Slip at the surface of an oscillating spheroidal particle in a micropolar fluid

H. H. Sherief[1] M. S. Faltas[2] E. I. Saad[3]

(Received 1 March 2013; revised 15 September 2013)

Abstract

The axisymmetric rectilinear and rotary oscillations of a spheroidal particle in an incompressible micropolar fluid are considered. Basset type linear slip boundary conditions on the surface of the solid spheroidal particle are used for velocity and microrotation. Under the assumption of small amplitude oscillations, analytical expressions for the fluid velocity field and microrotation components are obtained in terms of a first order small parameter characterizing the deformation. For the rectilinear oscillations, the drag acting on the particle is evaluated and expressed in terms of two real parameters for the prolate and oblate spheroids. Also, the couple exerted on the spheroid is evaluated for the prolate and oblate spheroids for the rotary oscillations. Their variations with respect to the frequency, deformity, micropolarity and slip parameters are tabulated and displayed graphically. Well-known results are deduced and comparisons are made between the classical

http://journal.austms.org.au/ojs/index.php/ANZIAMJ/article/view/6813 gives this article, Published September 26, 2013. ISSN 1446-8735.

viscous fluids and micropolar fluids. The results of this study serve to improve the accuracy of viscosity measurements for micropolar fluids.
Subject class: 76A05; 76B99
Keywords: rectilinear oscillations, rotary oscillations, prolate spheroids, oblate spheroids, micropolar fluid, slip condition

Contents

1 Introduction

The classical theory of continuum mechanics has proved to be inadequate to describe the behavior of complex fluids such as emulsions, solutions of polymers, polymer melts, liquid crystals, animal blood and fluid suspensions. There has been increasing interest in developing theories to model such complex fluids. One of the simplest fluid models is the micropolar fluid theory introduced by Eringen [1]. In micropolar fluid theory, the laws of classical continuum mechanics are augmented with additional equations that account for conservation of microinertia moments and balance of the first stress moments, that arise due to consideration of the microstructure in a material. Thus, new kinematic variables, for example the gyration tensor and microinertia moment tensors, and the concepts of body moments, stress moments, and microstress are combined with classical continuum mechanics. The books written by Łukaszewicz [2] and Eringen [3] provide a useful account of the theory and extensive surveys of the literature of the micropolar fluids theory.

In fluid mechanics, both no-slip and partial-slip boundary conditions were proposed in the nineteenth century when the proper boundary conditions were discussed in the first place [4]. Navier [5] gave the slip boundary condition where the tangential velocity of the fluid relative to the solid at a point on its surface is proportional to the tangential stress acting at that point. For gas flows, Maxwell [6] had shown that the surface slip is related to the non-continuous nature of the gas and the slip length is proportional to the mean-free path. For liquids, from experiments at that age, the no-slip boundary condition was accepted and since then has been treated as a fundamental law. However, from recent extensive studies on the surface slip in micro and nano scales, the physics of the fluid-solid slip is recognized to be much more complicated than that for gases. Apparent violations of the no-slip boundary condition at the fluid-solid interface on the nanoscale have been reported [4, 7, 8, 9]. The hydrodynamic slip boundary condition has also been studied in the context of nanofluidics [10, 11]. Basset [12] derived expressions for the force exerted by the surrounding fluid on a translating rigid sphere with a slip boundary condition at its surface (for example a settling aerosol sphere). The hydrodynamic effects of homogeneous and inhomogeneous slip boundary conditions for Newtonian fluids have been discussed extensively in the literature [13, 14, 15, 16]. Usually, slip exists to a degree between the fluid and the surface of the solid. Motivated by this understanding, we here analyze a micropolar flow problem using the slip boundary conditions for both the velocity and the microrotation. We propose a slip boundary condition for the microrotation by assuming that the tangential component of the microrotation vector of the fluid relative to the solid at a point on its surface is proportional to the corresponding tangential couple stress acting at that point. Slip boundary conditions for micropolar fluids have been used for the velocity but not for the microrotation by Sherief et al. [17] and Saad [18]. We think that it is physically more appropriate to use the slip boundary conditions for both the velocity and the microrotation because both conditions are applied at the same surface and the slip is mainly due to the nature of the surface and the fluid.

The movement of small particles in a continuous medium at low Reynolds numbers is of much fundamental and practical interest in the areas of chemical, biomedical, and environmental engineering and science. The majority of these moving phenomena are fundamental in nature, but permit one to develop rational understanding of many practical systems and industrial processes such as sedimentation, flotation, spray drying, agglomeration, and motion of blood cells in an artery or vein. Kanwal studied the translational oscillations of several axisymmetric bodies including a sphere, a prolate spheroid, and a thin cylindrical disk using the Stokes stream function [19] and eigenfunction expansion techniques [20]. On the other hand, the axisymmetric creeping flow of a viscous incompressible fluid past a spheroid which deforms slightly in shape from a sphere with the slip boundary condition has been investigated, and an explicit expression for the hydrodynamic drag force experienced by a slip spheroid was given by Palaniappan [21] and Ramkissoon [22] to the first order of a small parameter characterizing the deformation. Recently, the

Stokes translation and rotation of a rigid particle which departs but little in shape from a sphere with the slip boundary condition were also analyzed, and Senchenko, Chang and Keh [23, 24] obtained explicit expressions for the hydrodynamic drag force and couple acting on the slip spheroid to the second order in the small deformation parameter.

The value of the couple experienced by the various bodies of revolution, rotating steadily in a viscous and incompressible fluid has been evaluated. When inertial effects can be validly ignored, so that Stokes's linearized theory applies, the solutions have been found for some configurations, for example a sphere by Lamb [25], spheroids and a pair of spheres by Jeffery [26] and a spindle, a torus, a lens by Kanwal [27]. This value of the couple is needed in designing and calibrating viscometries and better predictions of the couple are essential in order to improve the accuracy of viscosity measurements. Numerical solutions for rotary oscillations of arbitrary axisymmetric bodies in an axisymmetric viscous flow has been investigated by Tekasakul et al. [28]. They evaluated numerically the local stresses and torques on a selection of free, oscillating, axisymmetric bodies in the continuum regime in an axisymmetric viscous incompressible flow. The accuracy of their technique is tested against known solutions for a sphere, a prolate spheroid, a thin disk and an infinitely long cylinder. Tekasakul and Loyalka [29] extended the work of Tekasakul et al. [28] into the slip regime. An accurate numerical result for local stress and torque on spheres and spheroids as function of the frequency parameter and the slip coefficients have been obtained. However, a realistic model is to consider these irregular shapes as approximate spheres. Happel and Brenner [30] found the couple experienced by a slightly deformed sphere in an incompressible viscous flow.

Lakshmana et al. [31] studied the slow steady rotation of a sphere about its diameter in a micropolar fluid. Lakshmana and Bhujanga [32] extended the work. They examined the rectilinear oscillations of a sphere along a diameter and the rotary oscillations of a sphere about its diameter in micropolar fluid. Lakshmana, Rao and Iyengar [33] utilized spheroidal coordinates to study the rectilinear oscillations of a spheroid in a micropolar fluid. Sran [34] obtained a general expression for the force exerted on a sphere performing longitudinal oscillations in an incompressible micropolar fluid. In some of the fluid mechanics applications such as sedimentation, particles of irregular shapes are encountered and it is difficult to evaluate the drag force or couple, therefore many authors modelled these irregular shapes as regular shapes and then evaluated the drag or couple with considerable ease, [35, 36, e.g.]. Hayakawa [37] discussed the slow viscous flow of micropolar fluid around a sphere and a cylinder, and a preliminary calculation of the steady flows inside a container. Hoffmann et al. [38] calculated the resistant force exerted on a sphere moving with a constant velocity in a micropolar fluid using a nonhomogeneous boundary condition for the microrotation vector. More recently, Sherief et al. [39] investigated the translational motion of an arbitrary body of revolution in a micropolar fluid by using a combined analytical-numerical method. They evaluated the drag force exerted on a prolate spheroid and a prolate Cassini oval particle.

The purpose of this work is to study the translational and rotational oscillatory motions of a spheroidal particle along and about its axis of revolution, respectively, in an infinite micropolar fluid medium which is at rest. This study is an extension to pervious work [35, 36] allowing for the slip boundary conditions for both velocity and microrotation. The amplitude of oscillations is assumed to be small so that the nonlinear terms in the equations of motion are neglected under the usual Stokesian assumption. The analytical expressions are deduced for flow fields, to the first order in a small parameter characterizing the deformation of the spheroidal surface from the spherical shape. The expressions for the hydrodynamic drag force and couple acting on the particle are derived in closed forms, for both prolate and oblate spheroids, and then expressed in terms of real parameters. The effects of the variation of frequency, deformity, micropolarity and slip parameters on the two parameters, as revealed by numerical studies, are shown through figures. Results for the drag force and couple are compared with earlier ones for some particular cases [24, 32].

2 Field equations

The equations of motion for an unsteady flow of an incompressible micropolar fluid under the Stokesian assumption in the absence of body force and body couples introduced by Eringen [1] are

$$\nabla \cdot \vec{q} = 0\,, \tag{1}$$
$$\rho\dot{\vec{q}} = -\nabla p + k\nabla \wedge \vec{\nu} - (\mu + k)\nabla \wedge \nabla \wedge \vec{q}\,, \tag{2}$$
$$\rho j\dot{\vec{\nu}} = -2k\vec{\nu} + k\nabla \wedge \vec{q} - \gamma\nabla \wedge \nabla \wedge \vec{\nu} + (\alpha + \beta + \gamma)\nabla\nabla \cdot \vec{\nu}\,, \tag{3}$$

in which $\vec{q}$ and $\vec{\nu}$ are the velocity and microrotation vectors, and p is the fluid pressure at any point. The symbols ρ and j are the density of the fluid and gyration parameters, respectively, and are assumed to be constants. The variable μ is the viscosity coefficient of the classical fluid, and $(k, \alpha, \beta, \gamma)$ are the new viscosity coefficients for micropolar fluids. A superposed dot indicates time material differentiation.

The equations for the stress tensor t_{ij} and the couple stress tensor m_{ij} are the constitutive equations

$$t_{ij} = -p\delta_{ij} + \mu(q_{i,j} + q_{j,i}) + k(q_{j,i} - \epsilon_{ijm}\nu_m), \tag{4}$$
$$m_{ij} = \alpha\nu_{m,m}\delta_{ij} + \beta\nu_{i,j} + \gamma\nu_{j,i}\,, \tag{5}$$

where the comma denotes partial differentiation, δ_{ij} and ϵ_{ijm} are the Kronecker delta and the alternating tensor, respectively.

3 Rectilinear oscillations of a slip spheroid in a micropolar fluid

Let (r, θ, ϕ) denote spherical polar coordinates with origin at a center of a sphere radius a. Consider an axisymmetric body that performs oscillations of velocity $U_z e^{i\sigma t}$ along the axis of symmetry $\theta = 0$ in an infinite expanse of an incompressible micropolar fluid which is otherwise at rest. Here σ is the frequency of oscillation. The generated flow is axially symmetric, and all the flow functions are independent of ϕ. We then choose the velocity and microrotation vectors as

$$\vec{q} = q_r(r,\theta)e^{i\sigma t}\vec{e}_r + q_\theta(r,\theta)e^{i\sigma t}\vec{e}_\theta\,, \tag{6}$$

$$\vec{\nu} = \nu_\phi(r,\theta)e^{i\sigma t}\vec{e}_\phi\,. \tag{7}$$

The hydrostatic pressure is also written in the form $p(r,\theta)e^{i\sigma t}$. Therefore, the velocity components of a solid spheroid body in the directions of the unit vectors $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$ are

$$V_r = U_z\cos\theta e^{i\sigma t}, \quad V_\theta = -U_z\sin\theta e^{i\sigma t}, \quad V_\phi = 0, \tag{8}$$

Since $\nabla\cdot\vec{q} = 0$, the velocity components q_r and q_θ in terms of Stokes' stream function ψ are

$$q_r = -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad q_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \tag{9}$$

Assuming the amplitude of rectilinear oscillations U_z to be sufficiently small, the assumption of the Stokesian flow applies. The problem is then governed by the following equations:

$$-\frac{i\rho\sigma}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = -\frac{\partial p}{\partial r} + \frac{k}{r\sin\theta}\frac{\partial}{\partial\theta}(\nu_\phi\sin\theta) - \frac{\mu+k}{r^2\sin\theta}\frac{\partial}{\partial\theta}(L_{-1}\psi), \tag{10}$$

$$\frac{i\rho\sigma}{r\sin\theta}\frac{\partial\psi}{\partial r} = -\frac{1}{r}\frac{\partial p}{\partial\theta} - \frac{k}{r}\frac{\partial}{\partial r}(r\nu_\phi) + \frac{\mu+k}{r\sin\theta}\frac{\partial}{\partial r}(L_{-1}\psi), \tag{11}$$

$$i\rho\sigma j\nu_\phi = -2k\nu_\phi + \gamma L\nu_\phi + \frac{k}{r\sin\theta}(L_{-1}\psi), \tag{12}$$

where L_{-1} and L are the differential operators defined by

$$L_{-1} = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}, \quad L = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r^2\sin^2\theta}.$$

The surface of a spheroid is assumed to be $r = a[1 + f(\theta)]$. This surface deviates slightly in shape from the sphere $r = a$. The orthogonality relations of the Gegenbauer functions $\mathfrak{I}_m(\zeta)$, $\zeta = \cos\theta$, permit us, under general circumstance, to assume the expansion $f(\theta) = \sum_{m=1}^{\infty} \alpha_m \mathfrak{I}_m(\zeta)$, where the Gegenbauer function is related to the Legendre functions $P_n(\zeta)$ by the relation

$$\mathfrak{I}_n(\zeta) = \frac{P_{n-2}(\zeta) - P_n(\zeta)}{2n-1}, \quad n \geqslant 2.$$

Therefore, the surface of a spheroid will be

$$r = a[1 + \alpha_m \mathfrak{I}_m(\zeta)], \tag{13}$$

and assume that the coefficients α_m are sufficiently small that their squares and higher powers may be neglected, that is $(r/a)^\rho \approx 1 + \rho\alpha_m \mathfrak{I}_m(\zeta)$, where ρ is positive or negative. The solution for the case $r = a[1 + \sum_m \alpha_m \mathfrak{I}_m(\zeta)]$ is to be found from the results of (13).

At a surface of the solid, we consider the slip boundary conditions on the surface of the solid particle [12, 30], which in this case take the forms

$$\beta_1(\vec{q} - \vec{V}) = (\mathbf{I} - \vec{n}\vec{n}) \cdot (\vec{n} \cdot \mathbf{t}), \tag{14}$$

$$\chi\vec{\nu} = (\mathbf{I} - \vec{n}\vec{n}) \cdot (\vec{n} \cdot \mathbf{m}), \tag{15}$$

where $\mathbf{I}$ is the unit dyadic, $\vec{n}$ is the unit normal vector at the particle surface pointing into the fluid, $\mathbf{t}$ and $\mathbf{m}$ are the stress and the couple stress tensors (dyadic) given by equations (4) and (5), respectively. The constants β_1 and χ are termed the coefficients of sliding friction. These coefficients are a measure of the degree of tangential slip existing between the fluid and solid at its surface. The slip coefficients are assumed to depend only on the nature of the fluid and solid surface. In the limiting case of $\beta_1 = \chi = 0$, there is a perfect slip at the surface of the body and the solid spheroid acts like a spheroidal gas bubble, while the standard no-slip boundary condition for solids is obtained by letting $\beta_1 = \chi \to \infty$.

