Estimating the error of a $H^1$-mixed finite element solution for the Burgers equation

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Abstract

We compute error estimations for a $H^1$-mixed finite element method for Burgers equation. By using a $H^1$-mixed finite element method, the problem is reformulated as a system of first order partial differential equations, which allows an approximation of the unknown function and its derivative. Local parabolic and elliptic methods approximate the true errors from the computed solutions; the so-called a posteriori error estimates. Numerical experiments show that the error estimations converge to the true errors.
1 Introduction

We consider the Burgers equation
\[ \partial_t u(x, t) - \nu \partial_{xx} u(x, t) + u(x, t) \partial_x u(x, t) = 0, \quad x \in \Omega, \quad t \in (0, T), \] (1)
with boundary and initial conditions
\[ u(0, t) = u(1, t) = 0, \quad t \in [0, T], \] (2)
\[ u(x, 0) = u_0(x), \quad x \in \Omega, \] (3)
where \( \partial_t := \partial / \partial t, \partial_x := \partial / \partial x, \partial_{xx} := \partial^2 / \partial x^2, \) \( T \) and \( \nu \) (viscosity coefficient) are positive constants and \( \Omega := (0, 1) [2]. \)

The aim is to design methods to compute a posteriori error estimations when the solution of (1)–(3) is approximated by a \( H^1 \)-mixed finite element method (\( H^1 \)-MFEM).

Using the \( H^1 \)-MFEM, the problem is reformulated as a system of first order partial differential equations, which allows an approximation for \( u \) and its
derivative $\partial_x u$. The $H^1$-MFEM considered in this article is based on an approach suggested by Pani for nonlinear parabolic equations [3]. Pany et al. [4] adapted the method to Burgers equation. Section 2 gives details of this $H^1$-MFEM.

The method considered in this article is closely related to least squares mixed finite element methods in that the second order partial differential equation is reformulated as a system of first order partial differential equations with a new unknown defined as the flux [7, 8, 9, 10, and references therein].

A posteriori error estimates are a fundamental component in the design of efficient adaptive algorithms for solving partial differential equations. In this study we consider an implicit type of a posteriori error estimation which is based on the procedure developed by Adjerid et al. [1] for one dimensional parabolic systems. This a posteriori error estimation with finite element methods of lines was studied for one dimensional nonlinear parabolic system and the Sobolev equation [5, 6]. For the approximation of the solution we use a mixed formulation of finite element methods of lines with an a posteriori error estimation computed using the procedure developed by Adjerid et al. [1]. To the best of our knowledge, this is the first time this a posteriori error estimation method is considered for the Burgers equation, where the approximate solution is computed using $H^1$-MFEM.

## 2 The $H^1$-mixed finite element method

Throughout this article, $\langle \cdot, \cdot \rangle$ and $\| \cdot \|_{H^0(\Omega)}$ denote the inner product and norm in $H^0(\Omega) = L^2(\Omega)$, respectively. As usual, the Sobolev space $H^1(\Omega)$ consists of functions $u$ for which

$$
\| u \|_{H^1(\Omega)} = \sqrt{\| u \|_{H^0(\Omega)}^2 + \| \partial_x u \|_{H^0(\Omega)}^2}
$$

exists. The space $H^1_0(\Omega)$ contains all functions in $H^1(\Omega)$ with zero trace at the endpoints of the domain $\Omega$, namely at $x = 0, 1$. For any $p \in [0, \infty]$ and
The \( H^1 \)-mixed finite element method

any normed vector space \( X \), \( L^p(X) \) is the space \( L^p(0, T; X) \) of all functions defined in \([0, T]\) with values in \( X \). The norm in this space \( \| \cdot \|_{L^p(X)} \) is defined as usual. We write \( L^p(L^\infty) \) and \( L^p(H^1) \) instead of \( L^p(L^\infty(\Omega)) \) and \( L^p(H^1(\Omega)) \), respectively.

