Implicit difference approximation of the Galilei invariant fractional advection diffusion equation

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Abstract

A Galilei invariant fractional advection diffusion equation with initial-boundary conditions is considered. An implicit difference approximation for solving the Galilei invariant fractional advection diffusion equation is presented. We introduce a new Fourier method for analyzing the stability and convergence of the implicit difference approximation. Finally, some numerical examples are given. The numerical results are in good agreement with our theoretical analysis. This method and supporting theoretical techniques can also be extended to a larger class of fractional integro-differential equations.

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1 Introduction

There has been increasing interest in the description of physical phenomena exhibiting anomalous diffusion, that is, diffusion not accurately modeled by the usual advection diffusion equation. An extension of the continuous time random walk approach to anomalous diffusion leads to fractional advection-dispersion equations (FADE). These equations have been used to describe transport in amorphous semiconductors, spread of contaminants in underground water, relaxation in polymer systems, and tracer dynamics in polymer networks (Sokolov et al. [7]).

We consider the numerical solution of the following initial/boundary value problem of the Galilei invariant FADE (GI-FADE) (Metzler et al. [4])

$$\frac{\partial W(x,t)}{\partial t} + v\frac{\partial W(x,t)}{\partial x} = {}_0D_t^{1-\gamma} K_\gamma \frac{\partial^2 W(x,t)}{\partial x^2} + f(x,t)\,, \qquad 0 < t \le T\,, \quad 0 < x < L\,, \tag{1}$$

where $0 < \gamma < 1$, $K_\gamma > 0$ and $v > 0$ are constants, the function $f(x,t)$ can be used to represent sources and sinks, and ${}_0D_t^{1-\gamma}V(x,t)$ is the Riemann–Liouille fractional derivative of order $1-\gamma$ defined by Podlubny [6]

$$ {}_0D_t^{1-\gamma}V(x,t) = \frac{1}{\Gamma(\gamma)}\frac{\partial}{\partial t}\int_0^t \frac{V(x,\eta)}{(t-\eta)^{1-\gamma}}\,d\eta\,. \tag{2} $$

We impose on (1) the following initial and nonhomogeneous Dirichlet boundary conditions:

$$ W(x,0) = \varphi(x)\,, \quad 0 \le x \le L\,, \tag{3} $$
$$ W(0,t) = \phi(t)\,, \quad 0 < t \le T\,, \tag{4} $$
$$ W(L,t) = \psi(t)\,, \quad 0 < t \le T. \tag{5} $$

The FADE has been recently treated by Liu et al. [2]. Yuste and Acedo [8] proposed an explicit finite difference method and a new von Neumann-type stability analysis for the fractional subdiffusion equation, that is, the GI-FADE without the advection term. However, they did not give the convergence analysis and pointed out the difficulty of this task when implicit methods are considered. Langlands and Henry [1] also investigated this problem and proposed an implicit numerical scheme (L1 approximation), and discussed its accuracy and stability. However, the global accuracy of the implicit numerical scheme was not derived and it seems that the unconditional stability for all γ in the range $0 < \gamma \le 1$ has not been established. The main purpose of this article is to address these issues for the GI-FADE. We analyze the problem via a Fourier method.

Section 2 proposes an implicit difference approximation (IDA) for GI-FADE. The stability and convergence of the IDA are discussed using Fourier analysis in Sections 3 and 4, respectively. Finally, some numerical results will be given to evaluate the accuracy of the method.

2 An implicit difference approximation for the GI-FADE

This section proposes an implicit difference approximation for solving the GI-FADE (1) with the initial and boundary conditions (3)–(5).

We take an equally spaced mesh of M points for the spatial domain $0 \leq x \leq L$, and N constant time steps for the temporal domain. We denote the spatial grid points and the temporal grid points by $x_j = jh$, $j = 0, 1, \ldots, M$, $t_k = k\tau$, $k = 0, 1, \ldots, N$, respectively, where the grid spacing is simply $h = L/M$ in the spatial domain and $\tau = T/N$ in the time domain.