Taking projections of (14) and (15) to the normal and arbitrary tangential direction $\vec{s}$ results in

$$(\vec{q}-\vec{V})\cdot\vec{n} = 0\,, \tag{16}$$
$$\beta_1(\vec{q}-\vec{V})\cdot\vec{s} = (\vec{n}\cdot\mathbf{t})\cdot\vec{s}\,, \tag{17}$$
$$\chi\vec{\nu}\cdot(\vec{s}\wedge\vec{n}) = (\vec{n}\cdot\mathbf{m})\cdot(\vec{s}\wedge\vec{n}). \tag{18}$$

On the boundary $r = a[1+\alpha_m\mathfrak{I}_m(\zeta)]$, the normal and tangential vectors are

$$\vec{n} = \vec{e}_r - \alpha_m a\nabla\mathfrak{I}_m(\zeta) = \vec{e}_r - \alpha_m(1-\zeta^2)^{1/2}P_{m-1}(\zeta)\vec{e}_\theta\,, \tag{19}$$
$$\vec{s} = -\alpha_m(1-\zeta^2)^{1/2}P_{m-1}(\zeta)\vec{e}_r - \vec{e}_\theta\,. \tag{20}$$

Substituting the expression for the unit normal and tangential vectors into (16) and (17) gives the approximate boundary conditions (up to $O(\alpha_m)$):

The impenetrability and slip boundary conditions are

$$0 = q_r - V_r - \alpha_m(q_\theta - V_\theta)(1-\zeta^2)^{1/2}P_{m-1}(\zeta), \tag{21}$$
$$0 = \beta_1(q_\theta - V_\theta) - t_{r\theta} + \alpha_m(t_{\theta\theta} - t_{rr})(1-\zeta^2)^{1/2}P_{m-1}(\zeta), \tag{22}$$
$$0 = \chi\nu_\phi - m_{r\phi} + \alpha_m m_{\theta\phi}(1-\zeta^2)^{1/2}P_{m-1}(\zeta). \tag{23}$$

The velocity components as well as the microrotation have to vanish as $r \to \infty$

The system of equations under consideration reduces to the following for ψ and ν_ϕ:

$$L_{-1}\left(L_{-1}-\ell^2\right)\left(L_{-1}-\kappa^2\right)\psi = 0\,, \tag{24}$$
$$\nu_\phi = \frac{1}{kr\sin\theta}\frac{i\rho\sigma}{\ell^2\kappa^2}\left[L_{-1}^2\psi - \left(\frac{i\ell^2\kappa^2(\mu+k)}{\rho\sigma} + \ell^2 + \kappa^2\right)L_{-1}\psi\right], \tag{25}$$

where ℓ and κ are such that

$$\ell^2+\kappa^2 = \frac{k(2\mu+k)+i\rho\sigma(\gamma+j\mu+jk)}{\gamma(\mu+k)}\,,$$
$$\ell^2\kappa^2 = \frac{\rho\sigma(2ik-j\rho\sigma)}{\gamma(\mu+k)}\,. \tag{26}$$

Using the separation of variables technique, the general solution of (24) is

$$\frac{\psi}{U_z a^2} = \left(\frac{a_2}{r} + b_2\sqrt{r}K_{\frac{3}{2}}(r\ell) + c_2\sqrt{r}K_{\frac{3}{2}}(r\kappa)\right)\mathfrak{I}_2(\zeta) + \sum_{n=3}^{\infty}\left(A_n r^{-n+1} + B_n\sqrt{r}K_{n-\frac{1}{2}}(r\ell) + C_n\sqrt{r}K_{n-\frac{1}{2}}(r\kappa)\right)\mathfrak{I}_n(\zeta). \quad (27)$$

In the expression (27), the values of ℓ and κ are to be such that the regularity of ψ at infinity is ensured and this is attained by selecting the roots ℓ and κ from (26) such that each of them has a positive real part. Substituting this in (25), we get the microroration component as

$$\frac{a\nu_\phi}{U_z} = \frac{1}{r\sqrt{1-\zeta^2}}\left[\left(b_2\sqrt{r}A_\ell K_{\frac{3}{2}}(r\ell) + c_2\sqrt{r}A_\kappa K_{\frac{3}{2}}(r\kappa)\right)\mathfrak{I}_2(\zeta) + \sum_{n=3}^{\infty}\left(B_n\sqrt{r}A_\ell K_{n-\frac{1}{2}}(r\ell) + C_n\sqrt{r}A_\kappa K_{n-\frac{1}{2}}(r\kappa)\right)\mathfrak{I}_n(\zeta)\right], \quad (28)$$

where K_m is a modified Bessel function of second kind of order m, and

$$A_\ell = \frac{\ell^2(\mu+k) - i\rho\sigma a^2}{k}, \quad A_\kappa = \frac{\kappa^2(\mu+k) - i\rho\sigma a^2}{k}.$$

In the equations (27) and (28), and in all subsequent expressions in this section, r is nondimensional with respect to the sphere radius a, as well as the parameters ℓ and κ. The only coefficients which contribute to the solution of the rectilinear oscillations of a perfect sphere, treated by Lakshmana and Bhujanga [32], are b_2, d_2 and e_2. Consequently, all other coefficients must be of $O(\alpha_m)$ and for these, to the first order in α_m, we take on the surface of the particle $r = 1$. The boundary conditions (21)–(23) in terms of ψ and ν_ϕ

lead to the following:

$$\frac{\partial\psi}{\partial\zeta} - r^2 U_z a^2 P_1(\zeta) - \alpha_m r^2\Big(\frac{1}{r}\frac{\partial\psi}{\partial r} + 2U_z a^2 \mathfrak{I}_2(\zeta)\Big)P_{m-1}(\zeta) = 0\,, \tag{29}$$

$$\lambda_1\Big(\frac{\partial\psi}{\partial r} + 2rU_z a^2\mathfrak{I}_2(\zeta)\Big) = r\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial\psi}{\partial r}\Big) - \frac{\omega}{\ell^2\kappa^2}L_{-1}(L_{-1}-\ell^2-\kappa^2)\psi$$
$$- L_{-1}\psi - \alpha_m(1-\zeta^2)\Big(\frac{3}{r^2}\frac{\partial\psi}{\partial\zeta} - \frac{1}{r}\frac{\zeta}{1-\zeta^2}\frac{\partial\psi}{\partial r} - \frac{2}{r}\frac{\partial^2\psi}{\partial r\partial\zeta}\Big)P_{m-1}(\zeta), \tag{30}$$

$$\chi\nu_\phi = \gamma\frac{\partial\nu_\phi}{\partial r} - \frac{\beta}{r}\nu_\phi + \alpha_m\left(\frac{\beta\zeta}{r}\nu_\phi + \frac{\gamma(1-\zeta^2)}{r}\frac{\partial\nu_\phi}{\partial\zeta}\right)P_{m-1}(\zeta), \tag{31}$$

where $\lambda_1 = \beta_1 a/(2\mu+k)$ and $\omega = i\rho\sigma a^2/(2\mu+k)$. Inserting expressions (27) and (28) into the approximate boundary conditions (29)–(31), we obtain three equations in the unknown constants, which are given in Appendix A. These equations are sufficient to determine the unknown constants to the desired order of approximation, $O(\epsilon)$. Therefore, the stream function and the microrotation component for the flow field are to be found up to $O(\epsilon)$.

3.1 Application to a slip spheroid

As an application of the above analysis, we now consider the particular case of the rectilinear oscillations of a slip prolate or oblate spheroid along its symmetry axis (see Figure 1). The surface of a spheroidal particle is represented in the Cartesian frame (x, y, z) by the equation

$$\frac{x^2+y^2}{c^2} + \frac{z^2}{c^2(1-\epsilon)^2} = 1\,, \tag{32}$$

where c is equatorial radius and ϵ is so small that squares and higher powers of it are neglected. The polar equation of the spheroidal surface (32) is

$$r = 1 + 2\epsilon\mathfrak{I}_2(\zeta),$$

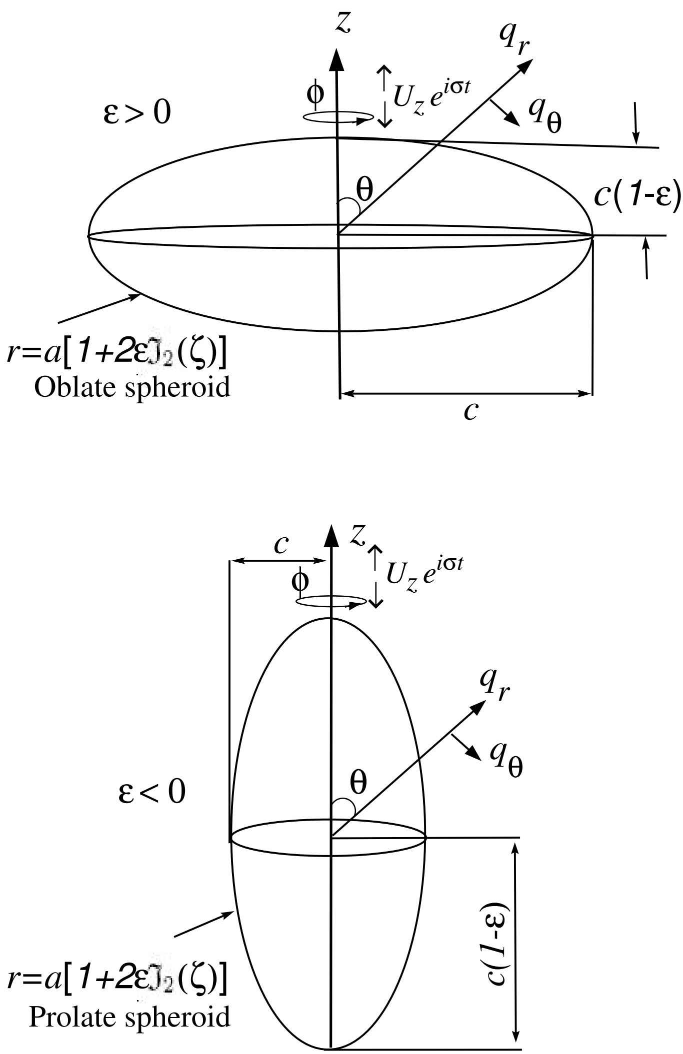

Figure 1: Physical model and coordinate system for an oblate or a prolate spheroid surface.

and $a = c(1-\epsilon)$. For the case $0 < \epsilon \leqslant 1$, the spheroid is oblate, and for the case $\epsilon < 0$, it is prolate. When $\epsilon = 0$, equation (32) describes a sphere of radius c.

To apply the above results, we must take $m = 2$, $\alpha_m = 2\epsilon$. Therefore, the stream function and the microrotation component are

$$\frac{\psi}{U_z a^2} = \left[\frac{a_2 + A_2}{r} + (b_2 + B_2)\sqrt{r}K_{\frac{3}{2}}(r\ell) + +(c_2 + C_2)\sqrt{r}K_{\frac{3}{2}}(r\kappa)\right]\mathfrak{I}_2(\zeta) + \left[\frac{A_4}{r^3} + B_4\sqrt{r}K_{\frac{7}{2}}(r\ell) + C_4\sqrt{r}K_{\frac{7}{2}}(r\kappa)\right]\mathfrak{I}_4(\zeta), \tag{33}$$

$$\frac{a\nu_\phi}{U_z} = \frac{1}{r\sqrt{1-\zeta^2}}\left\{\left[(b_2 + B_2)\sqrt{r}A_\ell K_{\frac{3}{2}}(r\ell) + (c_2 + C_2)\sqrt{r}A_\kappa K_{\frac{3}{2}}(r\kappa)\right] \times\mathfrak{I}_2(\zeta) + \left[B_4\sqrt{r}A_\ell K_{\frac{7}{2}}(r\ell) + C_4\sqrt{r}A_\kappa K_{\frac{7}{2}}(r\kappa)\right]\mathfrak{I}_4(\zeta)\right\}, \tag{34}$$

The corresponding pressure field $p(r,\theta)$, obtained by integration of the Stokes' flow equations (10) and (11), is

$$\frac{ap}{U_z} = -\omega(2\mu + k)\left[\frac{a_2 + A_2}{2r^2}P_1(\zeta) + \frac{A_4}{4r^4}P_3(\zeta)\right], \tag{35}$$

where a constant of integration has been neglected without loss of generality.

Now we discuss the force acting on a spheroid oscillating rectilinearly in an unbounded micropolar fluid. Because of the axial symmetry of the micropolar flow, the contribution of the surface stress to the couple on the spheroid is then zero. The couple vector on the spheroid arising from the couple stresses is found to be zero. However, the fluid will exert force on the spheroid. This force has only a component F_z in the direction of oscillations which is obtained by integrating the stresses on the surface of the spheroid as

$$F_z = \int_S (\vec{n}\cdot t)\cdot\vec{k}\,dS, \tag{36}$$

where $r = 1 + \epsilon\sin^2\theta$, $\vec{n} = \vec{e}_r - \epsilon\sin 2\theta\,\vec{e}_\theta$, $dS = 2\pi c^2(1 + 2\epsilon\sin^2\theta)\sin\theta\,d\theta$ to $O(\epsilon)$, and $\vec{k}$ is the unit vector in the direction of the axis of oscillations and the integral is taken over the surface of the body. Therefore,

$$F_z = 2\pi a^2 \int_0^\pi r^2 \big[(t_{rr} - \epsilon t_{\theta r} \sin 2\theta) \cos\theta - (t_{r\theta} - \epsilon t_{\theta\theta} \sin 2\theta) \sin\theta \big]\big|_{r=1+\epsilon \sin^2\theta} \sin\theta \, d\theta, \tag{37}$$

$$F_z = \frac{2\pi a U_z \omega(2\mu + k)}{3} \Big\{ a_2 + A_2 - 2(b_2 + B_2) K_{\frac{3}{2}}(\ell) - 2(c_2 + C_2) K_{\frac{3}{2}}(\kappa) + \frac{4\epsilon}{5} \Big[a_2 + \frac{b_2}{1+\ell}(2\ell^2 + \ell + 1) K_{\frac{3}{2}}(\ell) + \frac{c_2}{1+\kappa}(2\kappa^2 + \kappa + 1) K_{\frac{3}{2}}(\kappa) \Big] \Big\} e^{i\sigma t}. \tag{38}$$

Putting $a = c(1-\epsilon)$ in equation (38), the drag force

$$F_z = \frac{2}{3}\pi\rho\sigma c^3 U_z(-T' - iT)e^{i\sigma t}, \tag{39}$$

where T and T' are real force coefficients and the real part of this expression is seen to be

$$\Re F_z = \frac{2}{3}\pi\rho\sigma c^3 U_z(T \sin\sigma t - T' \cos\sigma t). \tag{40}$$

Physically the force coefficients T and T' represent, respectively, the in-phase and the out-of phase force oscillations.

1. For perfect sphere ($\epsilon = 0$) the expression for drag force becomes

$$F_z = \frac{-2\pi c U_z \omega_1 (2\mu + k)}{3\Delta_2} \Big[\Delta_2 + 9(1+\lambda_2) \big\{ (\mu + k) \big[(\ell_1 + \kappa_1)(1 + \kappa_1) \times \big(\delta(1+\ell_1) + \ell_1^2 \big) + \kappa_1^2 (\ell_1 + \ell_1\kappa_1 + \kappa_1) \big] - \omega_1 (2\mu + k)(\ell_1 + \ell_1\kappa_1 + \kappa_1) \big\} \Big] e^{i\sigma t}, \tag{41}$$

where $\lambda_2 = \beta_1 c/(2\mu + k)$, $\omega_1 = i\rho\sigma c^2/(2\mu + k)$, $\delta = (\chi c + 2\gamma + \beta)\gamma^{-1}$, and

$$\Delta_2 = \omega_1(\ell_1 + \kappa_1 + \ell_1\kappa_1) \Big\{ (\mu + k) \big[\kappa_1(\kappa_1 + \delta) + 2\lambda_2 + 3 \big] - k(1+\lambda_2) - (2\mu + k)\omega_1 \Big\} + \ell_1^2(\mu + k) \times \big[\omega_1(1+\kappa_1)(1 + \kappa_1 + \delta) + \kappa_1^2(1+\lambda_2)(\ell_1 + \kappa_1 + \delta) \big],$$

with ℓ_1 and κ_1 are such that

$$\ell_1^2+\kappa_1^2 = \frac{k(2\mu+k)+i\rho\sigma(\gamma+j\mu+jk)}{\gamma(\mu+k)}c^2, \tag{42}$$

$$\ell_1^2\kappa_1^2 = \frac{\rho\sigma(2ik-j\rho\sigma)}{\gamma(\mu+k)}c^4. \tag{43}$$

Moreover, in the case of no-slip ($\beta_1 \to \infty$, $\chi \to \infty$), we get the same value of the drag force calculated by Lakshmana and Bhujanga [32].