By \( H^1 \)-MFEM, (1) is reduced to a system of first order equations using a new variable defined as \( v = u_x \). As a consequence, (1) is reformulated as

\[
\partial_x u = v, \tag{4} \\
\partial_t u - \nu \partial_x v + uv = 0. \tag{5}
\]

We multiply (4) by \( \partial_x \chi \) and (5) by \( -\partial_x w \), where \( \chi \in H^1_0(\Omega) \) and \( w \in H^1(\Omega) \) are arbitrary test functions. Then, using integration by parts and applying the boundary conditions (2), we obtain a weak formulation of (1)–(3): given \( u_0 \in H^1_0(\Omega) \), find \( (u, v) : [0, T] \rightarrow H^1_0(\Omega) \times H^1(\Omega) \) satisfying, for \( t > 0 \),

\[
\langle \partial_x u(t), \partial_x \chi \rangle = \langle v(t), \partial_x \chi \rangle \quad \text{for all } \chi \in H^1_0(\Omega), \tag{6}
\]

\[
\langle \partial_t v(t), w \rangle + \nu \langle \partial_x v(t), \partial_x w \rangle = \langle u(t)v(t), \partial_x w \rangle \quad \text{for all } w \in H^1(\Omega), \tag{7}
\]

and, for \( t = 0 \),

\[
\langle v(0), w \rangle = \langle \partial_x u_0, w \rangle \quad \text{for all } w \in H^1(\Omega). \tag{8}
\]

**Remark 1.** Existence and uniqueness of the solution of (6)–(8) can be shown using the method of compactness [12, 11]. We will present this result in a future paper.

**Remark 2.** If \( u \in W^1_{\infty}(0, T; H^1_0(\Omega)) \cap L^\infty(0, T; H^2(\Omega)) \) and \( (u, v) \) satisfies (6)–(7) then \( (u, v) \) satisfies (4)–(5). Indeed, by using integration by parts we deduce from (6) that \( \partial_x (v - \partial_x u) = 0 \) which implies

\[
v(x, t) = \partial_x u(x, t) + g(t), \tag{9}
\]

for some function \( g \) depending on \( t \). Integrating (9) over \( \Omega \), noting (8), we infer \( g(0) = 0 \). Also, it follows from (9) and (7) (with \( w = 1 \)) that

\[
\int_\Omega [\partial_{tx} u + g'(t)] \, dx = 0, \tag{10}
\]
implying $g'(t) = 0$. Hence $g \equiv 0$, that is $(u, v)$ satisfies (4). This immediately gives (5).

Solutions to (6)–(8) are approximated using a high order finite element method defined as follows. We first partition the interval $\Omega$ into $0 = x_1 < x_2 < \cdots < x_{N+1} = 1$, and define $h_l := x_{l+1} - x_l$ for $l = 1, \ldots, N$ and $h := \max_l h_l$. The hat function on $(x_{l-1}, x_{l+1})$ for $l = 2, \ldots, N$ is defined as

$$
\phi_{l,1}(x) = \begin{cases} 
\frac{(x - x_{l-1})}{h_{l-1}}, & x \in [x_{l-1}, x_l), \\
\frac{(x_{l+1} - x)}{h_l}, & x \in [x_l, x_{l+1}), \\
0, & \text{otherwise}.
\end{cases}
$$

At the endpoints of $\Omega$ (namely, at $x = 0, 1$) we define

$$
\phi_{1,1}(x) = \begin{cases} 
\frac{(x_2 - x)}{h_1}, & x \in [x_1, x_2), \\
0, & \text{otherwise},
\end{cases}
\quad \phi_{N+1,1}(x) = \begin{cases} 
\frac{(x - x_N)}{h_N}, & x \in [x_N, x_{N+1}), \\
0, & \text{otherwise}.
\end{cases}
$$

The space of piecewise linear functions on $\Omega$ and its subspace consisting of functions vanishing at the endpoints of $\Omega$ are, respectively,

$$
S_h := \text{span}\{\phi_{1,1}, \phi_{2,1}, \ldots, \phi_{N+1,1}\} \quad \text{and} \quad \tilde{S}_h := \text{span}\{\phi_{2,1}, \ldots, \phi_{N,1}\}.
$$

