{x_i\}_{i=1}^{n_L}$ denote the set of nodes in γ_L which do not belong to γ_{L-1}. We decompose the index set $\{1, 2, \ldots, n_L\} = M_1 \cup M_2 \cup M_3$ into 3 disjoint sets M_k such that two indices $i, j \in \{1, 2, \ldots, n_L\}$ belong to the same set M_k if $|x_i - x_j|$ is an integer multiple of $2h_L$. The sets M_k are uniquely determined (up to permutation) (cf. Fig. 1). For $k \in \{1, 2, 3\}$ the nodes $\{x_i\}_{i \in M_k}$ are just the nodes of the coarse grid γ_{L-1} shifted by h_L in the x_1- and/or x_2-direction. There holds for $k \in \{1, 2, 3\}$

$$\mathrm{meas}(\mathrm{supp}\, b_{L,i} \cap \mathrm{supp}\, b_{L,j}) = 0 \quad \forall i, j \in M_k\,,\ i \neq j$$

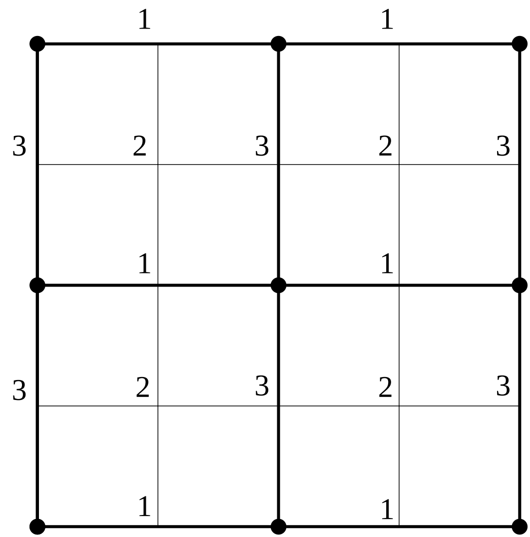

Figure 1: Decomposition of the nodes γ_L into disjoint subsets: Number $k \in \{1, 2, 3\}$ of the node x means $x \in M_k$. The nodes of the coarse mesh γ_{L-1} are marked with $\bullet$.

with the Lebesgue-measure meas$(\cdot)$. Hence

$$\sum_{i\in M_k} \|u_{L,i}\|_0^2 = \Big\|\sum_{i\in M_k} u_{L,i}\Big\|_0^2 \le \frac{16}{9}\|\tilde{u}\|_0^2 \tag{10}$$

where $\tilde{u} = u - u_{L-1}$ and $\|\cdot\|_s = \|\cdot\|_{\tilde{H}^s(\Gamma)}$, $s \in \mathbb{R}$.
Here the last inequality follows from Lemma 3 since $w = \sum_{i\in M_k} u_{L,i}$ and $\tilde{u}$ coincide at one node of each element in γ_L and w vanishes in all the other nodes. With the standard inverse inequality for finite elements and (10) and (6) we have

$$\begin{aligned}
\sum_{i=1}^{n_L} \|u_{L,i}\|_{1/2}^2 \le \quad & ch_L^{-1} \textstyle\sum_{i=1}^{n_L} \|u_{L,i}\|_0^2 && = ch_L^{-1} \sum_{k=1}^{3}\sum_{i\in M_k} \|u_{L,i}\|_0^2 \\
\le \quad & c' h_L^{-1} \|\tilde{u}\|_0^2 && \le c' h_L^{-1} \|u - I_{L-1}u\|_0^2 \\
\le \quad & ch_L \|u\|_1^2 && \le c\, \|u\|_{1/2}^2 \quad .
\end{aligned}$$

Together with (9) this implies the right inequality in (8). Thus the proof is complete due to the equivalence of the norms $\|u\|_{1/2}$ and $\langle Wu, u\rangle^{1/2}$. ♠

2 An a posteriori error estimate

Next we need the saturation assumption: (A_h) There exist constants $k_0 \in \mathrm{N}$ and $0 < \rho < 1$ such that

$$\|v - v_{k+1}\|_{1/2} \le \rho \|v - v_k\|_{1/2} \quad \forall k \ge k_0$$

where v_{k+1} denotes the Galerkin solution on the level $k+1$ and v the exact solution of (1).

