Optimising series solution methods for flow over topography—Part 1

P. J. Higgins* S. R. Belward* W. W. Read*

(received 22 November 2004, revised 7 November 2005)

Abstract

Series solution methods have recently been used to solve fully nonlinear flow over topography problems. These methods are iterative schemes that update an initial estimate of the fluid surface (a free boundary) using a cost function. Series solutions are obtained efficiently and accurately with exact error bounds immediately available. Critical to the speed of the procedure is the implementation of efficient computer code and numerical techniques. In this paper we discuss methods that improve the computational time of the original implementation by several orders of magnitude, without any loss of accuracy. The efficiency of the improved method is demonstrated by generating two dimensional solutions to subcritical flow over an isolated cosine shaped obstacle.

*School of Mathematical & Physical Sciences, James Cook University, Townsville, Queensland, Australia. mailto:Patrick.Higgins@jcu.edu.au

See http://anziamj.austms.org.au/V46/CTAC2004/Higg for this article, © Austral. Mathematical Soc. 2005. Published November 29, 2005. ISSN 1446-8735

Contents

1 Introduction

Many potential flow problems satisfy Laplace's equation. Often the location of one of the boundaries is unknown giving rise to a free boundary problem. Specific examples include groundwater seepage [6] and flow over topography [4]. Solution methods are typically iterative processes that consist of estimating the location of the free boundary, solving the known boundary problem at each step, then updating the free boundary location using a cost function. The known boundary problem is often solved numerically by boundary integral or boundary element methods [1]. However, these schemes can be computationally expensive.

Recently, it has been shown that series solution methods are efficient for solving Laplacian free boundary problems [3, 7]. At each step of the procedure, an analytic series solution is obtained with maximum error bounds immediately available. These error bounds may be used to examine the convergence behaviour of the solution. A comparison between boundary integral and analytic series methods shows that the analytic series approach has a superior efficiency and accuracy when calculating two dimensional flow over topography [5]. The series method has been successful in calculating supercritical, transcritical and subcritical solutions to flow over cosine shaped, asymmetric and arbitrary shaped obstacles.

The ultimate aim of this research is to determine three dimensional solutions to realistic fluid flow over topography. In order to achieve this, the efficiency of the two dimensional solution method must be optimised. In this two part paper, we discuss techniques used to improve the efficiency of the analytic series solution method for flow over topography problems. In Part 1 the methods include vectorising the code, accurately estimating integrals and using a proportional knot point positioning. Implementing these improvements we demonstrate a decrease in computational time by several orders of magnitude with no loss of accuracy. In Part 2 [2] we discuss the update method for the free surface and show that using information about the update of upstream knot points allows an improved update for downstream knot points. These improvements result in an order of magnitude decrease in the number of free surface updates required for computation of solutions.

A brief outline of Part 1 of the paper now follows. In Section 2 we describe the flow over topography problem, present a mathematical formulation and give a brief description of the series solution approach. Section 3 introduces the techniques we have used to optimise the efficiency of the solution method. Finally, in Section 4 we present and discuss the results showing the decrease in computational time as efficiencies are implemented on the procedure.

2 Problem description and solution method

The flow over topography model we use assumes that the fluid is inviscid, incompressible, of constant density and flows without rotation over an obstacle of arbitrary shape. We assume a uniform flow upstream and that flow quantities far downstream of the obstacle are bounded. All variables are nondimensionalised resulting in an upstream flow with unit depth and speed in the dimensionless system. With these assumptions, the stream function $\Psi(x, y)$ satisfies Laplace's equation:

$$\nabla^2\Psi = 0\,. \tag{1}$$

We also assume that the free surface $y = \eta(x)$, is a streamline, and that there is no penetration through the topography $y = f^b(x)$, giving the top and bottom boundary conditions

$$\Psi(x, \eta(x)) = 1 \quad \text{and} \quad \Psi(x, f^b(x)) = 0. \tag{2}$$

The flow domain is truncated to a finite length $x \in [-s, s]$, where s is chosen large enough such that the flow continues to satisfy the above conditions. With this truncation, the side boundary conditions are