The spaces of bubble functions in $\Omega$ are defined by $S_h^k := \text{span}\{\phi_{1,k}, \ldots, \phi_{N,k}\}$, where $\phi_{l,k}$ is an antiderivative of the Legendre polynomial $P_{k-1}$ of degree $k - 1$ scaled to the subinterval $[x_l, x_{l+1}]$. More precisely, for $l = 1, \ldots, N$ and $k = 2, 3, \ldots$, we define

$$
\phi_{l,k}(x) = \begin{cases} 
\left[\sqrt{2(2k - 1)}/h_l\right] \int_{x_l}^x P_{k-1}(y) \, dy, & x \in [x_l, x_{l+1}), \\
0, & \text{otherwise}.
\end{cases}
$$
For $p, q \in \mathbb{N}$ and $p, q \geq 2$, the finite dimensional subspaces of $H^1(\Omega)$ and $H_0^1(\Omega)$ are, respectively, 

\[
\mathcal{V}_h^q := S_h \cup \sum_{k=2}^{q} S_h^k \quad \text{and} \quad \mathcal{V}_h^p := S_h \cup \sum_{k=2}^{p} S_h^k.
\]

A semidiscrete approximation to (6)–(8) is to find $(\mathbf{U}, \mathbf{V}) : [0, T] \to \mathcal{V}_h^p \times \mathcal{V}_h^q$ such that for $t \in (0, T)$:

\[
\begin{align*}
\langle \partial_t \mathbf{U}(t), \partial_t \chi_h \rangle &= \langle \mathbf{V}(t), \partial_x \chi_h \rangle, \quad \text{for all } \chi_h \in \mathcal{V}_h^p, \\
\langle \partial_t \mathbf{V}(t), \mathbf{w}_h \rangle + \nu \langle \partial_x \mathbf{V}(t), \partial_x \mathbf{w}_h \rangle &= \langle \mathbf{U} \mathbf{V}(t), \partial_x \mathbf{w}_h \rangle, \quad \text{for all } \mathbf{w}_h \in \mathcal{V}_h^q,
\end{align*}
\]

and

\[
\langle \mathbf{V}(0), \mathbf{w}_h \rangle = \langle \partial_x \mathbf{u}_0, \mathbf{w}_h \rangle \quad \text{for all } \mathbf{w}_h \in \mathcal{V}_h^q.
\]

Let the errors in the approximation of (6)–(8) by (12)–(14) be $e(x, t) := u(x, t) - \mathbf{U}(x, t)$ and $f(x, t) := v(x, t) - \mathbf{V}(x, t)$. This leads to Proposition 3, the proof of which will be presented in a future paper.

**Proposition 3.** Assume that $u, \partial_t u \in L^\infty(H^1_0(\Omega) \cap H^{p+1}(\Omega))$ and $v, \partial_t v \in L^\infty(H^{q+1}(\Omega))$. Assume further that $\mathbf{U} \in L^\infty(\mathcal{V}_h^p)$ and $\mathbf{V} \in L^\infty(\mathcal{V}_h^q)$. Then, there exist positive constant $C > 0$ independent of $h$ such that

\[
\begin{align*}
\|e(t)\|_j &\leq Ch^{\min(p+1-j,q+1)} \left[ \|u\|_{L^\infty(\mathcal{H}^{p+1})} + \|v\|_{L^\infty(\mathcal{H}^{q+1})} + \|\partial_t v\|_{L^2(\mathcal{H}^{q+1})} \right], \\
\|f(t)\|_j &\leq Ch^{\min(p+1,q+1-j)} \left[ \|u\|_{L^\infty(\mathcal{H}^{p+1})} + \|v\|_{L^\infty(\mathcal{H}^{q+1})} + \|\partial_t v\|_{L^2(\mathcal{H}^{q+1})} \right].
\end{align*}
\]