Meerschaert et al. [5] showed that using the usual Grünwald formula to discretize the one dimensional fractional diffusion equation results in an unstable finite difference scheme. Thus, we start here with a right-shifted Grünwald approximation to the fractional derivative, which for $0 < \gamma \leq 1$ is

$$ {}_0D_t^{1-\gamma} V(x,t) = \tau^{\gamma-1} \sum_{l=0}^{[t/\tau]} w_l^{1-\gamma} V(x, t - l\tau) + O(\tau^p). \tag{6} $$

This formula is not unique because there are many different valid choices for $w_l^{1-\gamma}$ that lead to approximations of different order p (Ch. Lubich [3]).

We now present the following IDA for the initial/boundary value problem of the GI-FADE (1)–(5):

$$ \begin{aligned} & \frac{W_j^k - W_j^{k-1}}{\tau} + v \frac{W_{j+1}^k - W_{j-1}^k}{2h} \\ = \; & \tau^{\gamma-1} K_\gamma \sum_{m=0}^{k} w_m^{1-\gamma} \frac{W_{j-1}^{k-m} - 2W_j^{k-m} + W_{j+1}^{k-m}}{h^2} + f_j^k \,, \\ & j = 1, 2, \ldots, M-1\,, \quad k = 1, 2, \ldots, N\,, \end{aligned} \tag{7} $$

$$W_0^k = \phi(t_k)\,, \quad W_M^k = \psi(t_k)\,, \quad k = 1, 2, \ldots, N\,, \tag{8}$$

$$W_j^0 = \varphi(x_j)\,, \quad j = 0, 1, \ldots, M\,, \tag{9}$$

where $$w_m^{1-\gamma} = (-1)^m \begin{pmatrix} 1-\gamma \\ m \end{pmatrix}, \quad m = 0, 1, \ldots, k\,,$$

and $f_j^k \equiv f(x_j, t_k)$. We take $p = 1$. These coefficients can be evaluated as follows (Yuste et al. [8])

$$w_0^\alpha = 1\,, \quad w_m^\alpha = (-1)^m \frac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{m!}\,, \quad m = 1, 2, \ldots\,. \tag{10}$$

3 Stability of the IDA for the GI-FADE

This section analyzes the stability of the IDA using Fourier analysis. Firstly, we rewrite (7) as

$$\begin{aligned} W_j^k &= W_j^{k-1} + \mu_2 \sum_{m=0}^{k} w_m^{1-\gamma} \left(W_{j-1}^{k-m} - 2W_j^{k-m} + W_{j+1}^{k-m}\right) \\ &\quad - \mu_1 \frac{W_{j+1}^k - W_{j-1}^k}{2} + \tau f_j^k\,, \quad j = 1, 2, \ldots, M-1\,, \quad k = 1, 2, \ldots, N\,, \end{aligned} \tag{11}$$

where $\mu_1 = v\tau/h$ and $\mu_2 = K_\gamma \tau^\gamma / h^2$. The roundoff error ρ_j^k of the solution for the IDA (7)–(9) satisfy the difference equation

$$\begin{aligned} \rho_j^k = \rho_j^{k-1} - \mu_1 \frac{\rho_{j+1}^k - \rho_{j-1}^k}{2} + \mu_2 \sum_{m=0}^{k} w_m^{1-\gamma} \left(\rho_{j-1}^{k-m} - 2\rho_j^{k-m} + \rho_{j+1}^{k-m}\right)\,, \\ j = 1, 2, \ldots, M-1\,, \quad k = 1, 2, \ldots, N\,. \end{aligned} \tag{12}$$