$$\Psi(-s, y) = y \quad \text{and} \quad \Psi(s, y) = \frac{y}{\eta_s}\,, \tag{3}$$

where η_s is the height of the fluid at $x = s$. As the flow is irrotational, the Bernoulli equation evaluated along the free surface gives

$$\frac{1}{2}F^2u^2 + \eta = \frac{1}{2}F^2 + 1\,, \tag{4}$$

where u is the magnitude of the velocity and F is the Froude number which is defined in terms of three dimensional quantities: the upstream fluid speed U, depth H, and acceleration due to gravity g; and then $F = U/\sqrt{gH}$. The magnitude of the velocity at any point in the fluid is expressed in terms of the stream function as

$$u^2 = \left(\frac{\partial\Psi}{\partial x}\right)^2 + \left(\frac{\partial\Psi}{\partial y}\right)^2 .$$

Note that the function $\eta(x)$ which appears in two of the boundary conditions is unknown. This means we are solving a nonlinear free boundary problem.

The most difficult flow over topography problem to solve is the subcritical case, Froude number $F < 1$, characterised by a train of lee waves downstream of the obstacle. This is because the free surface possesses a more complicated profile with areas of high curvature. The success of any solution method is measured by its performance in obtaining subcritical solutions. Therefore this is the region of the parameter space in which we seek solutions.

2.1 Series solution method

The stream function is transformed to a related function $\psi(x,y)$ by

$$\Psi(x,y) = \psi(x,y) + y + \frac{(x+s)y}{2s}\left[\frac{1}{\eta_s} - 1\right]. \tag{5}$$

With this transformation equations (1) and (3) show that $\psi(x,y)$ will satisfy Laplace's equation with homogeneous side boundary conditions. The top and bottom boundary conditions (2) become

$$\psi[x,\eta(x)] = 1 - \eta(x) - \frac{(x+s)\eta(x)}{2s}\left[\frac{1}{\eta_s} - 1\right] = h^t(x)\,, \tag{6}$$

$$\psi[x,f^b(x)] = -f^b(x) - \frac{(x+s)f^b(x)}{2s}\left[\frac{1}{\eta_s} - 1\right] = h^b(x)\,. \tag{7}$$

The Bernoulli equation (4) is recast in terms of the partial derivatives of ψ. Once the transformed problem has been solved, a solution to the original problem is immediately available.

Using separation of variables, the general solution to equation (1) is represented by an infinite series; however, we truncate the series so that $\psi \approx \psi_N$ where

$$\psi_N(x,y) = \sum_{n=1}^{N} a_n u_n(x,y) + b_n v_n(x,y)\,. \tag{8}$$

Here u_n and v_n are eigenfunctions chosen to satisfy the partial differential equation (1) and side boundary conditions (3). Given an estimate of the location $y = \eta(x)$ of the free surface, the series coefficients a_n and b_n are calculated using the upper and lower boundary conditions (6) and (7). This process is covered in full by Read [6] and is not presented here.

Note that ψ_N is intended to approximate ψ and is computed from an estimate of the position of the free surface. We expect ψ_N to be a more accurate approximate of ψ as the value of N increases. In fact we have observed exponential convergence of $\psi_N \to \psi$ given a favourable choice for the representation of the free surface [3].

It is clear from the paragraphs above that the function $\psi(x, y)$ we compute changes as $\eta(x)$ in equation (6) changes. Note also a solution to the flow over topography problem must satisfy the Bernoulli equation (4). Therefore we update $\eta(x)$ (and recompute ψ) based on how closely this equation is satisfied.

2.2 Iterative update scheme

The iterative process updates the top boundary representation $\eta(x)$, at a set of knot points $\{x_j, j = 1, \ldots, M\}$ and has the form

$$\eta^{(i+1)}(x_j) = \eta^{(i)}(x_j) - c\delta\eta^{(i)}(x_j)\,, \tag{9}$$

where i is the iteration count and c is an update constant which is chosen to enhance the convergence rate.