2. The case of slow steady translation of a spheroid with slip is obtained also from the above analysis by allowing the period of oscillation $2\pi/\sigma$ tend to infinity. Using

$$\lim_{\sigma\to 0}(\ell_1^2+\kappa_1^2)=\ell_2^2 \quad \text{and} \quad \lim_{\sigma\to 0}(\ell_1^2\kappa_1^2)=0\,,$$

where $\ell_2^2 = kc^2(2\mu+k)/(\gamma(\mu+k))$, so that we may take, say $\ell_1=\ell_2$ and $\kappa_1=0$. Therefore, the drag reduces to

$$\begin{aligned} F_z = {} & -\frac{6\pi c U_z(2\mu+k)(\mu+k)}{\varpi}\Big\{\left[\ell_2^2+\delta(1+\ell_2)\right](1+\lambda_2) \\ & -\frac{\epsilon}{5\varpi}\Big((\mu+k)\left[\ell_2^2+\delta(1+\ell_2)\right]^2(2\lambda_2^2+6\lambda_2-3) \\ & -k(1+\lambda_2)^2\left[\delta^2(1+2\ell_2)+\ell_2^2(2\delta+4-\beta/\gamma)\right]\Big)\Big\}. \end{aligned} \tag{44}$$

where $\varpi=(\mu+k)(2\lambda_2+3)[\ell_2^2+\delta(1+\ell_2)]-k\delta(1+\lambda_2)$.

For viscous fluid, $k=0$, the drag will be

$$F_z = \frac{-6\pi\mu c U_z}{\beta_1 c+3\mu}\left(\beta_1 c+2\mu-\frac{\epsilon}{5}\frac{\beta_1^2c^2+6\beta_1 c\mu-6\mu^2}{\beta_1 c+3\mu}\right), \tag{45}$$

and this agrees with the result obtained by Chang and Keh [24].

3. The case of unsteady motion of a slip spheroidal particle in the classical viscous fluids, is recoverable from equation (38) by letting the viscosity coefficients k and γ tend to zero and carrying out the appropriate limiting process. Then in equation (26) one root (say κ_1) becomes infinity and the other finite root is

$$\ell_3^2 = \lim_{|\kappa_1|\to\infty} \frac{\ell_1^2\kappa_1^2}{\ell_1^2+\kappa_1^2} = \frac{i\rho\sigma c^2}{\mu}. \tag{46}$$

In this case the drag F_z reduces to

$$\begin{aligned} F_z = {} & \frac{-2i\pi c U_z \omega_1}{3\Delta_3}\Big[(\beta_2+2)(\ell_3^2+9\ell_3+9)+i\omega_1(1+\ell_3) \\ & -\frac{\epsilon}{5\Delta_3}\{i\omega_1(1+\ell_3)^2(i\omega_1-18\beta_2+90)+\ell_3^2(\beta_2+2) \\ & \times\left[(\beta_2+2)(\ell_3^2-18\ell_3-9)+2i\omega_1(\ell_3+1)\right]\}\Big]e^{i\sigma t}, \end{aligned} \tag{47}$$

where $\beta_2=\beta_1 c/\mu$, $\omega_1=\rho\sigma c^2/\mu$, $\ell_3=(1+i)\sqrt{\omega_1/2}$, $\Delta_3=i\omega_1(\ell_3+1)+\ell_3^2(\beta_2+2)$.

3.2 Numerical results

The in-phase and out-of phase real coefficients of the force oscillations T and T' introduced in the equation (40) are plotted in figures 2–6 versus the parameter of the frequency of the oscillations ω_1 and the sliding friction parameters β_2 and $\chi_1(=\chi/\mu c)$ for several different values of the micropolarity coefficient k/μ, and the deformity parameter ϵ when the parameters $j/c^2=0.2$, $\gamma/\mu c^2=0.3$, $\beta/\mu c^2=0.2$, and $\alpha/\mu c^2=0.1$.

Figure 2 indicates that over the range of the slip parameters $0\leqslant\chi_1=\beta_2<\infty$, the values of the force parameters monotonically decrease with the increase of the frequency parameter. Also for the entire range of the frequency parameter, the coefficients T and T' increase with the increase of the slip parameters

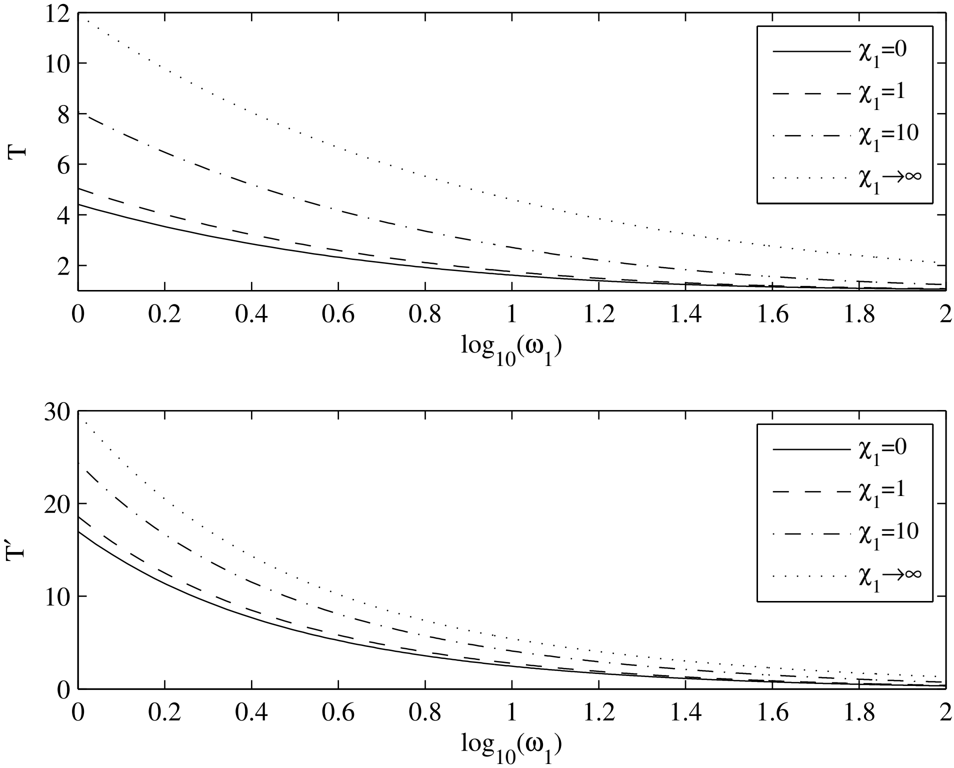

Figure 2: Variation of drag parameters versus the frequency parameter for various values of slip parameter for $\beta_2 = \chi_1$, $\epsilon = 0.1$ and $k/\mu = 2$.

$\chi_1 = \beta_2$. As seen from figure 3 for $\chi_1 = \beta_2 = 10$, over the entire range of frequency parameter, the coefficients T and T' increase with the increase of the micropolarity parameter. Figure 4 shows that for certain values of $\chi_1 = \beta_2 = 10$ and $k/\mu = 2$ over the entire range of frequency parameter, the coefficients T and T' decrease with the increase of the deformity parameter ϵ. For a spheroid of a given aspect ratio, the drag force parameters monotonically increase with the slip parameters (see figure 6). Figure 5 shows that the force coefficients are to be finite in both the perfect slip and no-slip limits. It indicates also that for the entire range of the slip parameters, the force

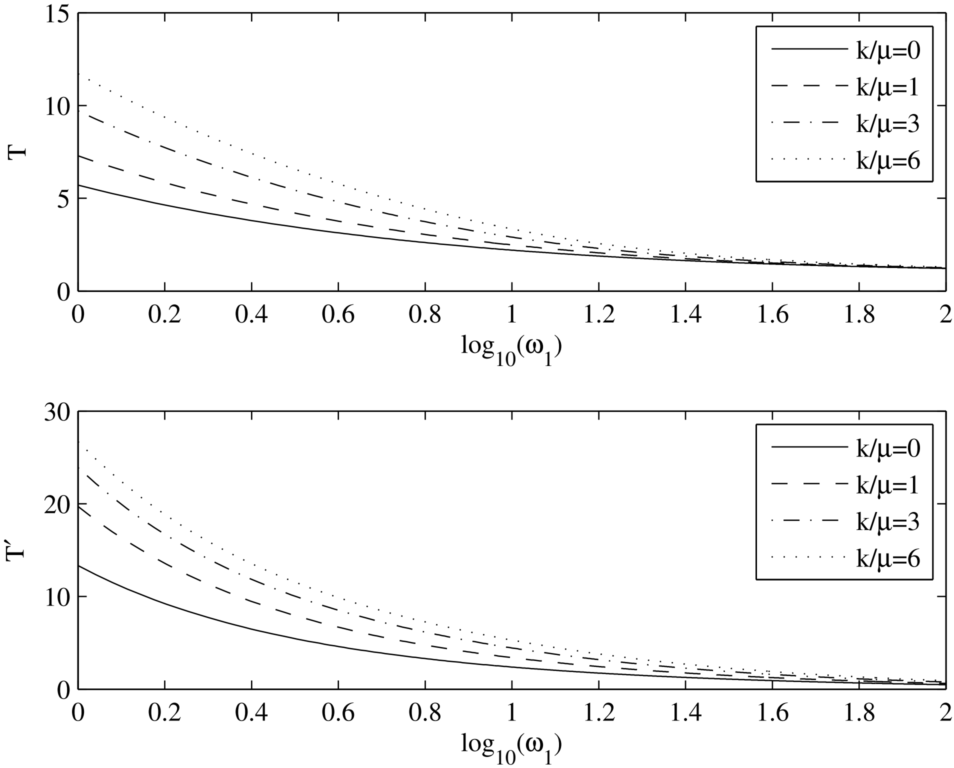

Figure 3: Variation of drag parameters versus the frequency parameter for various values of micropolarity coefficient for $\beta_2 = \chi_1 = 10$ and $\epsilon = 0.1$.

coefficients increases with the increase of micropolarity parameter. The lowest values of the force coefficients correspond to the case of viscous fluid. For $\epsilon < 0$ (aspect ratio exceeds one), the major portion of the fluid slip at the particle surface occurs in the direction of the particle's movement. However, for $\epsilon > 0$ (aspect ratio is smaller than one), the main component of the fluid slip at the surface of a spheroidal particle is in the direction normal to the motion of the spheroid.

Here, the drag parameters exerted on the oscillating slip sphere of radius

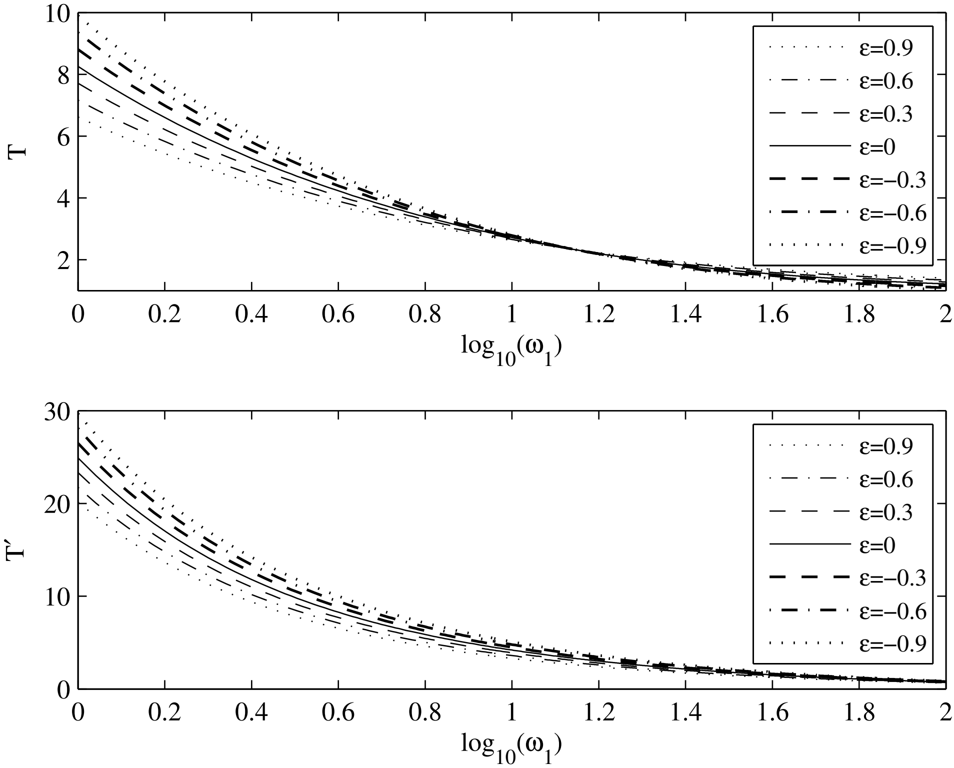

Figure 4: Variation of drag parameters versus the frequency parameter for various values of deformity parameter for $\beta_2 = \chi_1 = 10$ and $k/\mu = 2$.

equal to the equatorial radius of the spheroid is less than that experienced by an oscillating slip oblate spheroid and greater than that of the slip prolate spheroid with a small value of β_2 ($\beta_2 < 5$), and the reverse occurring with a large but finite value of β_2 ($\beta_2 > 5$). Also, that the in-phase and out-of phase values of T and T' take positive values. This does not contradict that the force should oppose the direction of particle motion. These quantities are only the coefficients of $\sin \sigma t$ and $\cos \sigma t$ in expression (40), the direction of the force is determined by the sign of the expression as a whole.

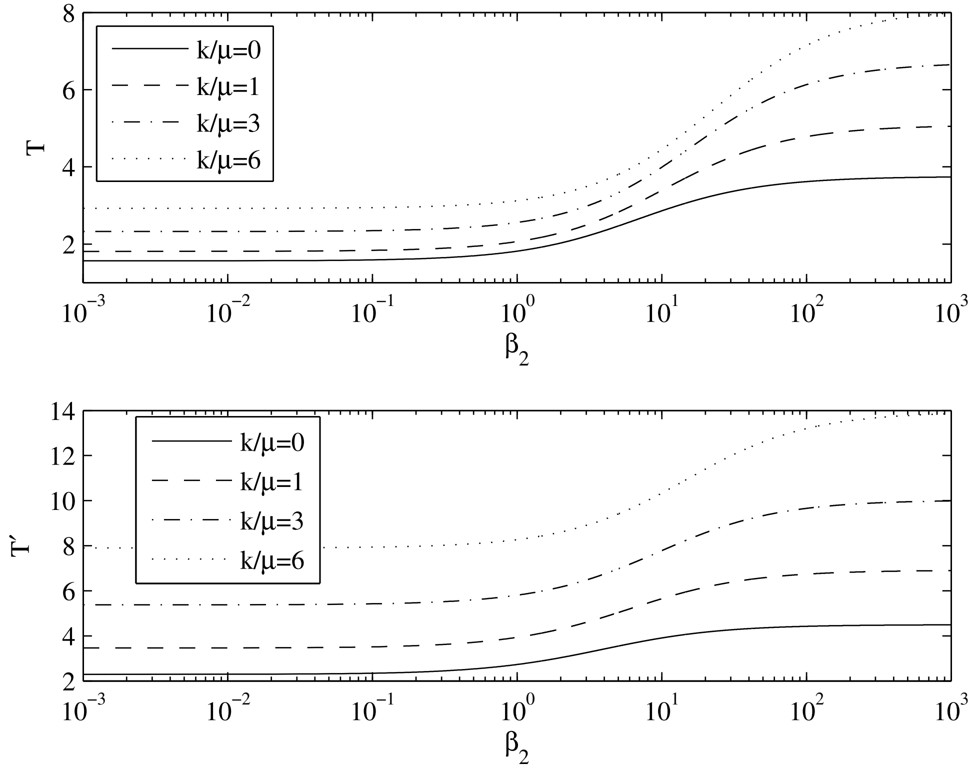

Figure 5: Variation of drag parameters versus the slip parameter for various values of micropolarity coefficient for $\omega_1 = 5$, $\epsilon = 0.1$ and $\chi_1 = 10$.