Theorem 5 *Suppose (A_h) holds. Then there exist constants $c_1, c_2 > 0$ such that there holds for $k \geq k_0$*

$$c_1 \sum_{j=1}^{n_L} \eta_{L,j}^2 \leq \|v - v_{L-1}\|_{1/2}^2 = c_2 \sum_{j=1}^{n_L} \eta_{L,j}^2 \tag{11}$$

with

$$\eta_{L,j} = \frac{|\langle f - W v_{L-1}, b_{L,j} \rangle|}{\langle W b_{L-1}, b_{L,j} \rangle^{1/2}} \quad (j = 1, 2, \ldots, n_L)\,. \tag{12}$$

Proof: The saturation assumption (A_h) yields the equivalence of norms

$$\|v_L - v_{L-1}\|_{1/2} \sim \|v - v_{L-1}\|_{1/2}\,.$$

Due to Theorem 1 we have

$$c_1 \|v_L - v_{L-1}\|_{1/2}^2 \leq \|P_{L-1}(v_L - v_{L-1})\|_{1/2}^2 + \sum_{i=1}^{n_L} \|P_{L,i}(v_L - v_{L-1})\|_{1/2}^2$$
$$\leq c_2 \|v_L - v_{L-1}\|_{1/2}^2\,.$$

Firstly, we observe that since v_{L-1} and v_L satisfy the Galerkin equation there holds for any $w \in X_{L-1}$

$$\langle W P_{L-1} v_L, w \rangle = \langle W v_L, w \rangle = \langle f, w \rangle = \langle W v_{L-1}, w \rangle = \langle W P_{L-1} v_{L-1}, w \rangle\,.$$

Hence

$$\|P_{L-1}(v_L - v_{L-1})\|^2_{1/2} = 0.$$

The error indicator $\eta_{L,j}$ in (12) is obtained by solving a linear problem in the space $X^0_{L,j}$. The function $v_{L,j} = P_{L,j}(v_L - v_{L-1}) \in X^0_{L,j}$ solves for any $v \in X^0_{L,j}$

$$\langle W v_{L,j}, v\rangle = \langle f - W v_{L-1}, v\rangle \tag{13}$$

Hence firstly one solves (13) for $1 \leq j \leq n_L$ and then one computes the terms $\eta_{L,j} = \langle W v_{L,j}, v_{L,j}\rangle^{1/2}$. Since $X^0_{L,j} = \operatorname{span}\{b_{L,j}\}$ is a one-dimensional space, we have $v_{L,j} = c b_{L,j}$ with coefficient

$$c = \frac{\langle f - W v_{L-1}, b_{L,j}\rangle}{\langle W b_{L,j}, b_{L,j}\rangle}\,.$$

Hence

$$\eta_{L,j} = |c| \langle W b_{L,j}, b_{L,j}\rangle^{1/2}\,.$$

♠

3 Numerical results

Algorithm 3.1 (Adaptive multilevel algorithm) *Let γ_0 denote an initial mesh on Γ and X_0 the corresponding space of continuous bilinear functions. Furthermore let $0 \leq \theta \leq 1$ and $\delta > 0$ be given.*

1. *Compute the Galerkin solution* $u_k \in X_k$ *of* (2).

2. *Compute the error indicators* $\eta_{k,j}$, $j = 1, \ldots, n_k$ *with* (12).