The first step at each iteration requires the location of the knot points to be determined. We have found it essential to position knot points in areas of high curvature. Thus we solve

$$\int_{-s}^{x_j} \sqrt{1 + p\,\ddot{\eta}(x)^2}\,dx = \left(\frac{j-1}{M-1}\right) \int_{-s}^{s} \sqrt{1 + p\,\ddot{\eta}(x)^2}\,dx \tag{10}$$

for x_j where $j = 2, 3, \ldots, M - 1$. We choose $x_1 = -s$ and $x_M = s$ for every iteration. The parameter p has the effect of clustering knots in regions of higher curvature as its value is increased.

The free boundary increment $\delta\eta^{(i)}(x)$ is calculated by a cost function. We use an integrated cost function which is based on the Bernoulli equation (4). In brief, we solve for the fluid speed u and integrate to determine a velocity potential. This is compared to a velocity potential we obtain when we use the Cauchy–Riemann equations on Ψ. Exact details are presented in Part 2 of this paper, see equation (3) there. For Part 1 of the paper, the important point to observe is that to evaluate the cost function at a knot point x_j, an integral along the free surface over $[-s, x_j]$ needs to be computed.

In order to compute these integrals we need to determine the position of the upper surface at locations between the knot points. We use an interpolant based on a Fourier sine series representation of the free surface. This provides the exponential convergence we discussed in Section 2.1.

The accuracy of the solution is immediately available at each iteration by examining the root mean squared (r.m.s.) error in the top and bottom boundary conditions and the cost function. The r.m.s. errors may be used in a convergence criterion for the update method. If the obstacle is small, the solution is stable and the errors decrease to a minimum and remain for all following iterations. For higher obstacles errors may fluctuate. Then we find that choosing the solution with the global minimum cost function error is appropriate.

3 Efficiency improvements

In order to determine solutions for three dimensional flow over topography, the efficiency of the two dimensional solution method must be optimised. In this section, we introduce techniques that we have used to achieve this.

With these methods, we have calculated accurate solutions to complicated flow problems in approximately 30 seconds.

Our first attempt to improve efficiency involved vectorising the code. For convenience we chose to code in Matlab as we required the program to be portable and without reliance on external subroutines. Therefore as a benchmark we use "unvectorised" Matlab code.

Increased efficiency was observed by vectorising the evaluation of the series solutions for the stream function and the velocity potential, the Fourier series approximations for the top and bottom boundaries and the first and second derivatives of each. See Table 1 for the percentage increase in efficiency after vectorising the code.

3.1 Evaluating the integrals

The next element of the solution method that was affecting the efficiency was the evaluation of the integrals required for the cost function and the r.m.s. errors. The fastest Matlab routine available was `quadl`, which uses adaptive Lobatto quadrature to approximate the integral to within a predefined tolerance. We assumed the recursive nature of this routine was taking the most time. Therefore, we estimated the integrals with Gaussian quadrature as implemented by Trefethen [8].

To use the cost function an integral over $[-s, x_j]$ is computed. The integral is evaluated between consecutive knot points (x_{j-1}, x_j), and these are summed giving a composite Guassian rule. To determine the number of quadrature points required between each knot point, we compare consecutive estimates of a typical integral $(I_n,\, n = 1, 2, \ldots, 20)$ where n is the number of points. In this case I_n is an integral required for the cost function. This comparison is displayed in Figure 1.