4 Rotary oscillations of a slip spheroid in a micropolar fluid

Consider an axisymmetric spheroidal particle performing rotary oscillations with speed $V_\phi e^{i\sigma t}$ about the axis of symmetry $\theta = 0$ in an infinite expanse of an incompressible micropolar fluid which is otherwise at rest. The generated flow is rotationally symmetric. We then choose in this case the velocity and

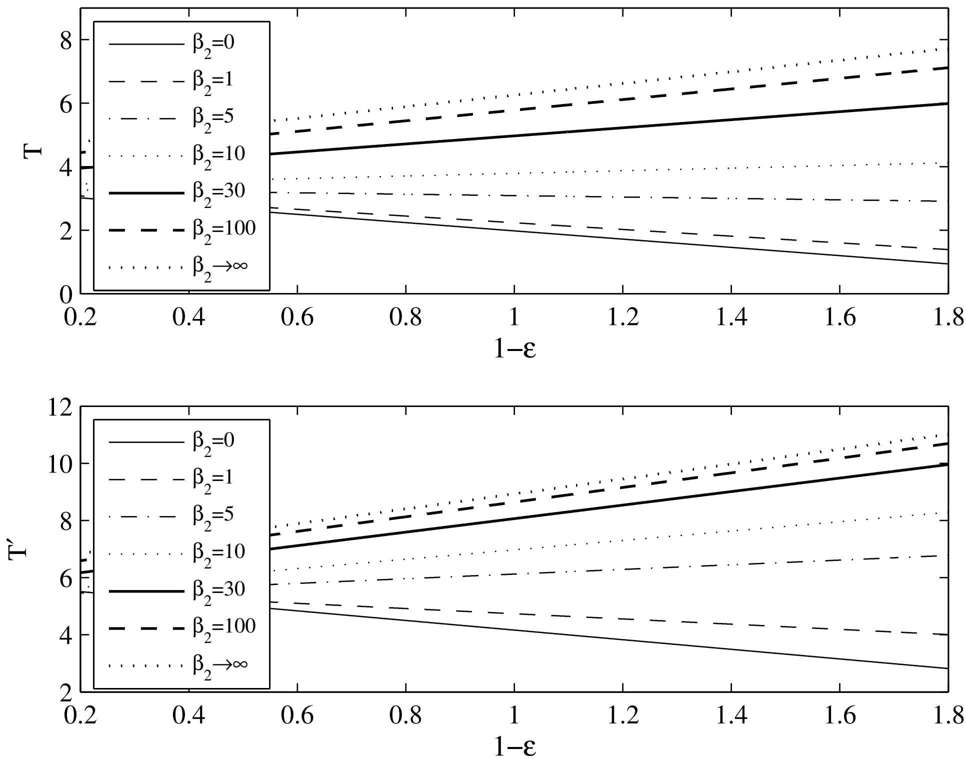

Figure 6: Variation of drag parameters versus the aspect ratio for various values of slip parameter for $\omega_1 = 5$, $k/\mu = 2$ and $\chi_1 = \beta_2$.

microrotation vectors as

$$\vec{q} = q_\phi(r,\theta)e^{i\sigma t}\vec{e}_\phi\,, \tag{48}$$

$$\vec{v} = \big(v_r(r,\theta)\vec{e}_r + v_\theta(r,\theta)\vec{e}_\theta\big)e^{i\sigma t}\,. \tag{49}$$

The velocity components of the particle are

$$V_r = 0\,, \quad V_\theta = 0\,, \quad V_\phi = \Omega_z r\sin\theta e^{i\sigma t}. \tag{50}$$

The field equations, in this case reduce to

$$\frac{\partial p}{\partial r} = 0\,, \quad \frac{\partial p}{\partial \theta} = 0\,, \tag{51}$$

$$\left(L - \frac{i\rho\sigma}{\mu + k}\right) q_\phi = -\frac{k}{\mu + k} G(r,\theta), \tag{52}$$

$$\begin{aligned} v_r = {} & \frac{1}{\eta^2}\frac{\partial F}{\partial r} - \frac{\gamma}{2k + i\rho\sigma j}\frac{1}{r}\left(\frac{\partial G}{\partial \theta} + G\cot\theta\right) \\ & + \frac{k}{2k + i\rho\sigma j}\frac{1}{r}\left(\frac{\partial q_\phi}{\partial \theta} + q_\phi \cot\theta\right), \end{aligned} \tag{53}$$

$$v_\theta = \frac{1}{\eta^2}\frac{1}{r}\frac{\partial F}{\partial \theta} + \frac{\gamma}{2k + i\rho\sigma j}\left(\frac{\partial G}{\partial r} + \frac{G}{r}\right) - \frac{k}{2k + i\rho\sigma j}\left(\frac{\partial q_\phi}{\partial r} + \frac{q_\phi}{r}\right), \tag{54}$$

where

$$F(r,\theta) = \operatorname{div}\vec{v}\,, \quad G(r,\theta)\vec{e}_\phi = \operatorname{curl}\vec{v}\,, \quad \eta^2 = \frac{2k + i\rho\sigma j}{\alpha + \beta + \gamma}\,.$$

Equation (51) gives a constant pressure p throughout the flow region.

Let the equation, in polar form, of the axisymmetric rotating body be of the form $r = b[1 + f(\theta)]$. The orthogonality relations of the Legendre functions $P_m(\zeta)$, $\zeta = \cos\theta$, permit us, under general circumstance, to assume the expansion $f(\theta) = \sum_{m=1}^{\infty} \alpha_m P_m(\zeta)$. We therefore, take the surface of a spheroid to be

$$r = b[1 + \alpha_m P_m(\zeta)]. \tag{55}$$

Also, the solution for the case $r = b[1 + \sum_m \alpha_m P_m(\zeta)]$ is to found from the results of (55). We use a form in terms of the Legendre polynomial as opposed to equation (13) where we take the surface of a spheroid in terms of the Gegenbauer function. This is mainly to simplify the calculations. The solution in either case is the same since the Gegenbauer function is expressed in terms of Legendre polynomials. On the surface boundary $r = b[1 + \alpha_m P_m(\zeta)]$ the normal vector

$$\vec{n} = \vec{e}_r - \alpha_m b \nabla P_m(\zeta) = \vec{e}_r + \alpha_m P_m^1(\zeta)\vec{e}_\theta\,.$$

The slip boundary conditions (14) and (15)

$$\beta_1(q_\phi - V_\phi) = t_{r\phi} + \alpha_m t_{\theta\phi} P_m^1(\zeta), \tag{56}$$

$$\nu_r + \alpha_m \nu_\theta P_m^1(\zeta) = 0\,, \tag{57}$$

$$\chi \nu_\theta = m_{r\theta} + \alpha_m (m_{\theta\theta} - m_{rr}) P_m^1(\zeta). \tag{58}$$

From equations (53) and (54), we see that

$$(\nabla^2 - \eta^2) F = 0\,, \tag{59}$$

where ∇^2 denotes the Laplacian operator. Also,

$$\left(L - \frac{2k + i\rho\sigma j}{\gamma}\right) G = \frac{k}{\gamma} L q_\phi\,. \tag{60}$$

From equations (52) and (60), the velocity q_ϕ satisfies the equation

$$\left[L^2 - \left(\frac{k(2\mu + k) + i\rho\sigma(\gamma + j\mu + jk)}{\gamma(\mu + k)}\right) L + \frac{\rho\sigma(2ik - j\rho\sigma)}{\gamma(\mu + k)}\right] q_\phi = 0\,. \tag{61}$$

This equation is factorized as

$$\left(L - \xi^2\right)\left(L - \varphi^2\right) q_\phi = 0\,, \tag{62}$$

where ξ and φ are such that

$$\xi^2 + \varphi^2 = \frac{k(2\mu + k) + i\rho\sigma(\gamma + j\mu + jk)}{\gamma(\mu + k)},$$

$$\xi^2 \varphi^2 = \frac{\rho\sigma(2ik - j\rho\sigma)}{\gamma(\mu + k)}. \tag{63}$$

Keeping only solutions which are regular at infinity in (59) and (62) results in

$$\frac{q_\phi}{b\Omega_z} = r^{-1/2}\left\{\left[a_1 K_{\frac{3}{2}}(\xi r) + b_1 K_{\frac{3}{2}}(\varphi r)\right] P_1^1(\zeta) + \sum_{n=2}^{\infty}\left(A_n K_{n+\frac{1}{2}}(\xi r) + B_n K_{n+\frac{1}{2}}(\varphi r)\right) P_n^1(\zeta)\right\}, \tag{64}$$

$$\frac{F}{b\Omega_z} = r^{-1/2}\left\{c_1 K_{\frac{3}{2}}(\eta r) P_1(\zeta) + \sum_{n=2}^{\infty} C_n K_{n+\frac{1}{2}}(\eta r) P_n(\zeta)\right\}, \tag{65}$$

where P_n^1 is the associated Legendre function. In the expressions (64) and (65) the values of ξ and φ are to be such that the regularity of q_ϕ and F at infinity is ensured and this is attained by selecting the roots ξ and φ of (63) having positive real parts. The microrotation components are therefore

$$\begin{aligned}\frac{\nu_r}{\Omega_z} &= r^{-3/2}\Bigg\{2\Big[A_\xi a_1 K_{\frac{3}{2}}(\xi r) + A_\varphi b_1 K_{\frac{3}{2}}(\varphi r)\\ &\quad - \frac{c_1}{\eta^2}\Big(K_{\frac{3}{2}}(\eta r) + \frac{\eta}{2} r K_{\frac{1}{2}}(\eta r)\Big)\Big]P_1(\zeta)\\ &\quad + \sum_{n=2}^{\infty}\Big[n(n+1)\big(A_\xi A_n K_{n+\frac{1}{2}}(\xi r) + A_\varphi B_n K_{n+\frac{1}{2}}(\varphi r)\big)\\ &\quad - \frac{C_n}{\eta^2}\big((1+n)K_{n+\frac{1}{2}}(\eta r) + \eta r K_{n-\frac{1}{2}}(\eta r)\big)\Big]P_n(\zeta)\Bigg\},\end{aligned} \tag{66}$$

$$\begin{aligned}\frac{\nu_\theta}{\Omega_z} &= r^{-3/2}\Bigg\{\Big[A_\xi a_1\big(K_{\frac{3}{2}}(\xi r) + \xi r K_{\frac{1}{2}}(\xi r)\big)\\ &\quad + A_\varphi b_1\big(K_{\frac{3}{2}}(\varphi r) + \varphi r K_{\frac{1}{2}}(\varphi r)\big) - \frac{c_1}{\eta^2}K_{\frac{3}{2}}(\eta r)\Big]P_1^1(\zeta)\\ &\quad + \sum_{n=2}^{\infty}\Big[A_\xi A_n\big(nK_{n+\frac{1}{2}}(\xi r) + \xi r K_{n-\frac{1}{2}}(\xi r)\big) + A_\varphi B_n\\ &\quad \times \big(nK_{n+\frac{1}{2}}(\varphi r) + \varphi r K_{n-\frac{1}{2}}(\varphi r)\big) - \frac{C_n}{\eta^2}K_{n+\frac{1}{2}}(\eta r)\Big]P_n^1(\zeta)\Bigg\},\end{aligned} \tag{67}$$

where

$$A_\xi = \frac{k^2 + \gamma\xi^2(\mu+k) - i\rho\sigma\gamma}{k(2k + i\rho\sigma j)}, \quad A_\varphi = \frac{k^2 + \gamma\varphi^2(\mu+k) - i\rho\sigma\gamma}{k(2k + i\rho\sigma j)}.$$

Here the expressions (64)–(67) and in all subsequent expressions in this section, r is nondimensional with respect to the sphere radius b, as well as the parameters ξ, φ and η. The only coefficients which contribute to the solution of the rotary oscillations of a sphere are a_1, b_1 and c_1 and as a result we expect all other coefficients in (64), (66) and (67) to be of $O(\alpha_m)$. Therefore, except where these coefficients a_1, b_1 and c_1 are encountered, we may take the surface to be $r = 1$ instead of the exact form (55).

The boundary conditions (56)–(58) in terms of the functions q_ϕ, F, v_r and v_θ, lead to

$$\lambda\big(q_\phi - \Omega_z r P_1^1(\zeta)\big) = \frac{k}{\eta^2(2\mu+k)}\frac{1}{r}\frac{\partial F}{\partial\theta} - \frac{q_\phi}{r} - \frac{\omega}{\xi^2\varphi^2}\frac{1}{r}\frac{\partial}{\partial r}$$
$$\times\left[r(L-\xi^2-\varphi^2)q_\phi\right] + \alpha_m\left\{\frac{\omega}{r\xi^2\varphi^2}\frac{\partial}{\partial\zeta}\left[(1-\zeta^2)^{1/2}(L-\xi^2-\varphi^2)q_\phi\right]\right.$$
$$\left.-\frac{\zeta}{r(1-\zeta^2)^{1/2}}q_\phi - \frac{k}{\eta^2(2\mu+k)}\frac{\partial F}{\partial r}\right\}P_m^1(\zeta), \tag{68}$$
$$v_r + \alpha_m v_\theta P_m^1(\zeta) = 0, \tag{69}$$
$$\chi v_\theta + \frac{\beta}{r}\left((1-\zeta^2)^{1/2}\frac{\partial v_r}{\partial\zeta} + v_\theta\right) - \gamma\frac{\partial v_\theta}{\partial r}$$
$$+\alpha_m\frac{\beta+\gamma}{r}\left((1-\zeta^2)^{1/2}\frac{\partial v_\theta}{\partial\zeta} + r\frac{\partial v_r}{\partial r} - v_r\right)P_m^1(\zeta) = 0, \tag{70}$$

where, here, $\lambda = \beta_1 b/(2\mu+k)$, $\omega = i\rho\sigma b^2/(2\mu+k)$. The unknown constants appearing in the equations (64), (66) and (67) determined from the conditions (68) and (70). We thus completely determined the velocity and microrotation components for the flow field which are also listed in Appendix B.

4.1 Application to a spheroid with slip effects

As a particular example of the above analysis, we now consider the rotary oscillations of a prolate or an oblate spheroid using the slip conditions at the surface, whose equation we take as

$$\frac{x^2+y^2}{b^2(1-\upsilon/2)^2} + \frac{z^2}{b^2(1+\upsilon)^2} = 1\,, \tag{71}$$

where υ is a small quantity ($\upsilon < 1$). For $\upsilon < 0$ the spheroid is oblate, and for $\upsilon > 0$ it is prolate. To $O(\upsilon)$, equation (71) in polar form, becomes

$r = 1 + \upsilon P_2(\zeta)$. Here, we must take $m = 2$ and $\alpha_m = \upsilon$. Therefore, the velocity and microrotation components are

$$\begin{aligned}\frac{q_\phi}{b\Omega_z} &= r^{-1/2}\Big\{\big[(a_1 + A_1)K_{\frac{3}{2}}(\xi r) + (b_1 + B_1)K_{\frac{3}{2}}(\varphi r)\big]P_1^1(\zeta) \\ &\quad + \big[A_3K_{\frac{7}{2}}(\xi r) + B_3K_{\frac{7}{2}}(\varphi r)\big]P_3^1(\zeta)\Big\}, \qquad (72)\\ \frac{\nu_r}{\Omega_z} &= r^{-3/2}\Big\{2\Big[A_\xi(a_1 + A_1)K_{\frac{3}{2}}(\xi r) + A_\varphi(b_1 + B_1)K_{\frac{3}{2}}(\varphi r) \\ &\quad - \frac{(c_1 + C_1)}{2\eta^2}\big(2K_{\frac{3}{2}}(\eta r) + \eta r K_{\frac{1}{2}}(\eta r)\big)\Big]P_1(\zeta) \\ &\quad + \Big[12\big(A_\xi A_3 K_{\frac{7}{2}}(\xi r) + A_\varphi B_3 K_{\frac{7}{2}}(\varphi r)\big) \\ &\quad - \frac{C_3}{\eta^2}\big(4K_{\frac{7}{2}}(\eta r) + \eta r K_{\frac{5}{2}}(\eta r)\big)\Big]P_3(\zeta)\Big\}, \qquad (73)\\ \frac{\nu_\theta}{\Omega_z} &= r^{-3/2}\Big\{\Big[A_\xi(a_1 + A_1)\big(K_{\frac{3}{2}}(\xi r) + \xi r K_{\frac{1}{2}}(\xi r)\big) \\ &\quad + A_\varphi(b_1 + B_1)\big(K_{\frac{3}{2}}(\varphi r) + \varphi r K_{\frac{1}{2}}(\varphi r)\big) \\ &\quad - \frac{(c_1 + C_1)}{\eta^2}K_{\frac{3}{2}}(\eta r)\Big]P_1^1(\zeta) + \Big[A_\xi A_3\big(3K_{\frac{7}{2}}(\xi r) + \xi r K_{\frac{5}{2}}(\xi r)\big) \\ &\quad + A_\varphi B_3\big(3K_{\frac{7}{2}}(\varphi r) + \varphi r K_{\frac{5}{2}}(\varphi r)\big) - \frac{C_3}{\eta^2}K_{\frac{7}{2}}(\eta r)\Big]P_3^1(\zeta)\Big\}. \qquad (74)\end{aligned}$$

