On second order duality for nondifferentiable minimax fractional programming problems involving type-I functions

S. K. Gupta\textsuperscript{1}  D. Dangar\textsuperscript{2}  I. Ahmad\textsuperscript{3}

(Received 17 December 2013; revised 11 September 2014)

Abstract

We introduce second order \((C, \alpha, \rho, d)\) type-I functions and formulate a second order dual model for a nondifferentiable minimax fractional programming problem. The usual duality relations are established under second order \((F, \alpha, \rho, d)/(C, \alpha, \rho, d)\) type-I assumptions. By citing a nontrivial example, it is shown that a second order \((C, \alpha, \rho, d)\) type-I function need not be \((F, \alpha, \rho, d)\) type-I. Several known results are obtained as special cases.

\hspace{1cm}

\url{http://journal.austms.org.au/ojs/index.php/ANZIAMJ/article/view/7809} gives this article, © Austral. Mathematical Soc. 2014. Published November 17, 2014, as part of the Proceedings of the 11th Biennial Engineering Mathematics and Applications Conference. issn 1446-8735. (Print two pages per sheet of paper.) Copies of this article must not be made otherwise available on the internet; instead link directly to this URL for this article.
1 Introduction

An optimization problem in which the objective function is the ratio of two functions is a fractional programming problem. It has a wide number of applications in engineering and economics where a ratio of physical or economic functions must be minimised to measure the efficiency or productivity of the system. In mathematical programming, optimization problems in which both a minimization and maximization process is performed are known as minimax (or minmax) problems. Du and Pardalos [5] provided theory, algorithms and applications of some minimax problems. Schmitendorf [13] formulated the following static minimax problem and established necessary optimality conditions:

\[
\text{minimise } f(x) = \sup_{y \in Y} \phi(x, y) \quad \text{subject to } x \in X \subset \mathbb{R}^n,
\]

where \( \phi : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}, \ g : \mathbb{R}^n \to \mathbb{R}^m \) are twice continuously differentiable functions, \( Y \) is a subset of \( \mathbb{R}^l \) and \( X = \{ x \in \mathbb{R}^n : g(x) \leq 0 \} \).

Several different minimax fractional programming problems have been studied and duality relations were obtained under various generalized convexity assumptions [3, 7, 8, 9]. Hachimi and Aghezzaf [6] introduced second order \((F, \alpha, \rho, d)\) type-I functions which generalize convexity. Later, Ahmad et al. [2] formulated a second order dual model for a nondifferentiable minimax
programming problem and proved duality relations under \((F, \alpha, \rho, d)\) type-I functions. Recently, Sharma and Gulati [14] discussed duality results for a minimax fractional programming problem using type-I univex functions.

We first introduce second order \((C, \alpha, \rho, d)\) type-I functions. A numerical non-trivial example illustrates the existence of such functions. We then formulate a second order dual model involving a vector \(r \in \mathbb{R}^n\) for a nondifferentiable multiobjective fractional programming problem and established weak, strong and strict converse duality theorems under second order \((F, \alpha, \rho, d)/(C, \alpha, \rho, d)\) type-I functions.

## 2 Preliminaries

Throughout this article, gradients and Hessian matrices of the functions \(f, g, h, \) and \(\phi\) are with respect to the variable \(x\). For instance, \(\nabla f(x, y)\) means \(\nabla_x f(x, y)\). Here, \(\mathbb{R}^n\) denotes the \(n\) dimensional Euclidean space, \(\mathbb{R}_+\) is the set of nonnegative real numbers and \(M = \{1, 2, \ldots, m\}\).

**Definition 1** (Ahmad et al. [2]). A functional \(F : X \times X \times \mathbb{R}^n \mapsto \mathbb{R}\), where \(X \subseteq \mathbb{R}^n\), is sublinear with respect to the third variable if for all \((x, z) \in X \times X\)

- \(F_{x,z}(a_1 + a_2) \leq F_{x,z}(a_1) + F_{x,z}(a_2)\) for all \(a_1, a_2 \in \mathbb{R}^n\); and
- \(F_{x,z}(\alpha a) = \alpha F_{x,z}(a)\) for all \(\alpha \in \mathbb{R}_+\) and \(a \in \mathbb{R}^n\).