This Figure shows that 6 or 7 quadrature points is sufficient. The distance

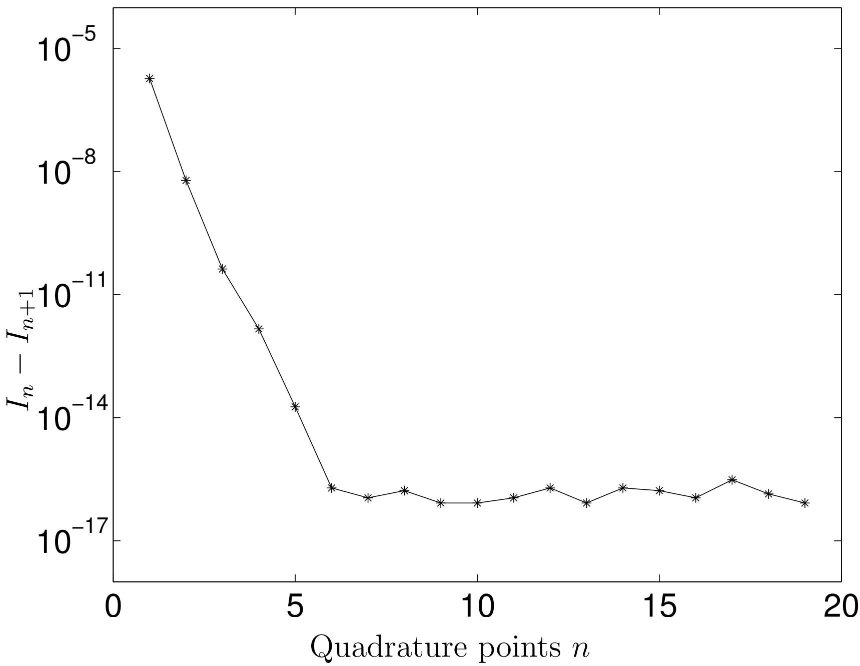

Figure 1: Comparison of consecutive estimates of a typical integral I_n for an increasing number of quadrature points n.

between knot points decreases as the number of knot points is increased, therefore fewer quadrature points are required if the number of knots increases. The r.m.s. errors are defined in terms of integrals over the entire solution domain $[-s, s]$ and therefore require significantly more quadrature points (≈ 50).

3.2 Proportional knot spacing

As discussed in Section 2.2 our initial approach is to cluster knot points around areas of higher curvature according to equation (10). This is another process that takes considerable time. Gaussian quadrature can be used to evaluate the integrals, but the zero finding routine cannot be vectorised. This routine is used to calculate the x coordinates of each knot point which is the upper limit of the integral on the left. Therefore, to decrease computational time, the x coordinates are approximated directly by

$$x_j = 2s \sum_{m=1}^{j} \frac{1}{\sqrt{1 + p\,\ddot{\eta}(x_m)^2}} \Bigg/ \sum_{m=1}^{M} \frac{1}{\sqrt{1 + p\,\ddot{\eta}(x_m)^2}} \,, \tag{11}$$

where the bottom summation is an approximation of the total modified arc length, and the top summation is the jth proportion of the total. We calculate the position of the new knot points x_j in terms of their positions from the previous iteration x_m.

4 Results and discussions

To test the efficiency of the solution process, and any improvements, we calculate a subcritical flow solution with Froude number $F = 0.5$. The obstacle is a standard cosine shape with maximum height $h = 0.1$, and a

Table 1: Times taken to calculate the subcritical flow solution in Figure 2 including different efficiency improvements.

Improvement	Time (secs)	Percentage of original
No improvements	4331	100.0 %
Vectorised code	1802	41.6 %
Estimating integrals	1239	28.6 %
Estimating knot spacing	663	15.3 %
All improvements	31	0.7 %

half base length $l = 2$. The bottom boundary is

$$f^b(x) = \begin{cases} \frac{h}{2}\left[\cos(\pi x/l) + 1\right] , & -l \leq x \leq l \,, \\ 0 \,. & -s \leq x \leq -l \text{ and } l \leq x \leq s \,. \end{cases} \tag{12}$$

where $s = 7$ in this case. The initial estimate of the free surface is the straight line $\eta(x) = 1$. We use the cost function given in equation (3) from Part 2 with an update constant $c = 0.2$. The number of terms in the series solution $N = 100$, and the number of update knot points $M = 100$. The arc length constant used in the knot point spacing $p = 5$. With these settings, convergence is achieved in 200 iterations. The solution is shown in Figure 2. See [3] and [7] for full details of this solution.