We now define the grid functions:

$$\rho^k(x) = \begin{cases} \rho_j^k\,, & \text{when} \quad x_j - \frac{h}{2} < x \le x_j + \frac{h}{2}\,, \quad j = 1, 2, \ldots, M-1\,, \\ 0\,, & \text{when} \quad 0 \le x \le \frac{h}{2} \quad \text{or} \quad L - \frac{h}{2} < x \le L\,, \end{cases}$$

and note that $\rho^k(x)$ extended in a Fourier series as

$$\rho^k(x) = \sum_{l=-\infty}^{\infty} d_k(l) e^{i2\pi lx/L}\,, \quad k = 1, 2, \dots, N\,,$$

where $d_k(l) = \frac{1}{L}\int_0^L \rho^k(x) e^{-i2\pi lx/L}\, dx$, $i = \sqrt{-1}$. We let

$$\rho^k = \left[\rho_1^k, \rho_2^k, \dots, \rho_{M-1}^k\right]^T$$

and introduce the following norm:

$$\|\rho^k\|_2 = \left(\sum_{j=1}^{M-1} h|\rho_j^k|^2\right)^{1/2} = \left[\int_0^L |\rho^k(x)|^2 dx\right]^{1/2}.$$

Based on the Parseval equality:

$$\int_0^L |\rho^k(x)|^2 dx = \sum_{l=-\infty}^{\infty} |d_k(l)|^2\,,$$

we have

$$\|\rho^k\|_2^2 = \sum_{l=-\infty}^{\infty} |d_k(l)|^2\,. \tag{13}$$

We now assume that the solution of equation (12) has the following form:

$$\rho_j^k = d_k e^{i\sigma jh}\,, \tag{14}$$

where $\sigma = 2\pi l/L$. Substituting the above expression into (12), we obtain

$$d_k = d_{k-1} - i\mu_1 \sin(\sigma h) d_k - 4\mu_2 \sin^2\frac{\sigma h}{2} \sum_{m=0}^{k} w_m^{1-\gamma} d_{k-m}\,, \quad k = 1, 2, \dots, N\,. \tag{15}$$

Lemma 1 *The coefficients* $w_m^{1-\gamma}$ $(m = 0, 1, \ldots)$ *satisfy*

1. $w_0^{1-\gamma} = 1\,;\quad w_1^{1-\gamma} = \gamma - 1\,;\quad w_m^{1-\gamma} < 0\,,\ m = 1, 2, \ldots\,;$
2. $\sum_{m=0}^{\infty} w_m^{1-\gamma} = 1$ *; for all* $n \in N$, $-\sum_{m=1}^{n} w_m^{1-\gamma} < 1$.

Zhuang et al. [9] gave the proof.

Applying Lemma 1, equation (15) is rewritten as

$$d_k = \frac{1 + (1-\gamma)\bar{\mu}}{1 + \bar{\mu} + i\tilde{\mu}} d_{k-1} - \frac{\bar{\mu}}{1 + \bar{\mu} + i\tilde{\mu}} \sum_{m=2}^{\infty} w_m^{1-\gamma} d_{k-m}\,, \quad k = 1, 2, \ldots, N\,, \tag{16}$$

where $\bar{\mu} = 4\mu_2 \sin^2 \frac{\sigma h}{2} \geq 0$ and $\tilde{\mu} = \mu_1 \sin \sigma h$.

Lemma 2 *Assuming that* d_k $(k = 1, 2 \ldots, N)$ *is the solution of equation* (16), *then*

$$|d_k| \leq |d_0|\,, \quad k = 1, 2, \ldots, N\,. \tag{17}$$

Proof: Mathematical induction proves this result. ♠

Theorem 3 *The* IDA (7)–(9) *for the* GI-FADE (1)–(5) *is unconditionally stable.*

Proof: Apply Lemma 2 and noting (13), we have

$$\|\rho^k\|_2 \leq \|\rho^0\|_2\,, \quad k = 1, 2, \ldots, N, \tag{18}$$

which implies that the IDA (7)–(9) for the GI-FADE (1)–(5) is unconditionally stable. ♠