We now rewrite the definition of second order \((F, \alpha, \rho, d)\) type-I functions introduced by Hachimi and Aghezzaf [6]. Let \(F\) be a sublinear functional with respect to the third variable, \(\alpha^1, \alpha^2 : X \times X \rightarrow \mathbb{R}_+ \setminus \{0\}\), \(d : X \times X \rightarrow \mathbb{R}_+\) and \(\rho^1, \rho^2_j \in \mathbb{R}\) for \(j \in M\). Let \(\phi : X \rightarrow \mathbb{R}\) and \(g_j : X \rightarrow \mathbb{R}\) for \(j \in M\) be twice differentiable functions.

**Definition 2** (Hachimi and Aghezzaf [6]). Function \((\phi, g)\) is second order \((F, \alpha, \rho, d)\) type-I at \(z \in X\) if for all \(x \in X\) there exists \(p \in \mathbb{R}^n\) such
that
\[
\phi(x) - \phi(z) + \frac{1}{2} p^T \nabla^2 \phi(z) p \geq F_{x,z}(\alpha^1(x, z)[\nabla \phi(z) + \nabla^2 \phi(z) p]) + \rho^1 d(x, z),
\]
\[
-g_j(z) + \frac{1}{2} p^T \nabla^2 g_j(z) p \geq F_{x,z}(\alpha^2(x, z)[\nabla g_j(z) + \nabla^2 g_j(z) p]) + \rho^2_j d(x, z),
\]
for each \( j \in M \).

**Definition 3.** Function \((\phi, g)\) is semistrictly second order \((F, \alpha, \rho, d)\) type-I at \( z \in X \) if for all \( x \in X \) there exists \( p \in \mathbb{R}^n \) such that
\[
\phi(x) - \phi(z) + \frac{1}{2} p^T \nabla^2 \phi(z) p \geq F_{x,z}(\alpha^1(x, z)[\nabla \phi(z) + \nabla^2 \phi(z) p]) + \rho^1 d(x, z),
\]
\[
-g_j(z) + \frac{1}{2} p^T \nabla^2 g_j(z) p \geq F_{x,z}(\alpha^2(x, z)[\nabla g_j(z) + \nabla^2 g_j(z) p]) + \rho^2_j d(x, z),
\]
for each \( j \in M \).

Yuan et al. [15] introduced \((C, \alpha, \rho, d)\) convexity and proved necessary and sufficient optimality conditions for a nondifferentiable multiobjective fractional programming problem. In the framework of this definition, Chinchuluun et al. [4] studied nonsmooth multiobjective fractional programming problems. Later, Long [12] established duality relations for a class of nondifferentiable multiobjective fractional programming problems involving \((C, \alpha, \rho, d)\) convex functions.

We now present \((C, \alpha, \rho, d)\) type-I functions, after defining convexity in the function \( C \).

**Definition 4** (Yuan et al. [15]). A function \( C : X \times X \times \mathbb{R}^n \to \mathbb{R} \) is convex on \( \mathbb{R}^n \) iff for any fixed \((x, z) \in X \times X\) and for any \( y_1, y_2 \in \mathbb{R}^n \),
\[
C_{x,z}[\lambda y_1 + (1 - \lambda) y_2] \leq \lambda C_{x,z}(y_1) + (1 - \lambda) C_{x,z}(y_2),
\]
for all \( \lambda \in (0, 1) \).

Suppose the real valued function \( d : X \times X \to \mathbb{R}_+ \) satisfies \( d(x, z) = 0 \) iff \( x = z \) and let \( C : X \times X \times \mathbb{R}^n \to \mathbb{R} \) be a convex function such that \( C_{x,z}(0) = 0 \) for any \((x, z) \in X \times X\).
**Definition 5.** Function \((\phi, g)\) is second order \((C, \alpha, \rho, d)\) type-I at \(z \in X\) if for all \(x \in X\) there exists \(p \in \mathbb{R}^n\) such that

\[
\frac{1}{\alpha^1(x, z)} [\phi(x) - \phi(z) + \frac{1}{2} p^T \nabla^2 \phi(z) p] \geq C_{x, z}[\nabla \phi(z) + \nabla^2 \phi(z) p] + \frac{\rho^1 d(x, z)}{\alpha^1(x, z)},
\]

\[
\frac{1}{\alpha^2(x, z)} [-g_j(z) + \frac{1}{2} p^T \nabla^2 g_j(z) p] \geq C_{x, z}[\nabla g_j(z) + \nabla^2 g_j(z) p] + \frac{\rho^2 d(x, z)}{\alpha^2(x, z)},
\]

for each \(j \in M\).