The couple acting on the spheroid has contributions from the surface stress tensor t_{ij} and couple stress tensor m_{ij} (given by equations (4) and (5), respectively). The couple due to the surface stress is

$$N_z^s = \int_S \vec{r} \wedge (\vec{n} \cdot t) \cdot \vec{k}\, dS\,, \qquad (75)$$

where $\vec{r} = b[1+\upsilon P_2(\zeta)]\vec{e}_r$, $\vec{n} = \vec{e}_r + \frac{3}{2}\upsilon \sin 2\theta \vec{e}_\theta$, $dS = 2\pi b^2[1+2\upsilon P_2(\zeta)] \sin\theta\, d\theta$ to $O(\upsilon)$, and $\vec{k}$ is the unit vector in the direction of the axis of rotation. The integral is taken over the surface of the boundary:

$$N_z^s = 2\pi b^3 \int_0^\pi r^3 \Big(t_{r\phi} + \frac{3}{2}\upsilon t_{\theta\phi}\sin 2\theta\Big)\Big|_{r=1+\upsilon P_2(\zeta)} \sin^2\theta \, d\theta, \tag{76}$$

$$\begin{aligned} N_z^s = {} & -\frac{8}{3}\pi b^3 \Omega_z (2\mu + k)\Big[(a_1 + A_1)\Big(\frac{\omega}{\xi^2}\big(K_{\frac{3}{2}}(\xi) + \xi K_{\frac{1}{2}}(\xi)\big) + K_{\frac{3}{2}}(\xi)\Big) \\ & + (b_1 + B_1)\Big(\frac{\omega}{\varphi^2}\big(K_{\frac{3}{2}}(\varphi) + \varphi K_{\frac{1}{2}}(\varphi)\big) + K_{\frac{3}{2}}(\varphi)\Big) + \frac{k(c_1 + C_1)}{\eta^2(2\mu + k)} K_{\frac{3}{2}}(\eta) \\ & + \frac{\upsilon}{5}\Big\{\Big(\omega K_{\frac{3}{2}}(\xi) - k\frac{1 - 2A_\xi}{2\mu + k}\big(\xi K_{\frac{1}{2}}(\xi) + 3K_{\frac{3}{2}}(\xi)\big)\Big)a_1 \\ & + \Big(\omega K_{\frac{3}{2}}(\varphi) - k\frac{1 - 2A_\varphi}{2\mu + k}\big(\varphi K_{\frac{1}{2}}(\varphi) + 3K_{\frac{3}{2}}(\varphi)\big)\Big)b_1 \\ & - \frac{2kc_1}{\eta^2(2\mu + k)}\big(\eta K_{\frac{1}{2}}(\eta) + 3K_{\frac{3}{2}}(\eta)\big)\Big\}\Big] e^{i\sigma t}. \end{aligned} \tag{77}$$

The couple due to the couple stress is

$$N_z^c = \int_S (\vec{n} \cdot \mathbf{m}) \cdot \vec{k} \, dS \,, \tag{78}$$

this becomes

$$\begin{aligned} N_z^c = {} & 2\pi b^2 \int_0^\pi r^2 \Big[\Big(m_{rr} + \frac{3}{2}\upsilon m_{\theta r}\sin 2\theta\Big)\cos\theta \\ & \qquad - \Big(m_{r\theta} + \frac{3}{2}\upsilon m_{\theta\theta}\sin 2\theta\Big)\sin\theta\Big]\Big|_{r=1+\upsilon P_2(\zeta)} \sin\theta \, d\theta, \end{aligned} \tag{79}$$

$$\begin{aligned} N_z^c = {} & \frac{8}{3}\pi b \Omega_z \gamma \Big\{(a_1 + A_1)\xi^2 A_\xi K_{\frac{3}{2}}(\xi) + (b_1 + B_1)\varphi^2 A_\varphi K_{\frac{3}{2}}(\varphi) \\ & + (c_1 + C_1)\frac{\alpha + \beta + \gamma}{2\gamma} K_{\frac{3}{2}}(\eta) + \frac{\upsilon}{5}\Big[a_1 \xi^2 A_\xi \big(3K_{\frac{3}{2}}(\xi) + \xi K_{\frac{1}{2}}(\xi)\big) \\ & + b_1 \varphi^2 A_\varphi \big(3K_{\frac{3}{2}}(\varphi) + \varphi K_{\frac{1}{2}}(\varphi)\big) \\ & - c_1 \frac{\alpha + \beta + \gamma}{\gamma}\big(3K_{\frac{3}{2}}(\eta) + \eta K_{\frac{1}{2}}(\eta)\big)\Big]\Big\} e^{i\sigma t}. \end{aligned} \tag{80}$$

Then the total couple is

$$N_z = N_z^s + N_z^c\,. \tag{81}$$

The couple N_z as given in (81)

$$N_z = \frac{8}{3}\pi\rho\sigma b^5\Omega_z(-R' - iR)e^{i\sigma t}, \tag{82}$$

where R and R' are real couple coefficients and the real part of this expression is

$$\Re N_z = \frac{8}{3}\pi\rho\sigma b^5\Omega_z(R\sin\sigma t - R'\cos\sigma t). \tag{83}$$

Physically the couple coefficients R and R' represent, respectively, the in-phase and the out-of phase couple oscillations.

1. The case of the rotary oscillations of a sphere is obtained from the above analysis by allowing $\upsilon = 0$. The expressions for N_z^s and N_z^c are then

$$\begin{aligned} N_z^s = {} & \frac{8\pi b^3\Omega_z\lambda\xi^2\varphi^2(2\mu+k)}{3\Delta_4}\Big[k\gamma A_\xi A_\varphi(\xi-\varphi)\big(\eta^2(1+\varphi)(1+\xi) \\ & \times(\xi+\varphi)+\xi^2\varphi^2(\eta^2+2\eta+2)\big)-\big(\xi^2(\eta^2+2\eta+2)[\gamma(1+\xi)+\tau] \\ & +\eta^2\tau(1+\xi)\big)\big((3\mu+2k)(1+\varphi)+\varphi^2(\mu+k)\big)A_\xi \\ & +\big((3\mu+2k)(1+\xi)+\xi^2(\mu+k)\big)\big(\varphi^2(\eta^2+2\eta+2)[\gamma(1+\varphi)+\tau] \\ & +\eta^2\tau(1+\varphi)\big)A_\varphi\Big], \end{aligned} \tag{84}$$

$$\begin{aligned} N_z^c = {} & \frac{8\pi b\Omega_z\lambda\xi^2\varphi^2 A_\xi A_\varphi(\xi-\varphi)}{3\Delta_4}\Big[\eta^2(1+\eta)(\alpha+\beta+\gamma)\big(\gamma(1+\xi) \\ & \times(1+\varphi)(\xi+\varphi)+\tau(\xi+\xi\varphi+\varphi)\big)-\big(\eta^2(1+\varphi)(1+\xi)(\xi+\varphi) \\ & +\xi^2\varphi^2(\eta^2+2\eta+2)\big)\tau\Big]. \end{aligned} \tag{85}$$

Moreover, in the case of no-slip ($\beta_1 \to \infty$ and $\chi \to \infty$), we get the same results as Lakshmana and Bhujanga [32].

2. The case of slow steady rotation of a spheroid with no-slip is obtained also from the above analysis by allowing the period of oscillations $2\pi/\sigma$ tend to infinity. Using

$$\lim_{\sigma\to 0}(\xi^2+\varphi^2)=\xi_1^2 \quad\text{and}\quad \lim_{\sigma\to 0}(\xi^2\varphi^2)=0\,,$$

where $\xi_1^2=kb^2(2\mu+k)/(\gamma(\mu+k))$, so that we take, say $\xi=\xi_1$ and $\varphi=0$. For $\upsilon=0$, the couple reduces to

$$N_z=-\frac{8\pi(2\mu+k)^2\lambda\Omega_z b^3}{\Delta_5}\left[\delta'(\Lambda^2+2\Lambda+2)+\Lambda^2\tau(\mu+k)(1+\xi_1)\right],\quad (86)$$

where $\delta'=\xi_1^2\tau(\mu+k)+k(1+\xi_1)(2\mu+k)$, and

$$\begin{aligned}\Delta_5 &= 2\delta'(\Lambda^2+2\Lambda+2)\big((2\mu+k)(\lambda+2)-\mu\big)-\Lambda^2\big[k\tau\xi_1^2(\mu+k)\\&\quad-(1+\xi_1)(2\mu+k)\{\tau\big((2\mu+k)(\lambda+2)-\mu\big)-k^2\}\big].\end{aligned}$$

For no-slip spheroid ($\beta_2\to\infty$ and $\chi\to\infty$), the hydrodynamic couple is

$$\begin{aligned}N_z &= -\frac{8\pi(2\mu+k)(\mu+k)\Omega_z b^3}{\Delta_6}\Big[\xi_1^2(\Lambda^2+2\Lambda+2)+\Lambda^2(1+\xi_1)\\&\quad-\frac{\upsilon}{5\Delta_6}\Big(3\Lambda^2(1+\xi_1)\Delta_2+6\xi_1^2(\mu+k)(\Lambda^2+2\Lambda+2)\big(\xi_1^2(\Lambda^2+2\Lambda+2)\\&\quad+\Lambda^2(1+\xi_1)\big)+\Lambda^2\xi_1^2k\big(2\xi_1(\Lambda^2+3\Lambda+1)+\Lambda(\Lambda+4)\big)\Big)\Big],\end{aligned}\quad (87)$$

in which

$$\Lambda^2=\frac{2kb^2}{\alpha+\beta+\gamma},\qquad \Delta_6=2\xi_1^2(\mu+k)(\Lambda^2+2\Lambda+2)+\Lambda^2(2\mu+k)(1+\xi_1).$$

3. The case of the unsteady oscillations of a slip spheroid in the classical viscous fluid, is recoverable from equations (81)–(83). The couple N_z^c

arising from the couple stresses is then zero in the limit and the limiting form of the couple is therefore

$$N_z = -\frac{8\pi\mu b^3\Omega_z\beta_3}{3\Delta_7}\Big[\xi_2^2+3\xi_2+3-\frac{\upsilon}{5\Delta_3}\Big(4(\xi_2^2+3\xi_2+3)^2 + \beta_3\big(2(2\xi_2+1)(\xi_2^2+3\xi_2+3)+3\big)\Big)\Big]e^{i\sigma t}, \tag{88}$$

where here

$$\beta_3 = \frac{\beta_1 b}{\mu}, \quad \omega_2 = \frac{\rho\sigma b^2}{\mu}, \quad \xi_2 = (1+i)\left(\frac{\omega_2}{2}\right)^{1/2},$$
$$\Delta_7 = \xi_2^2+3\xi_2+3+\beta_3(1+\xi_2).$$

Moreover, in the case of slow steady rotation the couple becomes

$$N_z = -\frac{8\pi\mu b^3\Omega_z\beta_3}{\beta_3+3}\left[1-\frac{3\upsilon(\beta_3+4)}{5(\beta_3+3)}\right], \tag{89}$$

and this agrees with the result of Chang and Keh [24]. From equation (89), the value of the couple for perfect slip ($\beta_3 = 0$) tends to zero because no fluid is displaced.

4.2 Numerical results

The in-phase and out-of phase real coefficients of the couple R and R' for the rotary oscillation motion of a prolate spheroid and an oblate spheroid are introduced in (83) and plotted in figures 7–11 versus the parameter of the frequency of the oscillations ω_2 and the slip parameters β_3, $\chi_2(=\chi/\mu b)$ for various values of k/μ and υ when the parameters $j/b^2 = 0.2$, $\alpha/\mu b^2 = 0.1$, $\beta/\mu b^2 = 0.2$, and $\gamma/\mu b^2 = 0.3$.

Figure 7 indicates that over the range of the slip parameters $0 \leqslant \chi_2 = \beta_3 < \infty$, the values of the couple coefficients monotonically decrease with the increase of

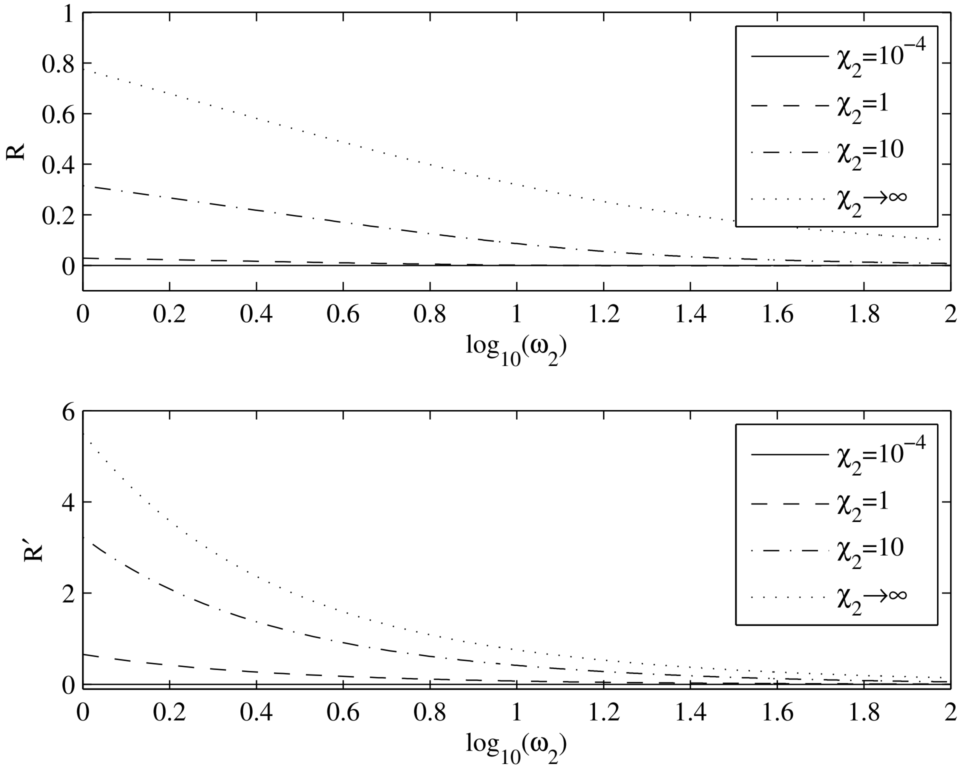

Figure 7: Variation of couple parameters versus the frequency parameter for various values of slip parameter for $\beta_3 = \chi_2$, $\upsilon = 0.1$ and $k/\mu = 2$.

the frequency parameter. Also for the entire range of the frequency parameter, the coefficients R and R' increase with the increase of the slip parameters $\chi_2 = \beta_3$. As seen from figure 8 for $\chi_2 = \beta_3 = 10$, over the entire range of frequency parameter, the coefficients R and R' increase with the increase of the micropolarity parameter. Figure 9 shows that for certain values of $\chi_2 = \beta_3 = 10$ and $k/\mu = 2$ over the entire range of frequency parameter, the coefficients R and R' decrease with the increase of the deformity parameter υ. For a spheroid of a given aspect ratio, the couple parameters monotonically increase with the slip parameters (see figure 11). Figure 10 shows that the