Now we show the computation of $(\mathbf{U}, \mathbf{V})$. With $\phi_{l,k}$ defined by (11), the solutions to (12)–(14) are

\[
\begin{align*}
\mathbf{U}(x, t) &= \sum_{l=1}^{N} u_{l,1}(t) \phi_{l,1}(x) + \sum_{l=1}^{N} \sum_{k=2}^{p} u_{l,k}(t) \phi_{l,k}(x), \\
\mathbf{V}(x, t) &= \sum_{l=1}^{N+1} v_{l,1}(t) \phi_{l,1}(x) + \sum_{l=1}^{N} \sum_{k=2}^{q} v_{l,k}(t) \phi_{l,k}(x).
\end{align*}
\]
Let

\[ \alpha_{k,k'}^{l,l'} = \langle \phi_{l,k}, \phi_{l',k'} \rangle, \quad \bar{\alpha}_{k,k'}^{l,l'} = \langle \partial_x \phi_{l,k}, \partial_x \phi_{l',k'} \rangle, \quad \beta_{k,k'}^{l,l'} = \langle \phi_{l,k}, \partial_x \phi_{l',k'} \rangle. \]  

(17)

For each \( l = 1, \ldots, N \) and \( r, r' = 2, 3, \ldots \) we define a \( 2 \times 2 \) matrix \( M_{l,1}^l \), a \( 2 \times (r - 1) \) matrix \( M_{l,r}^l \), and an \((r - 1) \times (r' - 1) \) matrix \( M_{r,r'}^l \) with entries

\[
[M_{l,1}^l]_{ij} = \alpha_{1,1}^{l+i-1}, \quad i, j = 1, 2, \\
[M_{l,r}^l]_{ij} = \alpha_{l,1}^{l+i-1}, \quad i = 1, 2, j = 2, \ldots, r, \\
[M_{r,r'}^l]_{ij} = \alpha_{r,1}^{l+i}, \quad i = 2, \ldots, r, j = 2, \ldots, r'.
\]

Similarly, we define matrices \( S_{l,1}^l, S_{l,r}^l, S_{r,r'}^l \) with \( \bar{\alpha}_{r,r'}^{l,l'} \), and \( B_{l,1}^l, B_{l,r}^l, B_{r,r'}^l \) with \( \beta_{r,r'}^{l,l'} \). We then define

\[
M_r^l = \begin{bmatrix} M_{l,1}^l & M_{l,r}^l \\ (M_{l,r}^l)^T & M_{r,r}^l \end{bmatrix}, \quad S_r^l = \begin{bmatrix} S_{l,1}^l & S_{l,r}^l \\ (S_{l,r}^l)^T & S_{r,r}^l \end{bmatrix},
\]

\[
B_{r,r'}^l = \begin{bmatrix} B_{l,1}^l & B_{l,r}^l \\ (B_{l,r}^l)^T & B_{r,r'}^l \end{bmatrix}.
\]

The matrices \( M_r^l \) and \( S_r^l \) have size \((r + 1) \times (r + 1)\), whereas the matrix \( B_{r,r'}^l \) has size \((r + 1) \times (r' + 1)\). The global matrices \( M_r, S_r \) and \( B_{r,r'} \) have elements \( M_r^1, S_r^1 \) and \( B_{r,r'}^1 \), respectively. The sizes of \( M_r \) and \( S_r \) are \((Nr + 1) \times (Nr + 1)\) and the size of \( B_{r,r'} \) is \((Nr + 1) \times (Nr' + 1)\).

For each \( l = 1, \ldots, N \) we also define vectors

\[
U_l^l = [u_{l,1}, u_{l+1,1}, u_{l,2}, \ldots, u_{l,p}]^T \quad \text{and} \quad V_l^l = [v_{l,1}, v_{l+1,1}, v_{l,2}, \ldots, v_{l,q}]^T,
\]

where the elements are defined in (15)–(16) and \( U_{1,1} \) and \( U_{N+1,1} \) are zero. The vectors \( U \) and \( V \) are of size \((Np + 1) \times 1\) and \((Nq + 1) \times 1\), respectively, and are assembled from the vectors \( U_l^l \) and \( V_l^l \).
With the matrices defined above, the matrix representation of (12)–(13) is

\begin{align*}
P_u(t) &= \mathbf{B}_{p,q} V(t), \quad (18) \\
M_q \partial_t V(t) + \nu S_q V(t) &= \mathbf{G}[U(t), V(t)]. \quad (19)
\end{align*}