4 Convergence of the IDA for the GI-FADE

This section discusses the convergence of the IDA. Let

$$\begin{aligned} R_j^k \;=\; & \frac{W(x_j,t_k)-W(x_j,t_{k-1})}{\tau}+v\frac{W(x_{j+1},t_k)-W(x_{j-1},t_k)}{2h} \\ & -\tau^{\gamma-1}K_\gamma\sum_{m=0}^{k}w_m^{1-\gamma}\frac{W(x_{j-1},t_{k-m})-2W(x_j,t_{k-m})+W(x_{j+1},t_{k-m})}{h^2} \\ & -f(x_j,t_k)\,,\quad j=1,2,\ldots,M-1\,,\quad k=1,2,\ldots,N. \end{aligned} \tag{19}$$

The following lemma holds.

Lemma 4 $\tau^{\gamma-1}\sum_{m=0}^{k}w_m^{1-\gamma}=\frac{1}{\Gamma(\gamma)}+O(\tau)\,.$

Applying the Taylor expansions, (6) and Lemma 4, we obtain

$$|R_j^k|\le C_1(\tau+h^2)\,,\quad j=1,2,\ldots,M-1\,,\quad k=1,2,\ldots,N\,, \tag{20}$$

where C_1 is a positive constant. From (19), we have

$$\begin{aligned} W(x_j,t_k) &= W(x_j,t_{k-1})-\mu_1\frac{W(x_{j+1},t_k)-W(x_{j-1},t_k)}{2} \\ &+\mu_2\sum_{m=0}^{k}w_m^{1-\gamma}\left[W(x_{j-1},t_{k-m})-2W(x_j,t_{k-m})+W(x_{j+1},t_{k-m})\right] \\ &+\tau f(x_j,t_k)+\tau R_j^k\,,\quad j=1,2,\ldots,M-1\,,\quad k=1,2,\ldots,N\,. \end{aligned} \tag{21}$$

Subtracting (11) from (21), we obtain

$$\begin{aligned} e_j^k \;=\; & e_j^{k-1}-\mu_1\frac{e_{j+1}^k-e_{j-1}^k}{2}+\mu_2\sum_{m=0}^{k}w_m^{1-\gamma}\left(e_{j-1}^{k-m}-2e_j^{k-m}+e_{j+1}^{k-m}\right) \\ & +\tau R_j^k\,,\quad j=1,2,\ldots,M-1\,,\quad k=1,2,\ldots,N\,, \end{aligned} \tag{22}$$

where $e_j^k = W(x_j, t_k) - W_j^k$, $j = 0, 1, \ldots, M$, $k = 0, 1, \ldots, N$.

We now analyze the convergence of the IDA by using Fourier analysis. We define the grid functions for $k = 0, 1, \ldots, N$ as

$$e^k(x) = \begin{cases} e_j^k \,, & \text{when} \quad x_j - \frac{h}{2} < x \le x_j + \frac{h}{2} \,, \quad j = 1, 2, \ldots, M-1 \,, \\ 0 \,, & \text{when} \quad 0 \le x \le \frac{h}{2} \quad \text{or} \quad L - \frac{h}{2} < x \le L \,, \end{cases} \tag{23}$$

and

$$R^k(x) = \begin{cases} R_j^k \,, & \text{when} \quad x_j - \frac{h}{2} < x \le x_j + \frac{h}{2} \,, \quad j = 1, 2, \ldots, M-1, \\ 0 \,, & \text{when} \quad 0 \le x \le \frac{h}{2} \quad \text{or} \quad L - \frac{h}{2} < x \le L \,, \end{cases} \tag{24}$$

respectively. Thus, $e^k(x)$ and $R^k(x)$ have extended Fourier series expansions