**Definition 6.** Function \((\phi, g)\) is semistrictly second order \((C, \alpha, \rho, d)\) type-I at \(z \in X\) if for all \(x \in X\) there exists \(p \in \mathbb{R}^n\) such that

\[
\frac{1}{\alpha^1(x, z)} [\phi(x) - \phi(z) + \frac{1}{2} p^T \nabla^2 \phi(z) p] > C_{x, z}[\nabla \phi(z) + \nabla^2 \phi(z) p] + \frac{\rho^1 d(x, z)}{\alpha^1(x, z)},
\]

\[
\frac{1}{\alpha^2(x, z)} [-g_j(z) + \frac{1}{2} p^T \nabla^2 g_j(z) p] \geq C_{x, z}[\nabla g_j(z) + \nabla^2 g_j(z) p] + \frac{\rho^2 d(x, z)}{\alpha^2(x, z)},
\]

for each \(j \in M\).

Function \((\phi, g)\) is (semistrictly) second order \((F, \alpha, \rho, d)/(C, \alpha, \rho, d)\) type-I over \(X\) iff it is (semistrictly) second order \((F, \alpha, \rho, d)/(C, \alpha, \rho, d)\) type-I at every point in \(X\).

**Remark 7.** If \(C\) is sublinear with respect to the third variable, then Definitions 5 and 6 are identical to Definitions 2 and 3, respectively.

**Remark 8.** Since the functional \(F\) is sublinear with respect to the third variable, it is convex, as defined in Definition 4. Further, since \(\alpha^1, \alpha^2 > 0\), every \((F, \alpha, \rho, d)\) type-I function is \((C, \alpha, \rho, d)\) type-I. But the converse need not be true. This is seen from the following example.

**Example 9.** Let \(X = \mathbb{R}\). Let \(\phi : X \to \mathbb{R}\) and \(g : X \to \mathbb{R}\) where \(\phi(x) = x^2 - 2 \sin^2 x\) and \(g(x) = \cos^2 x - 2x\). Suppose \(\alpha^1, \alpha^2 : X \times X \to \mathbb{R}_+ \setminus \{0\}\) and \(C : X \times X \times \mathbb{R}^n \to \mathbb{R}\) are \(\alpha^1(x, z) = 1/20\), \(\alpha^2(x, z) = 1/3\) and \(C_{x, z}(a) = \alpha^2/24\). Let \(d : X \times X \to \mathbb{R}_+\) be \(d(x, z) = (x - z)^2\). For \(p = -1\), \(\rho^1 = -1/20\), \(\rho^2 = -1\).
2 Preliminaries

and \( z = 0.5\pi \),
\[
\frac{1}{\alpha^1(x, z)} [\phi(x) - \phi(z) + \frac{1}{2}p^T \nabla^2 \phi(z)p] - C_{x,z} [\nabla \phi(z) + \nabla^2 \phi(z)p] - \frac{\rho^1 d(x, z)}{\alpha^1(x, z)}
\]
\[
= 20x^2 + 40 \cos^2 x + 60 - 5\pi^2 - \frac{1}{24} (\pi - 6)^2 + (x - 0.5\pi)^2 \geq 0,
\]
for all \( x \in X \), and
\[
\frac{1}{\alpha^2(x, z)} [-g(z) + \frac{1}{2}p^T \nabla^2 g(z)p] - C_{x,z} [\nabla g(z) + \nabla^2 g(z)p] - \frac{\rho^2 d(x, z)}{\alpha^2(x, z)}
\]
\[
= \frac{7}{3} + 3\pi + 3(x - 0.5\pi)^2 \geq 0,
\]
for all \( x \in X \). Hence, \((\phi, g)\) is second order \((C, \alpha, \rho, d)\) type-I but \((\phi, g)\) is not second order \((F, \alpha, \rho, d)\) type-I at \( z = 0.5\pi \) as \( C \) is not sublinear with respect to the third argument.