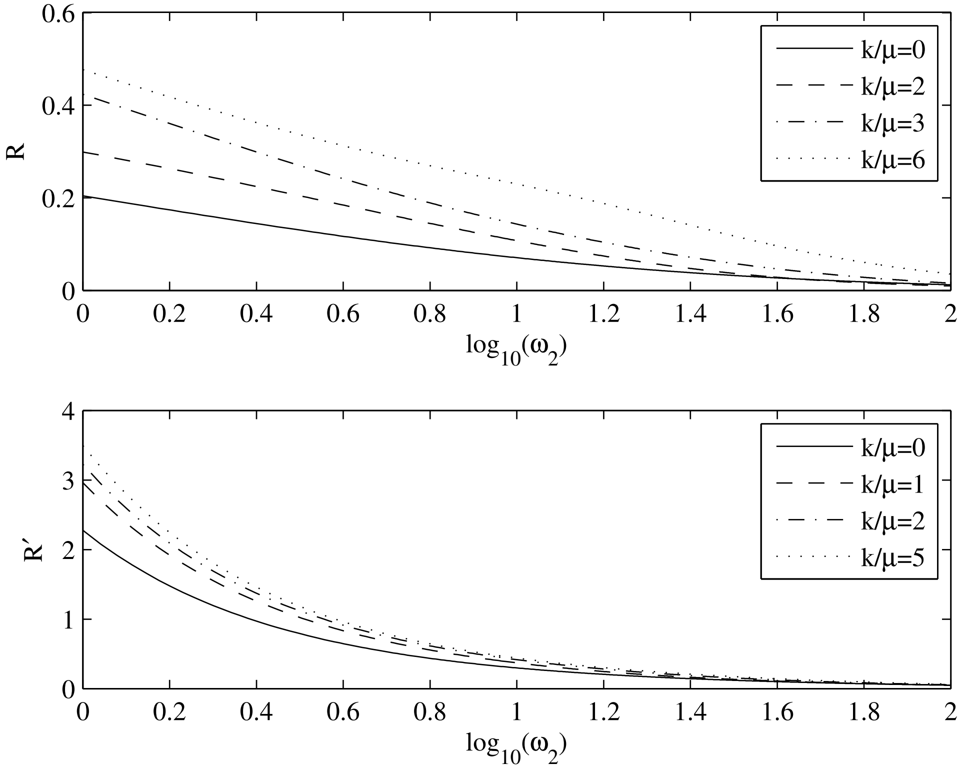

Figure 8: Variation of couple parameters versus the frequency parameter for various values of micropolarity coefficient for $\beta_3 = \chi_2 = 10$ and $\upsilon = 0.1$.

couple coefficients are to be finite in both the perfect slip and no-slip limits. It indicates also that for the entire range of the slip parameters, the couple parameters increases with the increase of micropolarity parameter. The lowest values of the couple coefficients correspond to the case of viscous fluid. For $\upsilon > 0$ (aspect ratio is large), the major portion of the fluid slip at the particle surface occurs in the direction of the particle's movement. However, for $\upsilon < 0$ (aspect ratio becomes small), the main component of the fluid slip at the surface of a spheroidal particle is in the direction normal to the motion of the spheroid. As expected, the couple parameters exerted on the oscillating

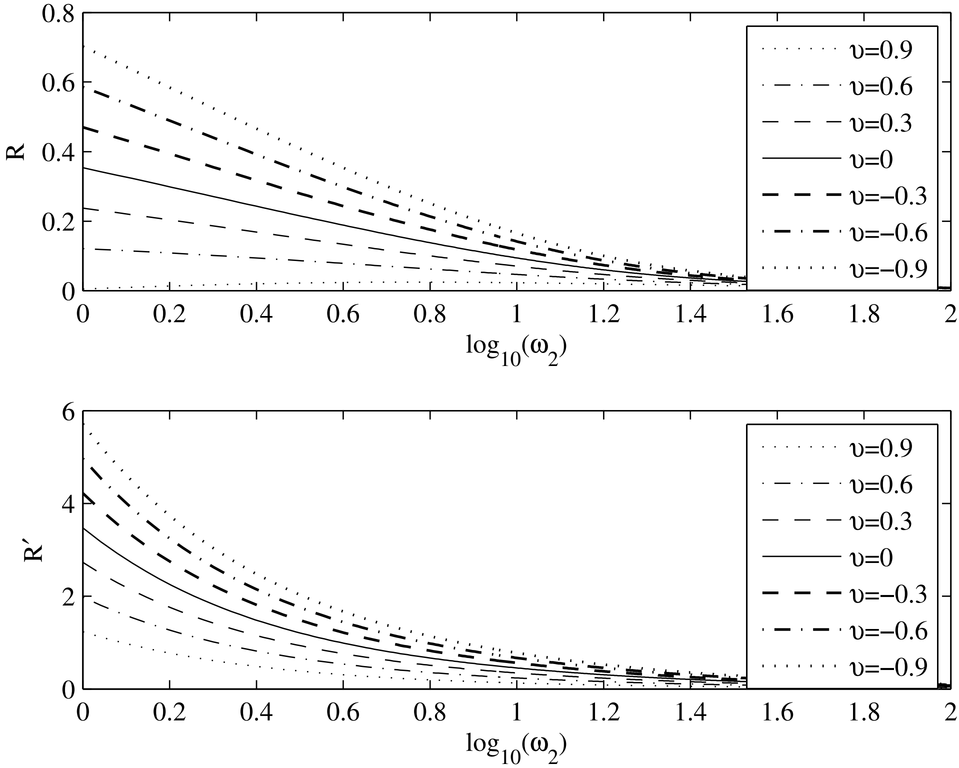

Figure 9: Variation of couple parameters versus the frequency parameter for various values of deformity parameter for $\beta_3 = \chi_2 = 10$ and $k/\mu = 2$.

sphere are smaller than those experienced by an oscillating oblate spheroid and larger than those on a prolate spheroid.

5 Conclusion

In this article, we presented the analytical solution for the hydrodynamically rectilinear and rotary oscillations of a spheroid (prolate and oblate) in a

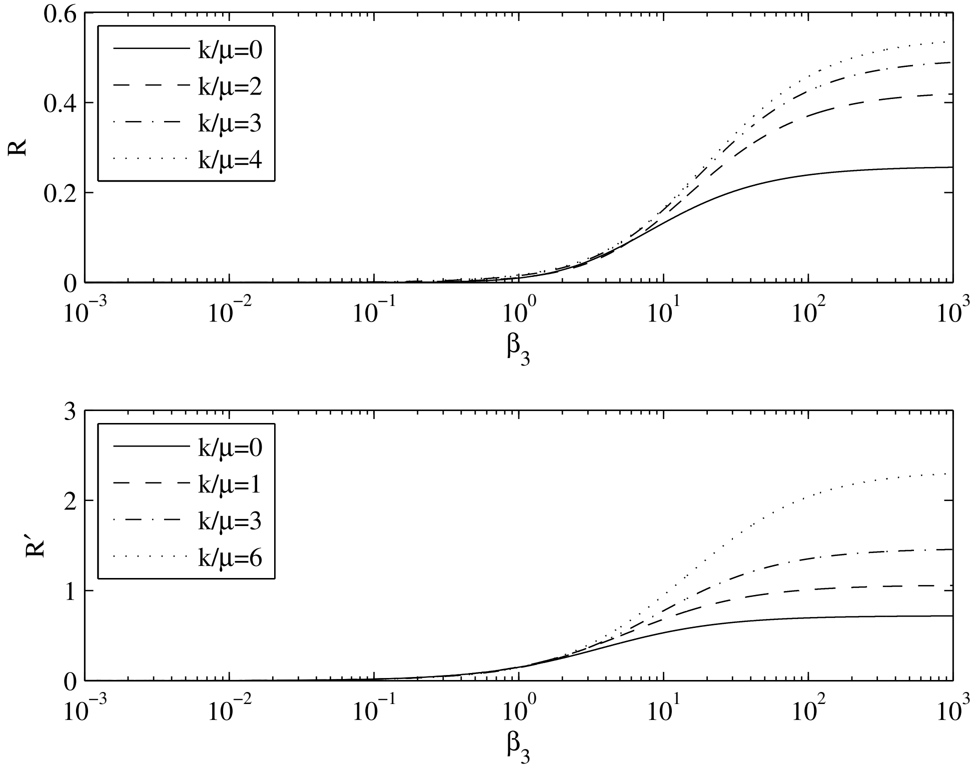

Figure 10: Variation of couple parameters versus the slip parameter for various values of micropolarity coefficient for $\omega_2 = 5$, $\upsilon = 0.1$ and $\chi_2 = 10$.

micropolar fluid, where the fluid may slip at the spheroid surface. For various values of the frequency, slip, micropolarity, and deformity parameters of the prolate or oblate spheroidal particle, our results of the hydrodynamic drag force and couple agree with the available values in the literature.

In addition, the drag force exerted on a spheroid is a monotonically increasing function of the slip and micropolarity parameters, but decreasing as the frequency and deformity parameters are increasing. These coefficients are increasing with an increase in the aspect ratio for a given value of the slip

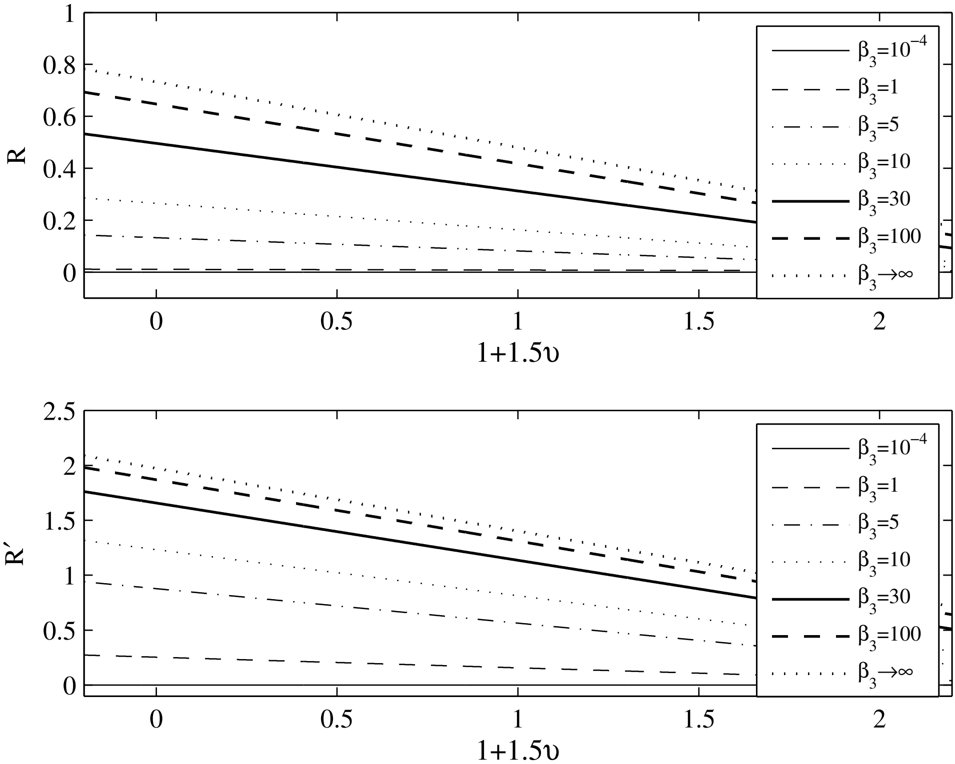

Figure 11: Variation of couple parameters versus the aspect ratio for various values of slip parameter for $\omega_2 = 5$, $k/\mu = 2$ and $\chi_2 = \beta_3$.

parameter. For a spheroid with a fixed aspect ratio, they are monotonically increasing function of the slip coefficients. For each β_2, the value of drag parameters is a monotonic function of the aspect ratio. However, this monotonic function is not the same for each β_2. That is, it may be either decreasing or increasing. As a consequence of our results, we see that the effects of the large but finite values of the slip parameters (greater than about five) on drag force experienced by the oscillating sphere are smaller (or larger) than those for an oscillating prolate (or oblate) spheroid and the reverse occurring for the small values of slip parameter (less than about five).

The hydrodynamic couple acting on the rotary oscillation spheroid decreases monotonically with an increase in the frequency parameter or the deformity parameter (or aspect ratio) for a no-slip or finite-slip spheroid and vanishes for a perfectly slip spheroid. For a spheroid with a fixed aspect ratio, its hydrodynamic couple is a monotonically increasing function of the slip and micropolarity parameters of the spheroid. As expected, the effects of slip on the couple experienced by the oscillating sphere are smaller (or larger) than those for an oscillating oblate (or prolate) spheroid. However, the drag and couple parameters of a micropolar fluid are larger than those of a classical fluid under all circumstances.

A Rectilinear oscillations case

Applying the boundary conditions (29)–(31) to the general solution (27) and (28), for the hydrodynamically rectilinear oscillations of a slip spheroid in a micropolar fluid and using the perturbation method, we obtain the following system of algebraic equations:

$$\begin{aligned}0 = {} & \left[1 + a_2 + b_2 K_{\frac{3}{2}}(\ell) + c_2 K_{\frac{3}{2}}(\kappa)\right] P_1(\zeta) + \Big[2 - a_2 - b_2\big(K_{\frac{3}{2}}(\ell) + \ell K_{\frac{1}{2}}(\ell)\big) \\ & - c_2\big(K_{\frac{3}{2}}(\kappa) + \kappa K_{\frac{1}{2}}(\kappa)\big)\Big] \alpha_m \big(\mathfrak{I}_m(\zeta) P_1(\zeta) + P_{m-1}(\zeta)\mathfrak{I}_2(\zeta)\big) \\ & + \sum_{n=3}^{\infty} \left[A_n + B_n K_{n-\frac{1}{2}}(\ell) + C_n K_{n-\frac{1}{2}}(\kappa)\right] P_{n-1}(\zeta), \end{aligned} \tag{90}$$

$$\begin{aligned}0 = {} & \Big[2\lambda_1 - (\lambda_1+3)a_2 - b_2\big((\omega+\lambda_1+3)K_{\frac{3}{2}}(\ell) + \ell(\lambda_1+1)K_{\frac{1}{2}}(\ell)\big) \\ & - c_2\big((\omega+\lambda_1+3)K_{\frac{3}{2}}(\kappa) + \kappa(\lambda_1+1)K_{\frac{1}{2}}(\kappa)\big)\Big]\mathfrak{I}_2(\zeta) + \alpha_m\Big[\Big(2\lambda_1 + (2\lambda_1 \\ & + 9)a_2 + b_2\big\{\big(\ell^2(\lambda_1+1) + 2\lambda_1 + \omega + 9\big)K_{\frac{3}{2}}(\ell) + \ell(\omega+3)K_{\frac{1}{2}}(\ell)\big\} \\ & + c_2\big\{\big(\kappa^2(\lambda_1+1) + 2\lambda_1 + \omega + 9\big)K_{\frac{3}{2}}(\kappa) + \kappa(\omega+3)K_{\frac{1}{2}}(\kappa)\big\}\Big)\mathfrak{I}_m(\zeta)\mathfrak{I}_2(\zeta) \\ & - 3\Big(3a_2 + b_2\big\{\big(3K_{\frac{3}{2}}(\ell) + \ell K_{\frac{1}{2}}(\ell)\big\} + c_2\big\{3K_{\frac{3}{2}}(\kappa) + \kappa K_{\frac{1}{2}}(\kappa)\big\}\Big)\end{aligned}$$

$$
\begin{aligned}
&\times P_1(\zeta)\mathfrak{I}_2(\zeta)P_{m-1}(\zeta)\Big] + \sum_{n=3}^{\infty}\Big[(1-n)(\lambda_1+n+1)A_n \\
&+ B_n\big\{\big(n(\lambda_1-n+2)-\omega\big)K_{n-\frac{1}{2}}(\ell) - \ell(\lambda_1+1)K_{n+\frac{1}{2}}(\ell)\big\} \\
&+ C_n\big\{\big(n(\lambda_1-n+2)-\omega\big)K_{n-\frac{1}{2}}(\kappa) - \kappa(\lambda_1+1)K_{n+\frac{1}{2}}(\kappa)\big\}\Big]\mathfrak{I}_n(\zeta),
\end{aligned} \tag{91}
$$

$$
\begin{aligned}
0 = {} & \big[b_2A_\ell\big(\gamma\ell K_{\frac{1}{2}}(\ell) + (2\gamma+\beta+\chi)K_{\frac{3}{2}}(\ell)\big) + c_2A_\kappa\big(\gamma\kappa K_{\frac{1}{2}}(\kappa) + (2\gamma \\
&+ \beta+\chi)K_{\frac{3}{2}}(\kappa)\big)\big]\mathfrak{I}_2(\zeta) - \alpha_m\Big\{\Big[b_2A_\ell\big[\gamma(1-\ell^2)K_{\frac{3}{2}}(\ell) - (2\gamma+\beta+\chi) \\
&\times \big(\ell K_{\frac{1}{2}}(\ell) + 3K_{\frac{3}{2}}(\ell)\big)\big] + c_2A_\kappa\big[\gamma(1-\kappa^2)K_{\frac{3}{2}}(\kappa) \\
&- (2\gamma+\beta+\chi)\big(\kappa K_{\frac{1}{2}}(\kappa) + 3K_{\frac{3}{2}}(\kappa)\big)\big]\Big]\mathfrak{I}_m(\zeta)\mathfrak{I}_2(\zeta) + (\gamma-\beta) \\
&\times \big(b_2A_\ell K_{\frac{3}{2}}(\ell) + c_2A_\kappa K_{\frac{3}{2}}(\kappa)\big)P_1(\zeta)\mathfrak{I}_2(\zeta)P_{m-1}(\zeta)\Big\} \\
&+ \sum_{n=3}^{\infty}\big[B_nA_\ell\big(\gamma\ell K_{n+\frac{1}{2}}(\ell) + (\gamma(1-n)+\beta+\chi)K_{n-\frac{1}{2}}(\ell)\big) \\
&+ C_nA_\kappa\big(\gamma\kappa K_{n+\frac{1}{2}}(\kappa) + (\gamma(1-n)+\beta+\chi)K_{n-\frac{1}{2}}(\kappa)\big)\big]\mathfrak{I}_n(\zeta).
\end{aligned} \tag{92}
$$