$$e^k(x) = \sum_{l=-\infty}^{\infty} \xi_k(l) e^{i2\pi lx/L} \,, \quad k = 0, 1, \ldots, N \tag{25}$$

and

$$R^k(x) = \sum_{l=-\infty}^{\infty} \eta_k(l) e^{i2\pi lx/L} \,, \quad k = 1, 2, \ldots, N \,, \tag{26}$$

respectively, where

$$\xi_k(l) = \frac{1}{L} \int_0^L e^k(x) e^{-i2\pi lx/L} \, dx \,, \quad \eta_k(l) = \frac{1}{L} \int_0^L R^k(x) e^{-i2\pi lx/L} \, dx \,. \tag{27}$$

We now let

$$e^k = \left[e_1^k, e_2^k, \ldots, e_{M-1}^k \right]^T \,, \quad k = 0, 1, \ldots \,, N \tag{28}$$

and

$$R^k = \left[R_1^k, R_2^k, \ldots, R_{M-1}^k \right]^T \,, \tag{29}$$

and introduce the following norms:

$$\|e^k\|_2 = \left(\sum_{j=1}^{M-1} h |e_j^k|^2 \right)^{1/2} = \left[\int_0^L |e^k(x)|^2 \, dx \right]^{1/2} \,, \quad k = 0, 1, \ldots \,, N \tag{30}$$

and

$$\|R^k\|_2 = \left(\sum_{j=1}^{M-1} h|R_j^k|^2\right)^{1/2} = \left[\int_0^L |R^k(x)|^2\,dx\right]^{1/2}, \quad k = 1, 2, \ldots, N\,, \quad (31)$$

respectively. Using Parseval's equality:

$$\int_0^L |e^k(x)|^2 dx = \sum_{l=-\infty}^{\infty} |\xi_k(l)|^2\,, \quad k = 0, 1, \ldots, N\,, \quad (32)$$

$$\int_0^L |R^k(x)|^2 dx = \sum_{l=-\infty}^{\infty} |\eta_k(l)|^2\,, \quad k = 1, 2, \ldots, N\,, \quad (33)$$

$$\|e_k\|_2^2 = \sum_{l=-\infty}^{\infty} |\xi_k(l)|^2\,, \quad k = 0, 1, \ldots, N\,, \quad (34)$$

$$\|R_k\|_2^2 = \sum_{l=-\infty}^{\infty} |\eta_k(l)|^2\,, \quad k = 1, 2, \ldots, N\,, \quad (35)$$

respectively. From the above analysis, we now suppose that

$$e_j^k = \xi_k e^{i\sigma jh} \quad (36)$$

and

$$R_j^k = \eta_k e^{i\sigma jh}\,, \quad (37)$$

where $\sigma = 2\pi l/L$. Substituting (36) and (37) into (22), we obtain

$$\begin{aligned}\xi_k \;&=\; \xi_{k-1} - i\mu_1 \sin\sigma h \cdot \xi_k - 4\mu_2 \sin^2\frac{\sigma h}{2} \sum_{m=0}^{k} w_m^{1-\gamma}\xi_{k-m} + \tau\eta_k\,, \quad (38)\\ &k = 1, 2, \ldots, N\,.\end{aligned}$$

Applying Lemma 1, rewrite equation (38) as

$$\xi_k \;=\; \frac{1+(1-r)\bar{\mu}}{1+\bar{\mu}+i\tilde{\mu}}\xi_{k-1} - \frac{\bar{\mu}}{1+\bar{\mu}+i\tilde{\mu}}\sum_{m=2}^{k} w_m^{1-\gamma}\xi_{k-m} \quad (39)$$

$$+\frac{1}{1+\bar{\mu}+i\tilde{\mu}}\tau\eta_k\,,\quad k=1,2,\ldots,N\,,$$

where $\bar{\mu}=4\mu_2\sin^2(\sigma h/2)\geq 0$ and $\tilde{\mu}=\mu_1\sin\sigma h\,.$

Similar to the proof of Lemma 2, we also obtain the following lemma.