Here,

\[ \mathbf{G}(U, V) = [\mathbf{G}^{(0)}, \mathbf{G}^{(1)}, \ldots, \mathbf{G}^{(N)}]^T \]

is an \((Nq + 1) \times 1\) vector with

\[ \mathbf{G}^{(0)} = [\langle UV, \phi_{1,1} \rangle, \langle UV, \phi_{2,1} \rangle, \ldots, \langle UV, \phi_{N+1,1} \rangle]^T \]

and

\[ \mathbf{G}^{(1)} = [\langle UV, \phi_{1,2} \rangle, \langle UV, \phi_{1,3} \rangle, \ldots, \langle UV, \phi_{1,q} \rangle]^T \]

for \(l = 1, \ldots, N\). We use the Matlab ODE solver to solve (18)–(19). The right hand side of (19) is computed by first solving (18) for a given \(V(t)\).

## 3 A posteriori error estimates and implementation issues

In this section we design methods to compute the error estimates. We infer that \(e\) and \(f\) satisfy

\begin{align*}
\langle \partial_x e, \partial_x \chi_h \rangle &= \langle f, \partial_x \chi_h \rangle \quad \text{for all } \chi_h \in \mathbb{V}_h^p, \\
\langle \partial_t f, w_h \rangle + \nu \langle \partial_x f, \partial_x w_h \rangle - \langle ef, \partial_x w_h \rangle - \langle Uf, \partial_x w_h \rangle - \langle eV, \partial_x w_h \rangle \\
&= -\nu \langle \partial_x V, \partial_x w_h \rangle + \langle UV, \partial_x w_h \rangle - \langle \partial_t V, w_h \rangle \quad \text{for all } w_h \in \mathbb{V}_h^q. \quad (21)
\end{align*}

At \(t = 0\), from (8) and (14),

\[ \langle f, w_h \rangle = 0 \quad \text{for all } w_h \in \mathbb{V}_h^q. \]
Due to (13), the right hand side of (21) vanishes. However, for the purpose of developing a posteriori error estimates, we keep these terms in the equation. We approximate the exact errors $e$ and $f$, respectively, by

$$E(x, t) = \sum_{l=1}^{N} E_l(t) \phi_{l,p+1}(x) \in S_{h}^{p+1},$$

$$F(x, t) = \sum_{l=1}^{N} F_l(t) \phi_{l,q+1}(x) \in S_{h}^{q+1}.$$

which are computed locally on each element $(x_l, x_{l+1})$, for $l = 1, \ldots, N$, from the approximate solutions $(U, V)$.

An accurate error estimation is one that satisfies

$$\lim_{h \to 0} \Theta(t) = 1, \quad t \in [0, T],$$

(22)

where

$$\Theta(t) := \frac{\hat{E}(t)}{\hat{e}(t)},$$

with

$$\hat{e}(t) := \|e(t)\|_{H^1(\Omega)} + \|f(t)\|_{H^1(\Omega)}, \quad \hat{E}(t) := \|E(t)\|_{H^1(\Omega)} + \|F(t)\|_{H^1(\Omega)}.$$

We propose four different methods to compute $E_l$ and $F_l$, $l = 1, \ldots, N$. The first equation to be solved for each method is:

1. **Nonlinear parabolic error estimate:** (cf. (21))

$$\langle \partial_t F_l, \phi_{l,q+1} \rangle_l + \nu \langle \partial_x F_l, \partial_x \phi_{l,q+1} \rangle_l - \langle E_l F_l, \partial_x \phi_{l,q+1} \rangle_l - \langle U F_l, \partial_x \phi_{l,q+1} \rangle_l$$