For \( f : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R} \), \( h : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R} \) and \( g : \mathbb{R}^n \to \mathbb{R}^m \) twice continuously differentiable functions, consider the nondifferentiable minimax fractional programming problem (PP):

\[
\text{minimise} \quad \psi(x) = \sup_{y \in Y} \frac{f(x, y) + (x^T Bx)^{1/2}}{h(x, y) - (x^T Dx)^{1/2}} \quad \text{subject to} \quad g(x) \leq 0,
\]

where \( Y \) is a compact subset of \( \mathbb{R}^l \), \( B \) and \( D \) are \( n \times n \) positive semidefinite matrices, \( f(x, y) + (x^T Bx)^{1/2} \geq 0 \) and \( h(x, y) - (x^T Dx)^{1/2} > 0 \) for each \( (x, y) \in \mathcal{J} \times Y \), where \( \mathcal{J} = \{ x \in \mathbb{R}^n : g(x) \leq 0 \} \). For each \( (x, y) \in \mathcal{J} \times Y \) we define

\[
J(x) = \{ j \in M : g_j(x) = 0 \},
\]

\[
Y(x) = \left\{ y \in Y : \frac{f(x, y) + (x^T Bx)^{1/2}}{h(x, y) - (x^T Dx)^{1/2}} = \sup_{z \in \tilde{Y}} \frac{f(x, z) + (x^T Bx)^{1/2}}{h(x, z) - (x^T Dx)^{1/2}} \right\},
\]

\[
K(x) = \left\{ (s, t, \tilde{y}) \in \mathbb{N} \times \mathbb{R}_+^s \times \mathbb{R}^{ls} : 1 \leq s \leq n + 1 \], \quad t = (t_1, t_2, \ldots, t_s) \in \mathbb{R}_+^s, \quad \sum_{i=1}^s t_i = 1, \quad \tilde{y} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_s), \tilde{y}_i \in Y(x), i = 1, 2, \ldots, s \right\}.
\]
3 Duality model

Consider the dual problem (DP) to the PP:

\[
\max_{(s,t,\tilde{y}) \in K(z)} \sup_{(z,\mu,\lambda,w,v,r,p) \in H_1(s,t,\tilde{y})} \lambda,
\]

where \( H_1(s, t, \tilde{y}) \) denotes the set of all \((z, \mu, \lambda, w, v, r, p) \in \mathbb{R}^n \times \mathbb{R}_+^m \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n\) satisfying

\[
\sum_{i=1}^s t_i I(z, \tilde{y}_i) + \nabla^2 \sum_{i=1}^s t_i [f(z, \tilde{y}_i) - \lambda h(z, \tilde{y}_i)]p + \sum_{j=1}^m \mu_j \nabla g_j(z)

+ \nabla^2 \sum_{j=1}^m \mu_j g_j(z)p = 0, \tag{1}
\]

\[
\sum_{i=1}^s t_i G(z, \tilde{y}_i) + \left[ \sum_{i=1}^s t_i I(z, \tilde{y}_i) \right]^T r - \frac{1}{2} p^T \nabla^2 \sum_{i=1}^s t_i [f(z, \tilde{y}_i) - \lambda h(z, \tilde{y}_i)]p \geq 0, \tag{2}
\]

\[
\sum_{j=1}^m \mu_j g_j(z) + \left[ \sum_{j=1}^m \mu_j \nabla g_j(z) \right]^T r - \frac{1}{2} p^T \nabla^2 \sum_{j=1}^m \mu_j g_j(z)p \geq 0, \tag{3}
\]

\[
\left[ \sum_{i=1}^s t_i I(z, \tilde{y}_i) \right]^T r + \left( \sum_{j=1}^m \mu_j \nabla g_j(z) \right)^T r \leq 0, \tag{4}
\]

\[
w^T Bw \leq 1 \quad \text{and} \quad v^T Dv \leq 1, \tag{5}
\]

where

\[
I(z, \tilde{y}_i) = \nabla f(z, \tilde{y}_i) + Bw - \lambda [\nabla h(z, \tilde{y}_i) - Dv],
\]

\[
G(z, \tilde{y}_i) = f(z, \tilde{y}_i) + z^T Bw - \lambda [h(z, \tilde{y}_i) - z^T Dv].
\]

If, for a triplet \((s, t, \tilde{y}) \in K(z)\), the set \( H_1(s, t, \tilde{y}) = \emptyset \), then we define the supremum over \( H_1 \) to be \(-\infty\). Now, we establish the duality relations between PP and DP.
Theorem 10 (Weak duality). Let \( x \) and \((z, \mu, \lambda, w, v, s, t, \tilde{y}, r, p)\) be feasible solutions of PP and DP, respectively. Assume that any one of the following four conditions hold:

1. \( \{G(\cdot, \tilde{y}_i), g_j(\cdot), i = 1, 2, \ldots, s, j = 1, 2, \ldots, m\} \) is second order \((F, \alpha, \rho, d)\) type-I at \( z \) and \( \sum_{i=1}^{s} t_i \rho_i^1 + \sum_{j=1}^{m} \mu_j \rho_j^2 \geq 0 \);