Solving the leading terms in the above system, we find

$$
a_2 = -\frac{\Delta_1 + 3(1+\lambda_1)\big(A_\ell\delta_1(1+\kappa) - A_\kappa\delta_2(1+\ell)\big)}{\Delta_1}, \tag{93}
$$

$$
b_2 = -\frac{3\ell A_\kappa\delta_2(1+\lambda_1)}{\Delta_1K_{\frac{1}{2}}(\ell)}, \tag{94}
$$

$$
c_2 = \frac{3\kappa A_\ell\delta_1(1+\lambda_1)}{\Delta_1K_{\frac{1}{2}}(\kappa)}, \tag{95}
$$

where

$$
\begin{aligned}
\delta_1 &= \gamma\ell^2 + (2\gamma+\beta+\chi)(1+\ell), \quad \delta_2 = \gamma\kappa^2 + (2\gamma+\beta+\chi)(1+\kappa), \\
\Delta_1 &= [(1+\lambda_1)\kappa^2 + \omega(1+\kappa)]\delta_1A_\ell - [(1+\lambda_1)\ell^2 + \omega(1+\ell)]\delta_2A_\kappa.
\end{aligned}
$$

Using these above values into (90)–(92), we find

$$
0 = \eta_1\alpha_m\big(\mathfrak{I}_m(\zeta)P_1(\zeta) + P_{m-1}(\zeta)\mathfrak{I}_2(\zeta)\big)
$$

$$
\begin{aligned}
&+\sum_{n=3}^{\infty}\left[A_n + B_n K_{n-\frac{1}{2}}(\ell) + C_n K_{n-\frac{1}{2}}(\kappa)\right] P_{n-1}(\zeta), && (96)\\
0 &= \eta_2 \alpha_m \mathfrak{I}_m(\zeta)\mathfrak{I}_2(\zeta) + \eta_3 \alpha_m P_1(\zeta)\mathfrak{I}_2(\zeta)P_{m-1}(\zeta)\\
&+\sum_{n=3}^{\infty}\Big[(1-n)(\lambda_1+n+1)A_n + B_n\big\{\big(n(\lambda_1-n+2)-\omega\big)K_{n-\frac{1}{2}}(\ell)\\
&-\ell(\lambda_1+1)K_{n+\frac{1}{2}}(\ell)\big\} + C_n\big\{\big(n(\lambda_1-n+2)-\omega\big)K_{n-\frac{1}{2}}(\kappa)\\
&-\kappa(\lambda_1+1)K_{n+\frac{1}{2}}(\kappa)\big\}\Big]\mathfrak{I}_n(\zeta), && (97)\\
0 &= \eta_4 \alpha_m \mathfrak{I}_m(\zeta)\mathfrak{I}_2(\zeta) + \eta_5 \alpha_m P_1(\zeta)\mathfrak{I}_2(\zeta)P_{m-1}(\zeta)\\
&+\sum_{n=3}^{\infty}\left[B_n A_\ell\big(\gamma\ell K_{n+\frac{1}{2}}(\ell) + (\gamma(1-n)+\beta+\chi)K_{n-\frac{1}{2}}(\ell)\big)\right.\\
&\left.+ C_n A_\kappa\big(\gamma\kappa K_{n+\frac{1}{2}}(\kappa) + (\gamma(1-n)+\beta+\chi)K_{n-\frac{1}{2}}(\kappa)\big)\right]\mathfrak{I}_n(\zeta). && (98)
\end{aligned}
$$

where

$$
\begin{aligned}
\eta_1 &= -\frac{3\omega\big(A_\ell\delta_1(1+\kappa) - A_\kappa\delta_2(1+\ell)\big)}{\Delta_1},\\
\eta_2 &= 3\omega + \frac{3}{\Delta_1}\Big(\omega(\lambda_1-\omega-2)\big(A_\ell\delta_1(1+\kappa) - A_\kappa\delta_2(1+\ell)\big)\\
&\qquad + (1+\lambda_1^2)\big[\kappa^2 A_\ell\delta_1(1+\kappa) - \ell^2 A_\kappa\delta_2(1+\ell)\big]\Big),\\
\eta_3 &= -3\eta_1, \quad \eta_5 = \frac{3A_\ell A_\kappa\gamma(\gamma-\beta)(1+\lambda_1)(\ell-\kappa)(\ell\kappa+\ell+\kappa)}{\Delta_1},\\
\eta_4 &= \frac{3A_\ell A_\kappa(1+\lambda_1)(\ell-\kappa)}{\Delta_1}\Big(\gamma^2\ell^2\kappa^2 + (\ell+\kappa+\ell\kappa)\big((\beta+\chi+\gamma)^2\\
&\qquad -\gamma(\beta+3\gamma)\big) + \gamma(\beta+\chi+2\gamma)(1+\ell)(1+\kappa)(\kappa+\ell)\Big).
\end{aligned}
$$

To obtain the remaining arbitrary constants in equations (96)–(98), we use the identities

$$
\mathfrak{I}_m(\zeta)\mathfrak{I}_2(\zeta) = \frac{-(m-2)(m-3)}{2(2m-1)(2m-3)}\mathfrak{I}_{m-2}(\zeta) + \frac{m(m-1)}{(2m+1)(2m-3)}\mathfrak{I}_m(\zeta)
$$

$$- \frac{(m+1)(m+2)}{2(2m-1)(2m+1)} \mathfrak{I}_{m+2}(\zeta), \tag{99}$$

$$\mathfrak{I}_m(\zeta)P_1(\zeta) + P_{m-1}(\zeta)\mathfrak{I}_2(\zeta) = \frac{-(m-2)(m-3)}{2(2m-1)(2m-3)} P_{m-3}(\zeta)$$
$$+ \frac{m(m-1)}{(2m+1)(2m-3)} P_{m-1}(\zeta) - \frac{(m+1)(m+2)}{2(2m-1)(2m+1)} P_{m+1}(\zeta), \tag{100}$$

$$\mathfrak{I}_2(\zeta)P_1(\zeta)P_{m-1}(\zeta) = \frac{-(m-1)(m-2)(m-3)}{2(2m-1)(2m-3)} \mathfrak{I}_{m-2}(\zeta)$$
$$+ \frac{m(m-1)}{2(2m+1)(2m-3)} \mathfrak{I}_m(\zeta) + \frac{m(m+1)(m+2)}{2(2m-1)(2m+1)} \mathfrak{I}_{m+2}(\zeta). \tag{101}$$

In solving the system of the equations (96)–(98), we see that

$$B_n = D_n = E_n = 0, \quad \text{if } n \neq m-2, m, m+2, \tag{102}$$

and when $n = m-2, m, m+2$, we have the following system

$$0 = \eta_1 \bar{a}_n + A_n + B_n K_{n-\frac{1}{2}}(\ell) + C_n K_{n-\frac{1}{2}}(\kappa), \tag{103}$$

$$\begin{aligned} 0 = {} & \eta_2 \bar{a}_n + \eta_3 \bar{b}_n + (1-n)(\lambda_1 + n + 1)A_n + B_n \big\{ \big(n(\lambda_1 - n + 2) - \omega\big) \\ & \times K_{n-\frac{1}{2}}(\ell) - \ell(\lambda_1+1)K_{n+\frac{1}{2}}(\ell) \big\} + C_n \big\{ \big(n(\lambda_1 - n + 2) - \omega\big) \\ & \times K_{n-\frac{1}{2}}(\kappa) - \kappa(\lambda_1+1)K_{n+\frac{1}{2}}(\kappa) \big\}, \end{aligned} \tag{104}$$

$$\begin{aligned} 0 = {} & \eta_4 \bar{a}_n + \eta_5 \bar{b}_n + B_n A_\ell \big(\gamma \ell K_{n+\frac{1}{2}}(\ell) + (\gamma(1-n) + \beta + \chi) K_{n-\frac{1}{2}}(\ell)\big) \\ & + C_n A_\kappa \big(\gamma \kappa K_{n+\frac{1}{2}}(\kappa) + (\gamma(1-n) + \beta + \chi) K_{n-\frac{1}{2}}(\kappa)\big), \end{aligned} \tag{105}$$

where

$$\bar{a}_{m-2} = \frac{-\alpha_m (m-3)(m-2)}{2(2m-1)(2m-3)}, \quad \bar{a}_m = \frac{\alpha_m m(m-1)}{(2m+1)(2m-3)},$$
$$\bar{a}_{m+2} = \frac{-\alpha_m (m+1)(m+2)}{2(2m-1)(2m+1)},$$
$$\bar{b}_{m-2} = (m-1)\bar{a}_{m-2}, \quad \bar{b}_m = \bar{a}_m/2, \quad \bar{b}_{m+2} = m\bar{a}_{m+2}.$$

Solving the equations (103)–(105), the individual expressions for B_n, D_n and E_n determined for $n = m-2, m, m+2$.

B Rotary oscillations case

Applying the boundary conditions (68) and (70) to the general solution (64), (66) and (67) for the rotary oscillations of a spheroid with slip and spin surface in a micropolar fluid using the perturbation method, we obtain the following system of algebraic equations:

$$
\begin{aligned}
0 = \Big[&\lambda - a_1\Big(\frac{\omega}{\xi^2}\big(K_{\frac{3}{2}}(\xi) + \xi K_{\frac{1}{2}}(\xi)\big) + (1+\lambda)K_{\frac{3}{2}}(\xi)\Big) - \Big(\frac{\omega}{\varphi^2}\big(K_{\frac{3}{2}}(\varphi) \\
&+ \varphi K_{\frac{1}{2}}(\varphi)\big) + (1+\lambda)K_{\frac{3}{2}}(\varphi)\Big) - \frac{kc_1}{\eta^2(2\mu+k)}K_{\frac{3}{2}}(\eta)\Big]P_1^1(\zeta) \\
&+ \alpha_m\Big\{\Big[a_1\Big(\frac{\omega}{\xi}\big(K_{\frac{5}{2}}(\xi) + \xi K_{\frac{3}{2}}(\xi)\big) + \xi(1+\lambda)K_{\frac{5}{2}}(\xi) - \lambda K_{\frac{3}{2}}(\xi)\Big) \\
&+ b_1\Big(\frac{\omega}{\varphi}\big(K_{\frac{5}{2}}(\varphi) + \varphi K_{\frac{3}{2}}(\varphi)\big) + \varphi(1+\lambda)K_{\frac{5}{2}}(\varphi) - \lambda K_{\frac{3}{2}}(\varphi)\Big) + \frac{kc_1}{\eta(2\mu+k)} \\
&\times K_{\frac{5}{2}}(\eta) + \lambda\Big]P_m(\zeta)P_1^1(\zeta) + \Big[a_1\Big(1 - \frac{2\omega}{\xi^2}\Big)K_{\frac{3}{2}}(\xi) + b_1\Big(1 - \frac{2\omega}{\varphi^2}\Big)K_{\frac{3}{2}}(\varphi) \\
&- \frac{kc_1}{\eta^2(2\mu+k)}\big(2K_{\frac{3}{2}}(\eta) + \eta K_{\frac{1}{2}}(\eta)\big)\Big]P_1(\zeta)P_m^1(\zeta)\Big\} - \sum_{n=2}^{\infty}\Big[\frac{kC_n}{\eta^2(2\mu+k)} \\
&\times K_{n+\frac{1}{2}}(\eta) + A_n\Big(\frac{\omega}{\xi^2}\big(nK_{n+\frac{1}{2}}(\xi) + \xi K_{n-\frac{1}{2}}(\xi)\big) + (1+\lambda)K_{n+\frac{1}{2}}(\xi)\Big) \\
&+ B_n\Big(\frac{\omega}{\varphi^2}\big(nK_{n+\frac{1}{2}}(\varphi) + \varphi K_{n-\frac{1}{2}}(\varphi)\big) + (1+\lambda)K_{n+\frac{1}{2}}(\varphi)\Big)\Big]P_n^1(\zeta), \qquad (106)
\end{aligned}
$$

$$
\begin{aligned}
0 = 2\Big[&A_\xi a_1 K_{\frac{3}{2}}(\xi) + A_\varphi b_1 K_{\frac{3}{2}}(\varphi) - \frac{c_1}{\eta^2}\Big(K_{\frac{3}{2}}(\eta) + \frac{\eta}{2}K_{\frac{1}{2}}(\eta)\Big)\Big]P_1(\zeta) \\
&- \alpha_m\Big\{\Big[2A_\xi a_1\xi K_{\frac{5}{2}}(\xi) + 2A_\varphi b_1\varphi K_{\frac{5}{2}}(\varphi) - \frac{c_1}{\eta}\big(2K_{\frac{5}{2}}(\eta) + \eta K_{\frac{3}{2}}(\eta)\big)\Big] \\
&\times P_m(\zeta)P_1(\zeta) - \Big[A_\xi a_1\big(K_{\frac{3}{2}}(\xi) + \xi K_{\frac{1}{2}}(\xi)\big) + A_\varphi b_1\big(K_{\frac{3}{2}}(\varphi) + \varphi K_{\frac{1}{2}}(\varphi)\big) \\
&- \frac{c_1}{\eta^2}K_{\frac{3}{2}}(\eta)\Big]P_m^1(\zeta)P_1^1(\zeta)\Big\} + \sum_{n=2}^{\infty}\Big[n(n+1)\big(A_\xi A_n K_{n+\frac{1}{2}}(\xi) + A_\varphi B_n
\end{aligned}
$$

$$\times K_{n+\frac{1}{2}}(\varphi)\big) - \frac{C_n}{\eta^2}\big((1+n)K_{n+\frac{1}{2}}(\eta) + \eta K_{n-\frac{1}{2}}(\eta)\big)\Big]P_n(\zeta), \tag{107}$$