$$- \langle \nabla E_l, \partial_x \phi_{l,q+1} \rangle_l$$

$$= -\nu \langle \partial_x V, \partial_x \phi_{l,q+1} \rangle_l + \langle U V, \partial_x \phi_{l,q+1} \rangle_l - \langle \partial_t V, \phi_{l,q+1} \rangle_l.$$
2. **Nonlinear elliptic error estimate:** We neglect the time rate of change in Method 1 so that
\[
\nu \langle \partial_x F_1, \partial_x \phi_{1,q+1} \rangle_l - \langle E_1 F_1, \partial_x \phi_{1,q+1} \rangle_l - \langle U F_1, \partial_x \phi_{1,q+1} \rangle_l - \langle V E_1, \partial_x \phi_{1,q+1} \rangle_l \\
= -\nu \langle \partial_x V, \partial_x \phi_{1,q+1} \rangle_l + \langle U V, \partial_x \phi_{1,q+1} \rangle_l - \langle \partial_t V, \phi_{l,q+1} \rangle_l.
\]

3. **Linear parabolic error estimate:** An additional reduction in the computation cost is obtained by neglecting the nonlinear term \( \langle E_1 F_1, \partial_x \phi_{1,q+1} \rangle_l \) in Method 1:
\[
\langle \partial_t F_1, \phi_{l,q+1} \rangle_l + \nu \langle \partial_x F_1, \partial_x \phi_{1,q+1} \rangle_l - \langle U F_1, \partial_x \phi_{1,q+1} \rangle_l - \langle V E_1, \partial_x \phi_{1,q+1} \rangle_l \\
= -\nu \langle \partial_x V, \partial_x \phi_{1,q+1} \rangle_l + \langle U V, \partial_x \phi_{1,q+1} \rangle_l - \langle \partial_t V, \phi_{l,q+1} \rangle_l. \tag{23}
\]

4. **Linear elliptic error estimate:** We neglect the nonlinear term in Method 2 so that
\[
\nu \langle \partial_x F_1, \partial_x \phi_{l,q+1} \rangle_l - \langle U F_1, \partial_x \phi_{l,q+1} \rangle_l - \langle V E_1, \partial_x \phi_{l,q+1} \rangle_l \\
= -\nu \langle \partial_x V, \partial_x \phi_{l,q+1} \rangle_l + \langle U V, \partial_x \phi_{l,q+1} \rangle_l - \langle \partial_t V, \phi_{l,q+1} \rangle_l.
\]

Each equation in Methods 1–4 is coupled with (cf. (20))
\[
\langle \partial_x E_1, \partial_x \phi_{l,p+1} \rangle_l = \langle F_1, \partial_x \phi_{l,p+1} \rangle_l, \tag{24}
\]
and an initial condition \( \langle F_1, \phi_{l,q+1} \rangle = \langle \partial_x u_0, \phi_{l,q+1} \rangle - \langle V, \phi_{l,q+1} \rangle \).

We finish this section with a discussion on implementation issues for the linear parabolic case. From (17), we have
\[
\langle \partial_x V, \partial_x \phi_{l,q+1} \rangle_l = V_{l+1,1} \bar{\alpha}^{l+1,l}_{1,q+1} + \sum_{k'=1}^{q} V_{l,k'} \bar{\alpha}^{l,l}_{k',q+1} := T_1,
\]
and
\[
\langle \partial_t V, \phi_{l,q+1} \rangle_l = \partial_t V_{l+1,1} \alpha^{l+1,l}_{1,q+1} + \sum_{k'=1}^{q} \partial_t V_{l,k'} \alpha^{l,l}_{k',q+1} := T_2.
\]
Also
\[ \alpha_{p+1,p+1}^{l,l} = h_l / ((2p + 3)(2p - 1)), \quad \bar{\alpha}_{p+1,p+1}^{l,l} = 2 / h_l, \]
and
\[ \beta_{p+1,q+1}^{l,l} = \begin{cases} 1 / \sqrt{(2q + 3)(2q + 1)}, & p = q + 1, \\ -1 / \sqrt{(2q + 1)(2q - 1)}, & p = q - 1, \\ 0, & \text{otherwise}. \end{cases} \]