Lemma 5 *Supposing that ξ_k $(k=1,2\ldots,N)$ is the solution of equation (39), then there is a positive constant C_2 such that*

$$|\xi_k|\leq C_2k\tau|\eta_1|\,,\quad k=1,2,\ldots,N. \tag{40}$$

Theorem 6 *The IDA (7)–(9) for the GI-FADE (1)–(5) is convergent, and the convergence order is $O(\tau+h^2)$.*

Proof: Noting (34) and (35), and applying Lemma 5, we obtain the result

$$\|e^k\|_2\leq C_2k\tau\|R^1\|_2\leq C_1C_2k\tau\sqrt{L}\left(\tau+h^2\right). \tag{41}$$

As $k\tau\leq T$,

$$\|e^k\|_2\leq C\left(\tau+h^2\right)\,, \tag{42}$$

where $C=C_1C_2T\sqrt{L}\,.$ The result then follows. ♠

5 Numerical results

This section gives a numerical example that confirms our theoretical analysis. We consider the initial-boundary value problem of the GI-FADE with a non-homogeneous source term:

$$\frac{\partial W(x,t)}{\partial t}+\frac{\partial W(x,t)}{\partial x}={}_0D_t^{1-\gamma}\frac{\partial^2 W(x,t)}{\partial x^2} \tag{43}$$

Table 1: The maximum error E_∞.

τ	h	$\gamma = 0.4$	$\gamma = 0.5$	$\gamma = 0.6$
$\frac{1}{64}$	$\frac{1}{8}$	9.7E-04	1.3E-03	1.6E-03
$\frac{1}{1024}$	$\frac{1}{32}$	1.4E-04	9.0E-05	1.9E-04

$$+ e^x \left[(1+\gamma)t^\gamma + t^{1+\gamma} - \frac{\Gamma(2+\gamma)}{\Gamma(1+2\gamma)} t^{2\gamma} \right], \quad 0 < t \le 1, \quad 0 < x < 1,$$

$$W(0,t) = t^{1+\gamma}, \quad W(1,t) = et^{1+\gamma}, \quad 0 < t \le 1, \tag{44}$$

$$W(x,0) = 0, \quad 0 \le x \le 1. \tag{45}$$

The exact solution of the problem (43)–(45) is

$$W(x,t) = e^x t^{1+\gamma}. \tag{46}$$

The maximum error of the exact and numerical solutions is defined as

$$E_\infty = \max_{0\le j\le M} \max_{0\le k\le N} |W(x_j,t_k) - W_j^k|. \tag{47}$$

Table 1 shows the maximum error at all mesh points for different γ using $\tau = \frac{1}{64}$, $h = \frac{1}{8}$ and $\tau = \frac{1}{1024}$, $h = \frac{1}{32}$ respectively, and the effect of τ and h. Physical conditions impose the range $0 < \gamma < 1$. In this example we select three typical values $\gamma = 0.4$, 0.5 and 0.6 in this range. Our analysis indicates a convergence order of $O(\tau + h^2)$ for small τ and h^2 for the IDA scheme. The small values $\tau = 1/64$, $1/1024$ and $h = 1/8$, $1/32$ have been used in this example.

Table 1 indeed indicates that the maximum error is $O(\tau + h^2)$ and IDA is unconditionally stable. These numerical results are in good agreement with our theoretical analysis.

6 Conclusions

Numerical treatment of fractional subdiffusion equations is known to be difficult. We consider the Galilei invariant fractional advection diffusion equation which covers fractional subdiffusion as a special case. We developed an implicit difference approximation for solving the GI-FADE. We introduced a Fourier method to successfully analyze the stability and convergence of the IDA. We proved that the IDA is unconditionally stable and convergent. This method and supporting theoretical techniques can also be extended to a larger class of fractional integro-differential equations.

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