$$\begin{aligned}
0 = {} & \Big[A_\xi a_1\big((\tau+2\beta+2\gamma+\gamma\xi^2)K_{\frac{3}{2}}(\xi) + \xi\tau K_{\frac{1}{2}}(\xi)\big) + A_\varphi b_1\big((\tau+2\beta+2\gamma \\
& + \gamma\varphi^2)K_{\frac{3}{2}}(\varphi) + \varphi\tau K_{\frac{1}{2}}(\varphi)\big) - \frac{c_1}{\eta^2}\big((\tau+2\beta+2\gamma)K_{\frac{3}{2}}(\eta) + \eta(\beta+\gamma)K_{\frac{1}{2}}(\eta)\big)\Big] \\
& \times P_1^1(\zeta) - \alpha_m\Big\{\Big[A_\xi a_1\big(\xi(\beta+\gamma)K_{\frac{1}{2}}(\xi) + \big[\xi^2(\tau-\gamma)+3\beta+3\gamma\big]K_{\frac{3}{2}}(\xi) \\
& + \xi(\tau+2\beta+2\gamma+\xi^2\gamma)K_{\frac{5}{2}}(\xi)\big) + A_\varphi b_1\big(\varphi(\beta+\gamma)K_{\frac{1}{2}}(\varphi) + \big[\varphi^2(\tau-\gamma) \\
& + 3\beta+3\gamma\big]K_{\frac{3}{2}}(\varphi) + \varphi(\tau+2\beta+2\gamma+\varphi^2\gamma)K_{\frac{5}{2}}(\varphi)\big) - \frac{c_1}{\eta^2}\big(\eta(\beta+\gamma) \\
& \times K_{\frac{1}{2}}(\eta) + (3+\eta^2)(\beta+\gamma)K_{\frac{3}{2}}(\eta) + \eta(\tau+2\beta+2\gamma)K_{\frac{5}{2}}(\eta)\big)\big)\Big] \\
& \times P_m(\zeta)P_1^1(\zeta) + 3(\beta+\gamma)\Big[A_\xi a_1\big(3K_{\frac{3}{2}}(\xi) + \xi K_{\frac{1}{2}}(\xi)\big) + A_\varphi b_1\big(3K_{\frac{3}{2}}(\varphi) \\
& + \varphi K_{\frac{1}{2}}(\varphi)\big) - \frac{c_1}{\eta^2}\big(\big[3+\frac{\eta^2}{3}\big]K_{\frac{3}{2}}(\eta) + \eta K_{\frac{1}{2}}(\eta)\big)\Big]P_1(\zeta)P_m^1(\zeta)\Big\} \\
& + \sum_{n=2}^{\infty}\Big[A_\xi A_n\big(\big[n[\tau+(n+1)(\beta+\gamma)] + \xi^2\gamma\big]K_{n+\frac{1}{2}}(\xi) + \xi\tau K_{n-\frac{1}{2}}(\xi)\big) \\
& + A_\varphi B_n\big(\big[n[\tau+(n+1)(\beta+\gamma)] + \varphi^2\gamma\big]K_{n+\frac{1}{2}}(\varphi) + \varphi\tau K_{n-\frac{1}{2}}(\varphi)\big) \\
& - \frac{C_n}{\eta^2}\big([\tau+(n+1)(\beta+\gamma)]K_{n+\frac{1}{2}}(\eta) + \eta(\beta+\gamma)K_{n-\frac{1}{2}}(\eta)\big)\Big]P_n^1(\zeta),
\end{aligned} \tag{108}$$

where $\tau = \chi+\beta+\gamma$. Solving the leading terms in the above system, we find

$$a_1 = -\frac{A_\varphi\xi^3\varphi^2\lambda(2\mu+k)}{\Delta_4 K_{\frac{1}{2}}(\xi)}\big[\tau\big(\eta^2(1+\varphi)+\varphi^2(\eta^2+2\eta+2)\big) + \gamma\varphi^3(\eta^2+2\eta+2)\big)\big], \tag{109}$$

$$b_1 = \frac{A_\xi\xi^2\varphi^3\lambda(2\mu+k)}{\Delta_4 K_{\frac{1}{2}}(\varphi)}\big[\tau\big(\eta^2(1+\xi)+\xi^2(\eta^2+2\eta+2)\big) + \gamma\xi^3(\eta^2+2\eta+2)\big)\big], \tag{110}$$

$$c_1 = -\frac{2A_\xi A_\varphi \eta^3 \xi^2 \varphi^2 \lambda(2\mu+k)}{\Delta_4 K_{\frac{1}{2}}(\eta)}\Big[\tau(\varphi-\xi)(\xi+\varphi+\xi\varphi) - \gamma(1+\varphi)(1+\xi)(\xi-\varphi)(\xi+\varphi)\Big], \tag{111}$$

where

$$\Delta_4 = 2k\xi^2\varphi^2 A_\xi A_\varphi(\xi-\varphi)(1+\eta)\big(\gamma(1+\varphi)(1+\xi)(\xi+\varphi)+\tau(\xi+\varphi + \xi\varphi)\big) + (2\mu+k)\Big[\big(\omega(1+\varphi+\varphi^2)+\varphi^2(1+\varphi)(1+\lambda)\big)\big(\xi^2(\eta^2+2\eta+2) \times[\gamma(1+\xi)+\tau]+\eta^2\tau(1+\xi)\big)\xi^2 A_\xi - \big(\omega(1+\xi+\xi^2)+\xi^2(1+\xi) \times(1+\lambda)\big)\big(\varphi^2(\eta^2+2\eta+2)[\gamma(1+\varphi)+\tau]+\eta^2\tau(1+\varphi)\big)\varphi^2 A_\varphi\Big].$$

Using these above values into (106)–(108), we obtain

$$0 = \vartheta_1\alpha_m P_m(\zeta)P_1^1(\zeta) + \vartheta_2\alpha_m P_1(\zeta)P_m^1(\zeta) + \sum_{n=2}^{\infty}\Big[\frac{kC_n}{\eta^2(2\mu+k)}K_{n+\frac{1}{2}}(\eta) + A_n\Big(\frac{\omega}{\xi^2}\big(nK_{n+\frac{1}{2}}(\xi)+\xi K_{n-\frac{1}{2}}(\xi)\big)+(1+\lambda)K_{n+\frac{1}{2}}(\xi)\Big) + B_n\Big(\frac{\omega}{\varphi^2} \times\big(nK_{n+\frac{1}{2}}(\varphi)+\varphi K_{n-\frac{1}{2}}(\varphi)\big)+(1+\lambda)K_{n+\frac{1}{2}}(\varphi)\Big)\Big]P_n^1(\zeta), \tag{112}$$

$$0 = \vartheta_3\alpha_m P_m(\zeta)P_1(\zeta) + \vartheta_4\alpha_m P_m^1(\zeta)P_1^1(\zeta) + \sum_{n=2}^{\infty}\Big[n(n+1)\big(A_\xi A_n K_{n+\frac{1}{2}}(\xi) + A_\varphi B_n K_{n+\frac{1}{2}}(\varphi)\big) - \frac{C_n}{\eta^2}\big((1+n)K_{n+\frac{1}{2}}(\eta)+\eta K_{n-\frac{1}{2}}(\eta)\big)\Big]P_n(\zeta), \tag{113}$$

$$0 = \vartheta_5\alpha_m P_m(\zeta)P_1^1(\zeta) + \vartheta_6\alpha_m P_1(\zeta)P_m^1(\zeta) + \sum_{n=2}^{\infty}\Big[A_\xi A_n\Big(\Big[(n+\chi_1+2)n + \frac{\xi^2\gamma}{\beta+\gamma}\Big]K_{n+\frac{1}{2}}(\xi)+\xi(\chi_1+1)K_{n-\frac{1}{2}}(\xi)\Big) + A_\varphi B_n\Big(\Big[(n+\chi_1+2)n + \frac{\varphi^2\gamma}{\beta+\gamma}\Big]K_{n+\frac{1}{2}}(\varphi)+\varphi(\chi_1+1)K_{n-\frac{1}{2}}(\varphi)\Big) - \frac{C_n}{\eta^2}\big((n+\chi_1+2)K_{n+\frac{1}{2}}(\eta)+\eta K_{n-\frac{1}{2}}(\eta)\big)\Big]P_n^1(\zeta), \tag{114}$$

where

$$\begin{aligned}
\vartheta_1 &= \frac{-\lambda}{\Delta_4}\Big\{2k\xi^2\varphi^2A_\xi A_\varphi(\xi-\varphi)(2+\eta)^2\big(\gamma(1+\varphi)(1+\xi)(\xi+\varphi)\\
&\quad+\tau(\xi+\varphi+\xi\varphi)\big)+(2\mu+k)\Big[\big(\varphi^4(1+\lambda)+\varphi^2(4+3\lambda+\omega)(1+\varphi)\\
&\quad+4\omega(1+\varphi)\big)\big(\xi^2(\eta^2+2\eta+2)[\gamma(1+\xi)+\tau]+\eta^2\tau(1+\xi)\big)\xi^2A_\xi\\
&\quad-\big(\xi^4(1+\lambda)+\xi^2(4+3\lambda+\omega)(1+\xi)+4\omega(1+\xi)\big)\big(\varphi^2(\eta^2+2\eta+2)\\
&\quad\times[\gamma(1+\varphi)+\tau]+\eta^2\tau(1+\varphi)\big)\varphi^2A_\varphi\Big]\Big\},\\
\vartheta_2 &= \frac{\lambda}{\Delta_4}\Big\{2k\xi^2\varphi^2A_\xi A_\varphi(\xi-\varphi)(\eta^2+2\eta+2)\big(\gamma(1+\varphi)(1+\xi)(\xi+\varphi)\\
&\quad+\tau(\xi+\varphi+\xi\varphi)\big)+(2\mu+k)\Big[(2\omega-\varphi^2)(1+\varphi)\big(\xi^2(\eta^2+2\eta+2)[\gamma\\
&\quad\times(1+\xi)+\tau]+\eta^2\tau(1+\xi)\big)\xi^2A_\xi-(2\omega-\xi^2)(1+\xi)\big(\varphi^2(\eta^2+2\eta\\
&\quad+2)[\gamma(1+\varphi)+\tau]+\eta^2\tau(1+\varphi)\big)\varphi^2A_\varphi\Big]\Big\},
\end{aligned}$$

$$\begin{aligned}
\vartheta_3 &= \frac{2\lambda}{\Delta_4}\xi^2\varphi^2A_\xi A_\varphi(\xi-\varphi)(2\mu+k)\big(\eta^3\gamma(1+\varphi)(1+\xi)(\xi+\varphi)+\eta^2\tau\\
&\quad\times(1+\eta)(\xi+\varphi+\xi\varphi)-\gamma\xi^2\varphi^2(\eta^2+2\eta+2)\big),\\
\vartheta_4 &= \frac{\gamma\lambda}{\Delta_4}\xi^2\varphi^2A_\xi A_\varphi(\xi-\varphi)(2\mu+k)\big(\eta^2(1+\varphi)(1+\xi)(\xi+\varphi)\\
&\quad+\xi^2\varphi^2(\eta^2+2\eta+2)\big),\\
\vartheta_5 &= \frac{\lambda}{\Delta_4}\xi^2\varphi^2A_\xi A_\varphi(\xi-\varphi)(2\mu+k)\Big(\xi^2\varphi^2(\eta^2+2\eta+2)\big(\gamma^2(\xi+\varphi+\xi\varphi)\\
&\quad-3\gamma(\beta+\gamma)+\tau[\gamma(\xi+\varphi)+\tau+\gamma]\big)+\eta^2(\beta+\gamma)\big(\gamma(1+\varphi)(1+\xi)\\
&\quad\times(\xi+\varphi)(2\eta-1)+2\tau(1+\eta)(\xi+\varphi+\xi\varphi)\big)+\eta^2\tau\big((1+\varphi)(1+\xi)\\
&\quad\times(\xi+\varphi)(\tau+\gamma)+\xi(1+\varphi)(\xi^2+\varphi^2+\xi\varphi)+\varphi^3\big)\Big),\\
\vartheta_6 &= \frac{\lambda}{\Delta_4}\xi^2\varphi^2A_\xi A_\varphi(\xi-\varphi)(2\mu+k)(\beta+\gamma)\big(2\eta\tau(1+\eta)(\xi+\varphi+\xi\varphi)
\end{aligned}$$

$$+\eta^2\gamma(1+\varphi)(1+\xi)(\xi+\varphi)(2\eta-1)-3\gamma\xi^2\varphi^2(\eta^2+2\eta+2)\big).$$

To solve the above equations for A_n, B_n and C_n ($n \geqslant 2$), we require the identities

$$P_1'(\zeta)P_m(\zeta) = \frac{1}{2m+1}P_{m+1}'(\zeta) - \frac{1}{2m+1}P_{m-1}'(\zeta), \tag{115}$$

$$P_1(\zeta)P_m'(\zeta) = \frac{m}{2m+1}P_{m+1}'(\zeta) + \frac{m+1}{2m+1}P_{m-1}'(\zeta), \tag{116}$$

$$P_1(\zeta)P_m(\zeta) = \frac{m+1}{2m+1}P_{m+1}(\zeta) + \frac{m}{2m+1}P_{m-1}(\zeta), \tag{117}$$

$$P_1^1(\zeta)P_m^1(\zeta) = \frac{-m(m+1)}{2m+1}P_{m+1}(\zeta) + \frac{m(m+1)}{2m+1}P_{m-1}(\zeta), \tag{118}$$

and note that $P_n^1(\zeta) = (1-\zeta^2)^{1/2}P_n'(\zeta)$.

Comparing the terms in equations (112)–(114), and taking all the coefficient A_n, B_n, and C_n are zero except at $n = m-1$ or $n = m+1$, we get

$$\begin{aligned} 0 &= \vartheta_1\bar{c}_n + \vartheta_2\bar{d}_n + \frac{kC_n}{\eta^2(2\mu+k)}K_{n+\frac{1}{2}}(\eta) + A_n\Big(\frac{\omega}{\xi^2}\big(nK_{n+\frac{1}{2}}(\xi) \\ &\quad + \xi K_{n-\frac{1}{2}}(\xi)\big) + (1+\lambda)K_{n+\frac{1}{2}}(\xi)\Big) + B_n\Big(\frac{\omega}{\varphi^2}\big(nK_{n+\frac{1}{2}}(\varphi) \\ &\quad + \varphi K_{n-\frac{1}{2}}(\varphi)\big) + (1+\lambda)K_{n+\frac{1}{2}}(\varphi)\Big), \end{aligned} \tag{119}$$

$$\begin{aligned} 0 &= \vartheta_3\bar{e}_n + \vartheta_4\bar{f}_n + n(n+1)\big(A_\xi A_n K_{n+\frac{1}{2}}(\xi) + A_\varphi B_n K_{n+\frac{1}{2}}(\varphi)\big) - \frac{C_n}{\eta^2} \\ &\quad \times \big((1+n)K_{n+\frac{1}{2}}(\eta) + \eta K_{n-\frac{1}{2}}(\eta)\big), \end{aligned} \tag{120}$$

$$\begin{aligned} 0 &= \vartheta_5\bar{c}_n + \vartheta_6\bar{d}_n + A_\xi A_n\Big(\big((n+\chi_1+2)n + \frac{\xi^2\gamma}{\beta+\gamma}\big)K_{n+\frac{1}{2}}(\xi) + \xi(\chi_1+1) \\ &\quad \times K_{n-\frac{1}{2}}(\xi)\Big) + A_\varphi B_n\Big(\big((n+\chi_1+2)n + \frac{\varphi^2\gamma}{\beta+\gamma}\big)K_{n+\frac{1}{2}}(\varphi) + \varphi(\chi_1+1) \\ &\quad \times K_{n-\frac{1}{2}}(\varphi)\Big) - \frac{C_n}{\eta^2}\big((n+\chi_1+2)K_{n+\frac{1}{2}}(\eta) + \eta K_{n-\frac{1}{2}}(\eta)\big), \end{aligned} \tag{121}$$

where

$$\bar{c}_{m-1} = \frac{-\alpha_m}{2m+1},\ \bar{c}_{m+1} = \frac{\alpha_m}{2m+1}, \quad \bar{d}_{m-1} = -(m+1)\bar{c}_{m-1},\ \bar{d}_{m+1} = m\bar{c}_{m+1},$$

$$\bar{e}_{m-1} = \frac{m\alpha_m}{2m+1},\ \bar{e}_{m+1} = \frac{(m+1)\alpha_m}{2m+1},\ \bar{f}_{m-1} = (m+1)\bar{e}_{m-1},\ \bar{f}_{m+1} = -m\bar{e}_{m+1}.$$

Finally, solving the above equations, we get the expressions for A_n, B_n and C_n when $n = m-1, m+1$.

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Author addresses

1. **H. H. Sherief**, Department of Mathematics, Faculty of Science, Alexandria University, Alexandria, Egypt
mailto:hanysherief@gmail.com
2. **M. S. Faltas**, Department of Mathematics, Faculty of Science, Alexandria University, Alexandria, Egypt
mailto:faltas50@yahoo.com
3. **E. I. Saad**, Department of Mathematics, Faculty of Science, Damanhour University, Damanhour, Egypt
mailto:elsayedsaad74@yahoo.com