By defining
\[ \bar{\beta}_{k,k',q}^{l} = \langle \phi_{l,k} \phi_{l,k'}, \partial_x \phi_{l,q+1} \rangle, \quad \tilde{\beta}_{k,k',q}^{l} = \langle \phi_{l+1,k} \phi_{l,k'}, \partial_x \phi_{l,q+1} \rangle \]
and
\[ \hat{\beta}_{k,k',q}^{l} = \langle \phi_{l+1,k} \phi_{l+1,k'}, \partial_x \phi_{l,q+1} \rangle, \]
we have
\[ \langle UF_l, \partial_x \phi_{l,q+1} \rangle_l = F_l \left[ U_{l+1,l} \hat{\beta}_{1,q+1}^{l} + \sum_{k=1}^{p} U_{l,k} \bar{\beta}_{k,q+1}^{l} \right] := T_3 F_l, \]
\[ \langle VE_l, \partial_x \phi_{l,q+1} \rangle_l = E_l \left[ V_{l+1,l} \tilde{\beta}_{k,p+1}^{l} + \sum_{k'=1}^{q} V_{l,k'} \bar{\beta}_{k',p+1}^{l} \right] := T_4 E_l, \]
and
\[ \langle UV, \partial_x \phi_{l,q+1} \rangle_l = U_{l+1,l} \left[ V_{l+1,l} \hat{\beta}_{l,q+1}^{l} + \sum_{k'=1}^{q} V_{l,k'} \bar{\beta}_{k,q+1}^{l} \right] \\
+ \sum_{k=1}^{p} U_{l,k} \left[ V_{l+1,l} \hat{\beta}_{k,l,q+1}^{l} + \sum_{k'=1}^{q} V_{l,k'} \bar{\beta}_{k,k',q}^{l} \right] := T_5. \]

The values of \( \bar{\beta}_{k,k',q}^{l}, \tilde{\beta}_{k,k',q}^{l} \) and \( \hat{\beta}_{k,k',q}^{l} \) can be computed using Maple.

By using the above definitions of \( T_1, \ldots, T_5 \), (23) is rewritten as
\[ \frac{h_l}{(2q + 3)(2q - 1)} \partial_t F_l(t) + \left( \frac{2 \nu}{h_l} - T_3 \right) F_l(t) - T_4 E_l(t) = -\nu T_1 + T_5 - T_2. \]
Moreover, (24) is rewritten as

\[ 2E_l(t) = h_l \beta_{p+1,q+1} \phi_1(t). \]

Then, by using the backward Euler formulation, we compute \( F_l(t_j) \) recursively using

\[
\left( d + \frac{2\nu}{h_l} - T_3 - T_4 \beta_{p+1,q+1} \frac{h_l}{2} \right) F_l(t_j) = -\nu T_1 + T_5 - T_2 + dF_l(t_{j-1}),
\]

where \( d = h_l/(2q + 3)(2q - 1)(t_j - t_{j-1}) \) and \( t_j = j\Delta t \) for \( j = 1, 2, 3, \ldots \). The time step \( \Delta t \) is chosen to be not less than \( h \).

### 4 Numerical results

In this section, we present the numerical results obtained when solving (1)–(3) whose exact solutions are

\[
\begin{align*}
    u(x, t) &= \frac{2\nu \pi a \sin(\pi x)}{2 + a \cos(\pi x)}, \\
    v(x, t) &= \frac{2\nu \pi^2 a \cos(\pi x)}{2 + a \cos(\pi x)} + 2\nu \frac{[\pi a \sin(\pi x)]^2}{[2 + a \cos(\pi x)]^2},
\end{align*}
\]

where \( a = \exp(-\pi^2 \nu t) \). The initial value is

\[ u_0(x) = 2\nu \pi \sin(\pi x)/(2 + \cos(\pi x)). \]

In the following, we choose \( \nu = 0.05 \) and \( p = q + 1 \). The numerical results are also satisfactory for a larger \( \nu \), such as \( \nu = 0.5 \). We present the numerical results for \( \nu = 0.05 \) only.