3. *Compute the row error indicators* $\eta_{R,m}$ *and the column error indicators* $\eta_{C,n}$ *as weighted sums of the single error indicators of one row and one column of the mesh, respectively. The weighting is done by dividing the quadratic mean of the local error indicators of a row or a column by the respective number of terms.*

4. *Compute* $\eta_{max} := \max\{\eta_{R,m}, \eta_{C,n}\}$. *Refine all elements of the* m^{th} *row in* x_1*-direction if*

$$\eta_{R,m} \geq \theta \eta_{max} \quad ,$$

refine all elements of the n^{th} *column in* x_2*-direction if*

$$\eta_{C,n} \geq \theta \eta_{max} \quad .$$

Here refining of the element with index i *in* x_k*-direction means halving the element in* x_k*-direction.*

5. *Thus obtain the space* X_{k+1}. *Check whether* $\eta_{max} < \delta$ *is satisfied. Otherwise go to 1.*

Remark 6 *Due to the row and column error indicators the above refinement strategy secures the continuity of the trial functions. No hanging nodes are obtained.*

Table 1: Adaptive h-refinement with $\theta = 0.6$.

L	N_L	E_L	η_L	η_L/E_L	$\kappa(B_hW_h)$	$\kappa(W_h)$
1	5	0.559700	0.335091	0.598697	$1.24 \cdot 10^0$	$1.24 \cdot 10^0$
2	16	0.466277	0.313657	0.672684	$4.86 \cdot 10^0$	$2.36 \cdot 10^0$
3	48	0.311299	0.201438	0.647088	$1.39 \cdot 10^1$	$3.74 \cdot 10^0$
4	96	0.214100	0.130477	0.609423	$1.65 \cdot 10^1$	$6.99 \cdot 10^0$
5	160	0.149149	0.085045	0.570201	$1.77 \cdot 10^1$	$1.37 \cdot 10^1$
6	240	0.105005	0.057504	0.547632	$1.80 \cdot 10^1$	$2.86 \cdot 10^1$
7	336	0.074666	0.041280	0.552871	$1.83 \cdot 10^1$	$6.13 \cdot 10^1$
8	448	0.053782	0.032152	0.597816	$1.84 \cdot 10^1$	$1.34 \cdot 10^2$

Next we choose the model problem Γ to be the L-shaped surface piece in the (x_1, x_2)-plane with vertices $(0,0,0)$, $(1,0,0)$, $(1,-1,0)$, $(-1,-1,0)$, $(-1,1,0)$, $(0,1,0)$, and the right hand side $f = 1$ in (1). In Table 1 the results are presented for the adaptive refinement strategy. N_L denotes the number of unknowns on level L; E_L the relative error in the energy norm; $\kappa(B_hW_h)$ and $\kappa(W_h)$ the condition numbers of the preconditioned and unpreconditioned Galerkin matrix of (2), respectively; B_h the preconditioner according to the additive Schwarz operator in (4); η_L the sum of the error indicators and η_L/E_L the efficiency index of the algorithm. The sequence of corresponding mesh refinements is given in Figure 2.

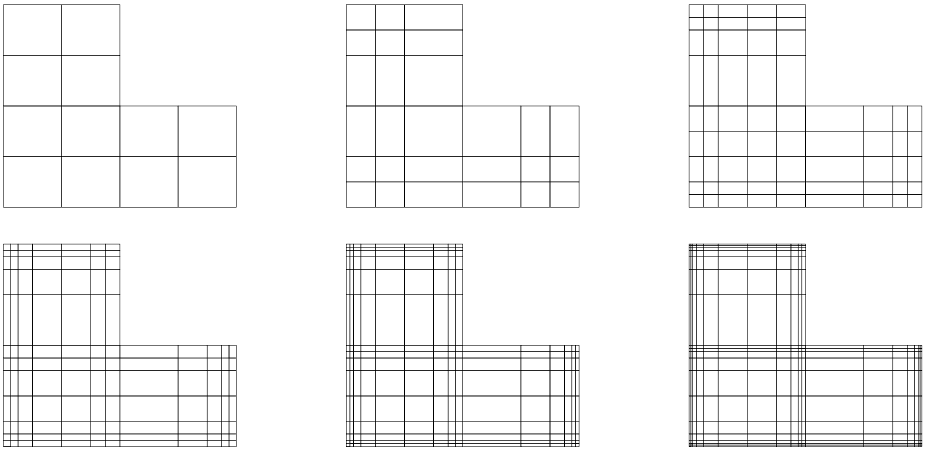

Figure 2: The sequence of refined meshes for $\theta = 0.6$.

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