2. \( \{\sum_{i=1}^{s} t_i G(\cdot, \tilde{y}_i), g_j(\cdot), j = 1, 2, \ldots, m\} \) is second order \((F, \alpha, \rho, d)\) type-I at \( z \) and \( \rho^1 + \sum_{j=1}^{m} \mu_j \rho_j^2 \geq 0 \);

3. \( \{G(\cdot, \tilde{y}_i), g_j(\cdot), i = 1, 2, \ldots, s, j = 1, 2, \ldots, m\} \) is second order \((C, \alpha, \rho, d)\) type-I at \( z \) and \( \sum_{i=1}^{s} t_i \rho_i^1 + \sum_{j=1}^{m} \mu_j \rho_j^2 \geq 0 \);

4. \( \{\sum_{i=1}^{s} t_i G(\cdot, \tilde{y}_i), g_j(\cdot), j = 1, 2, \ldots, m\} \) is second order \((C, \alpha, \rho, d)\) type-I at \( z \) and \( \rho^1 + \sum_{j=1}^{m} \mu_j \rho_j^2 \geq 0 \).

Furthermore, suppose \( \alpha^1(x, z) = \alpha^2(x, z) \), then

\[
\sup_{\tilde{y} \in Y} \frac{f(x, \tilde{y}) + (x^T B x)^{1/2}}{h(x, \tilde{y}) - (x^T D x)^{1/2}} \geq \lambda.
\]

Proof: Suppose, contrary to the theorem,

\[
\sup_{\tilde{y} \in Y} \frac{f(x, \tilde{y}) + (x^T B x)^{1/2}}{h(x, \tilde{y}) - (x^T D x)^{1/2}} < \lambda,
\]

then,

\[
f(x, \tilde{y}_i) + (x^T B x)^{1/2} - \lambda [h(x, \tilde{y}_i) - (x^T D x)^{1/2}] < 0,
\]

for all \( \tilde{y}_i \in Y(x) \) with \( i = 1, 2, \ldots, s \). It follows from \( t_i \geq 0, i = 1, 2, \ldots, s \), that

\[
t_i \{f(x, \tilde{y}_i) + (x^T B x)^{1/2} - \lambda [h(x, \tilde{y}_i) - (x^T D x)^{1/2}]\} \leq 0,
\]

with at least one strict inequality, since \( t = (t_1, t_2, \ldots, t_s) \neq 0 \). Taking the summation over \( i \) and using (5),

\[
\sum_{i=1}^{s} t_i \{f(x, \tilde{y}_i) + x^T B w - \lambda [h(x, \tilde{y}_i) - x^T D v]\} = \sum_{i=1}^{s} t_i G(x, \tilde{y}_i) < 0. \tag{6}
\]
3 Duality model

**Condition 1:** By the second order \((F, \alpha, \rho, d)\) type-I assumption on \(\{G(\cdot, \tilde{y}_i), g_j(\cdot) : i = 1, 2, \ldots, s, j = 1, 2, \ldots, m\}\) at \(z\), for \(i = 1, 2, \ldots, s\),

\[
G(x, \tilde{y}_i) - G(z, \tilde{y}_i) + \frac{1}{2} p^T \nabla^2 [f(z, \tilde{y}_i) - \lambda h(z, \tilde{y}_i)] p
\]

\[
\geq F_{x,z} (\alpha^1(x, z)[I(z, \tilde{y}_i) + \nabla^2 [f(z, \tilde{y}_i) - \lambda h(z, \tilde{y}_i)] p]) + \rho^1_i d(x, z),
\]

and, for \(j = 1, 2, \ldots, m\),

\[
g_j(z) - \frac{1}{2} p^T \nabla^2 g_j(z) p \geq F_{x,z} (\alpha^2(x, z)[\nabla g_j(z) + \nabla^2 g_j(z) p]) + \rho^2_j d(x, z).
\]