In the numerical experiment, we computed the approximate solution \((U, V)\) by solving (18)–(19). We then computed the errors \( e \) and \( f \) to check on the convergence rate given by Proposition 3. Finally, we computed the error estimations \( E \) and \( F \) by using the linear parabolic and linear elliptic a posteriori error estimate methods 3 and 4 introduced in Section 3.
Table 1: The orders of convergence for \((u, v)\) at \(t = 0.8\).

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>N</th>
<th>(|e_h(t)|_{H^1(\Omega)})</th>
<th>(\kappa_u)</th>
<th>(|f_h(t)|_{H^1(\Omega)})</th>
<th>(\kappa_v)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>20</td>
<td>1.1338E-3</td>
<td>6.6472E-2</td>
<td>6.3245E-2</td>
<td>0.999</td>
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<td></td>
<td></td>
<td>40</td>
<td>2.8487E-4</td>
<td>1.993</td>
<td>3.3245E-2</td>
<td>1.000</td>
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<td></td>
<td></td>
<td>80</td>
<td>7.1305E-5</td>
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<td>1.6624E-2</td>
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<td></td>
<td></td>
<td>160</td>
<td>1.7831E-5</td>
<td>1.999</td>
<td>8.3120E-3</td>
<td>1.000</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>20</td>
<td>2.2153E-5</td>
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<td>2.8620E-3</td>
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<td></td>
<td>40</td>
<td>2.7675E-6</td>
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<td>7.1684E-4</td>
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<td></td>
<td></td>
<td>80</td>
<td>3.4587E-7</td>
<td>3.000</td>
<td>1.7929E-4</td>
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<tr>
<td></td>
<td></td>
<td>100</td>
<td>1.7708E-7</td>
<td>3.000</td>
<td>1.1476E-4</td>
<td>1.999</td>
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</table>

Table 2: The effectivity indexes \(\Theta\) at \(t = 0.8\).

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>(h)</th>
<th>(\hat{\varepsilon}(t))</th>
<th>Method 3</th>
<th>(\hat{E}(t))</th>
<th>(\Theta(t))</th>
<th>Method 4</th>
<th>(\hat{E}(t))</th>
<th>(\Theta(t))</th>
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</thead>
<tbody>
<tr>
<td>2</td>
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<td>1/20</td>
<td>6.7606E-2</td>
<td>6.6984E-2</td>
<td>0.991</td>
<td>6.6543E-2</td>
<td>0.984</td>
<td>6.6543E-2</td>
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<td></td>
<td>1/40</td>
<td>3.3530E-2</td>
<td>3.3364E-2</td>
<td>0.995</td>
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<tr>
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<td></td>
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<td>1.6652E-2</td>
<td>0.997</td>
<td>1.6645E-2</td>
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<td>8.3188E-3</td>
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<tr>
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<td>7.1961E-4</td>
<td>7.1833E-4</td>
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<td>1.1484E-4</td>
<td>0.999</td>
<td>1.1484E-4</td>
<td>0.999</td>
</tr>
</tbody>
</table>

Table 1 presents the exact errors \(\|e(t)\|_{H^1(\Omega)}\) and \(\|f(t)\|_{H^1(\Omega)}\) for \(t = 0.8\). As predicted by Proposition 3, the convergence rate is \(\|e(t)\|_{H^1(\Omega)} = O(h^p)\) and \(\|f(t)\|_{H^1(\Omega)} = O(h^{p-1})\). Table 2 presents the computed a posteriori error estimate \(\hat{E}\) and the effectivity index \(\Theta(t)\), at \(t = 0.8\). For Method 3, when solving (25) we chose \(\Delta t = 0.4\). The results show that our a posteriori error estimations are efficient.
5 Conclusion

We designed algorithms to estimate the true error when a model problem is solved using $H^1$-MFEM. Our numerical experiments support our theoretical claims in Proposition 3 and (22). We emphasise that the computation of the error estimations $(E_l, F_l)$ for $l = 1, \ldots, N$ can be carried out in parallel on each element $(x_l, x_{l+1})$. A theoretical study to show $\lim_{h \to 0} \Theta(t) = 1$ is the subject of a future paper.

References


References


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