Multiplying (7) by \(t_i \geq 0, i = 1, 2, \ldots, s\), multiplying (8) by \(\mu_j \geq 0, j = 1, 2, \ldots, m\), taking summations over \(i\) and \(j\) and using the sublinearity of \(F\), we obtain

\[
\sum_{i=1}^{s} t_i G(x, \tilde{y}_i) - \sum_{i=1}^{s} t_i G(z, \tilde{y}_i) + \frac{1}{2} p^T \nabla^2 \sum_{i=1}^{s} t_i [f(z, \tilde{y}_i) - \lambda h(z, \tilde{y}_i)] p
\]

\[
\geq F_{x,z} \left[ \alpha^1(x, z) \left( \sum_{i=1}^{s} t_i I(z, \tilde{y}_i) + \nabla^2 \sum_{i=1}^{s} t_i [f(z, \tilde{y}_i) - \lambda h(z, \tilde{y}_i)] p \right) \right]
\]

\[
+ \sum_{i=1}^{s} t_i \rho^1_i d(x, z),
\]

\[
- \sum_{j=1}^{m} \mu_j g_j(z) + \frac{1}{2} p^T \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z) p
\]

\[
\geq F_{x,z} \left[ \alpha^2(x, z) \left( \sum_{j=1}^{m} \mu_j \nabla g_j(z) + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z) p \right) \right] + \sum_{j=1}^{m} \mu_j \rho^2_j d(x, z).
\]
Now, using (2), (4) and (6) in (9) and (3) in (10),

$$ F_{x,z} \left[ \alpha^1(x, z) \left( \sum_{i=1}^{s} t_i I(z, \tilde{y}_i) + \nabla^2 \sum_{i=1}^{s} t_i[f(z, \tilde{y}_i) - \lambda h(z, \tilde{y}_i)]p \right) \right] $$

$$ + \sum_{i=1}^{s} t_i \rho_i^1 d(x, z) < - \left[ \sum_{j=1}^{m} \mu_j \nabla g_j(z) \right]^T r, \quad (11) $$

and

$$ F_{x,z} \left[ \alpha^2(x, z) \left( \sum_{j=1}^{m} \mu_j \nabla g_j(z) + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z)p \right) \right] + \sum_{j=1}^{m} \mu_j \rho_j^2 d(x, z) $$

$$ \leq \left[ \sum_{j=1}^{m} \mu_j \nabla g_j(z) \right]^T r. \quad (12) $$

Finally, using $\alpha^1(x, z) = \alpha^2(x, z) > 0$, in the addition of (11) and (12) and from the sublinearity of $F$, $\sum_{i=1}^{s} t_i \rho_i^1 + \sum_{j=1}^{m} \mu_j \rho_j^2 \geq 0$ and (1), we have

$$ 0 = F_{x,z}(0) = F_{x,z} \left( \sum_{i=1}^{s} t_i I(z, \tilde{y}_i) + \nabla^2 \sum_{i=1}^{s} t_i[f(z, \tilde{y}_i) - \lambda h(z, \tilde{y}_i)]p \right) $$

$$ + \sum_{j=1}^{m} \mu_j \nabla g_j(z) + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z)p \right) $$

$$ < - \left( \sum_{i=1}^{s} t_i \rho_i^1 + \sum_{j=1}^{m} \mu_j \rho_j^2 \right) \frac{d(x, z)}{\alpha^1(x, z)} \leq 0, $$

which is a contradiction. Hence the theorem is proved. Similarly, the proof of the theorem can be obtained using Condition 2.
3 Duality model

Condition 3: Since \( \{G(\cdot, \tilde{y}_i), i = 1, 2, \ldots, s, j = 1, 2, \ldots, m\} \) is second order \((C, \alpha, \rho, d)\) type-I at \( z \), for \( i = 1, 2, \ldots, s \),

\[
\frac{1}{\alpha^1(x, z)} \{ G(x, \tilde{y}_i) - G(z, \tilde{y}_i) + \frac{1}{2} p^T \nabla^2 [f(z, \tilde{y}_i) - \lambda h(z, \tilde{y}_i)] p \}
\geq C_{x,z} (I(z, \tilde{y}_i) + \nabla^2 [f(z, \tilde{y}_i) - \lambda h(z, \tilde{y}_i)] p) + \frac{\rho^1_i d(x, z)}{\alpha^1(x, z)}, \quad (13)
\]

and, for \( j = 1, 2, \ldots, m \),

\[
\frac{1}{\alpha^2(x, z)} [-g_j(z) + \frac{1}{2} p^T \nabla^2 g_j(z) p] \geq C_{x,z} [\nabla g_j(z) + \nabla^2 g_j(z) p] + \frac{\rho^2_j d(x, z)}{\alpha^2(x, z)}. \quad (14)
\]

Multiplying (13) by \( t_i/\tau \geq 0 \) for \( i = 1, 2, \ldots, s \), and (14) by \( \mu_j/\tau \geq 0 \) for \( j = 1, 2, \ldots, m \), where \( \tau = 1 + \sum_{j=1}^{m} \mu_j \), we obtain, for \( i = 1, 2, \ldots, s \),

\[
\frac{1}{\tau \alpha^1(x, z)} ( t_i \{ G(x, \tilde{y}_i) - G(z, \tilde{y}_i) + \frac{1}{2} p^T \nabla^2 [f(z, \tilde{y}_i) - \lambda h(z, \tilde{y}_i)] p \} )
\geq \frac{t_i}{\tau} C_{x,z} (I(z, \tilde{y}_i) + \nabla^2 [f(z, \tilde{y}_i) - \lambda h(z, \tilde{y}_i)] p) + \frac{t_i \rho^1_i d(x, z)}{\tau \alpha^1(x, z)}, \quad (15)
\]

\[
\frac{\mu_j}{\tau \alpha^2(x, z)} [-g_j(z) + \frac{1}{2} p^T \nabla^2 g_j(z) p]
\geq \frac{\mu_j}{\tau} C_{x,z} [\nabla g_j(z) + \nabla^2 g_j(z) p] + \frac{\mu_j \rho^2_j d(x, z)}{\tau \alpha^2(x, z)}. \quad (16)
\]
Summing (15) over \( i \) and (16) over \( j \), using \( \alpha^1(x, z) = \alpha^2(x, z) \) and the convexity of \( C \),

\[
\frac{1}{\tau \alpha^1(x, z)} \left[ \sum_{i=1}^{s} t_i \{ G(x, \tilde{y}_i) - G(z, \tilde{y}_i) + \frac{1}{2} p^T \nabla^2 [ f(z, \tilde{y}_i) - \lambda h(z, \tilde{y}_i) ] p \} \\
- \sum_{j=1}^{m} \mu_j g_j(z) + \frac{1}{2} p^T \nabla^2 \left[ \sum_{j=1}^{m} \mu_j g_j(z) p \right] \right] > C_{x,z} \left[ \frac{1}{\tau} \left( \sum_{i=1}^{s} t_i I(z, \tilde{y}_i) + \nabla^2 \sum_{i=1}^{s} t_i [ f(z, \tilde{y}_i) - \lambda h(z, \tilde{y}_i) ] p + \sum_{j=1}^{m} \mu_j \nabla g_j(z) \right) + \left( \sum_{i=1}^{s} t_i \rho_1^i + \sum_{j=1}^{m} \mu_j \rho_j^2 \right) \right] \frac{d(x, z)}{\alpha^1(x, z) \tau}.
\]

(17)

Now, inequalities (2)–(4) yield

\[
- \sum_{j=1}^{m} \mu_j g_j(z) + \frac{1}{2} p^T \nabla^2 \left[ \sum_{j=1}^{m} \mu_j g_j(z) p \right] - \sum_{i=1}^{s} t_i G(z, \tilde{y}_i)
\]

\[
+ \frac{1}{2} p^T \nabla^2 \sum_{i=1}^{s} t_i [ f(z, \tilde{y}_i) - \lambda h(z, \tilde{y}_i) ] p \leq 0.
\]

(18)

Finally, using (1), (6), (18) and \( \sum_{i=1}^{s} t_i \rho_1^i + \sum_{j=1}^{m} \mu_j \rho_j^2 \geq 0 \) in (17),

\[
0 = C_{x,z}(0) = C_{x,z} \left[ \frac{1}{\tau} \left( \sum_{i=1}^{s} t_i I(z, \tilde{y}_i) + \nabla^2 \sum_{i=1}^{s} t_i [ f(z, \tilde{y}_i) - \lambda h(z, \tilde{y}_i) ] p \right.ight.
\]

\[
+ \sum_{j=1}^{m} \mu_j \nabla g_j(z) + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z) p \left. \right] < 0,
\]

which is a contradiction. Hence the theorem is proved. Similarly, the proof of the theorem can be obtained using Condition 4.
Theorem 11 (Strong duality). Assume that $x^*$ is an optimal solution of PP and $\nabla g_j(x^*)$ for $j \in J(x^*)$ are linearly independent. Then there exist $(s^*, t^*, \tilde{y}^*) \in K(x^*)$ and $(x^*, \mu^*, \lambda^*, w^*, v^*, r^* = 0, p^* = 0) \in H_1(s^*, t^*, \tilde{y}^*)$ such that $(x^*, \mu^*, \lambda^*, w^*, v^*, s^*, t^*, \tilde{y}^*, r^* = 0, p^* = 0)$ is a feasible solution of DP and the two objectives have the same values. If, in addition, the assumptions of Theorem 10 hold for all feasible solutions $(x, \mu, \lambda, w, v, s, t, \tilde{y}, r, p)$ of DP, then $(x^*, \mu^*, \lambda^*, w^*, v^*, s^*, t^*, \tilde{y}^*, r^* = 0, p^* = 0)$ is an optimal solution of DP.

Proof: Since $x^*$ is an optimal solution of PP and $\nabla g_j(x^*)$ for $j \in J(x^*)$ are linearly independent, then by Theorem 10 and Lai et al. [10] there exist $(s^*, t^*, \tilde{y}^*) \in K(x^*)$ and $(x^*, \mu^*, \lambda^*, w^*, v^*, r^* = 0, p^* = 0) \in H_1(s^*, t^*, \tilde{y}^*)$ such that $(x^*, \mu^*, \lambda^*, w^*, v^*, s^*, t^*, \tilde{y}^*, r^* = 0, p^* = 0)$ is a feasible solution of DP and the two objectives have same values. Optimality of $(x^*, \mu^*, \lambda^*, w^*, v^*, s^*, t^*, \tilde{y}^*, r^* = 0, p^* = 0)$ for DP thus follows from Theorem 10.

Theorem 12 (Strict Converse Duality). Let $x^*$ be an optimal solution of PP and $(z^*, \mu^*, \lambda^*, w^*, v^*, s^*, t^*, \tilde{y}^*, r^*, p^*)$ be an optimal solution of DP. Assume that any one of the following four conditions holds.

1. $\{G(\cdot, \tilde{y}^*_i) , g_j(\cdot) , i = 1, 2, \ldots, s^*, j = 1, 2, \ldots, m\}$ is semistrictly second order $(F, \alpha, \rho, d)$ type-I at $z^*$ and $\sum_{i=1}^{s^*} t_i^* \rho_i^1 + \sum_{j=1}^{m} \mu_j^* \rho_j^2 \geq 0$.

2. $\left\{ \sum_{i=1}^{s^*} t_i^* G(\cdot, \tilde{y}^*_i) , g_j(\cdot) , j = 1, 2, \ldots, m \right\}$ is semistrictly second order $(F, \alpha, \rho, d)$ type-I at $z^*$ and $\rho^1 + \sum_{j=1}^{m} \mu^*_j \rho_j^2 \geq 0$.

3. $\{G(\cdot, \tilde{y}^*_i) , g_j(\cdot) , i = 1, 2, \ldots, s^*, j = 1, 2, \ldots, m\}$ is semistrictly second order $(C, \alpha, \rho, d)$ type-I at $z^*$ and $\sum_{i=1}^{s^*} t_i^* \rho_i^1 + \sum_{j=1}^{m} \mu_j^* \rho_j^2 \geq 0$.

4. $\left\{ \sum_{i=1}^{s^*} t_i^* G(\cdot, \tilde{y}^*_i) , g_j(\cdot) , j = 1, 2, \ldots, m \right\}$ is semistrictly second order $(C, \alpha, \rho, d)$ type-I at $z^*$ and $\rho^1 + \sum_{j=1}^{m} \mu^*_j \rho_j^2 \geq 0$. 
Furthermore, suppose the set of vectors \( \{ \nabla g_j(x^*), j \in J(x^*) \} \) is linearly independent and \( \alpha^1(x^*, z^*) = \alpha^2(x^*, z^*) \). Then \( z^* = x^* \), that is, \( z^* \) is an optimal solution of PP.

**Proof:** The proof follows similarly to the proof of Theorem 10 and Theorem 3.3 of Ahmad et al. [2].

**Remark 13.** Let \( B \) and \( D \) be zero matrices of order \( n \), then the model DP becomes the dual models discussed by Hu et al. [8]. Further, if \( r = 0 \), then our dual models reduce to the problems of Husain et al. [7] and Sharma and Gulati [14]. In addition, if \( p = 0 \), then DP becomes the dual model considered by Liu and Wu [11]. If \( r = 0 \) and \( p = 0 \), then the model DP reduces to the model of Ahmad and Husain [1].

**References**


**Author addresses**

1. **S. K. Gupta**, Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee 247 667, India  
   mailto:guptashiv@gmail.com

2. **D. Dangar**, Department of Mathematics and Humanities, IT, Nirma University, Ahmedabad-382481, India  
   mailto:debasisiitp@gmail.com

   mailto:drizhar@kfupm.edu.sa