Note that upon implementing the procedures mentioned above there was no significant differences in the profiles, $\eta(x)$, or the r.m.s. errors of the solutions. The errors in the cost function, top boundary and bottom boundary for this solution are $9.2 \times 10^{-6}, 1.3 \times 10^{-5}$ and 1.4×10^{-6} respectively.

The original implementation with no efficiency improvements took approximately 72 minutes to converge [7]. We use this length of time as our benchmark for all subsequent speed comparisons. Table 1 displays the time taken to converge to the solution in Figure 2 after including the efficiency improvements discussed in the previous section.

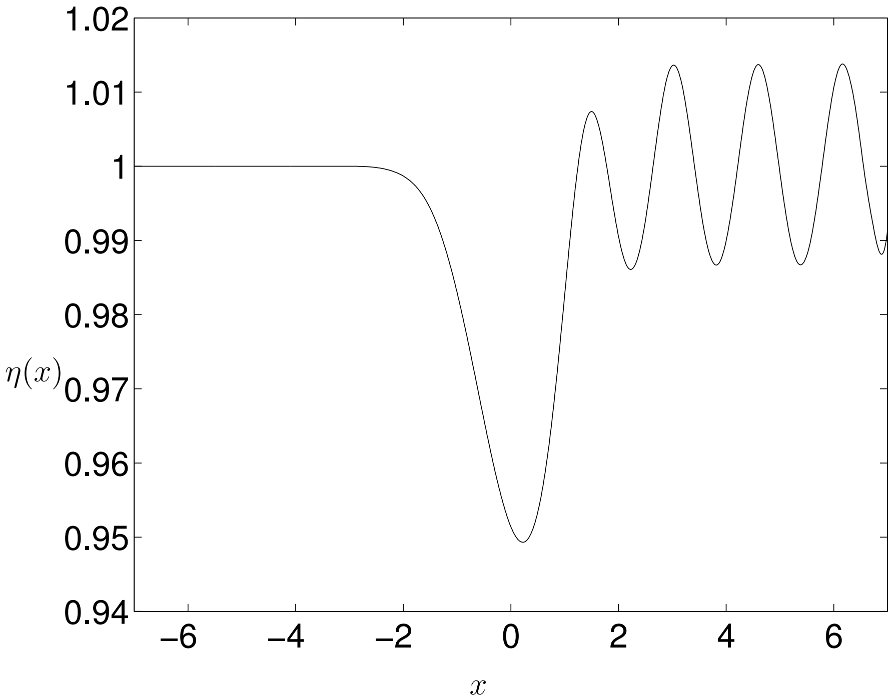

FIGURE 2: Subcritical flow solution after 200 iterations over a cosine shaped obstacle with maximum height $h = 0.1$ and half base length $l = 2$. The Froude number $F = 0.5$.

These times were obtained with the `tic` and `toc` Matlab commands. The program was run in Matlab version 7 on a pentium 4 laptop computer. Table 1 shows that with the above improvements, the computational time of the solution method has been decreased by 99.3%.

We have tested these efficiency improvements on a range of obstacle heights. Choosing the optimal update constant c as the obstacle height increases becomes more difficult. However, once the correct update constant is used, a similar decrease in computational time is observed for all obstacle heights. See [5] for the full range of solutions calculated with the analytical series method.

With the efficiencies demonstrated in this paper, we are able to routinely calculate accurate, two dimensional, subcritical flow solutions in approximately half a minute. These improvements are a major step towards obtaining three dimensional solutions to complicated flow over topography problems. However, the further refinement of the update procedure presented in Part 2 of this paper enables even faster computation of solutions.

References

[1] S. R. Belward and L. K. Forbes. Fully non-linear two-layer flow over arbitrary topography. *J. Engin. Math.*, 27:419–432, 1993. http://dx.doi.org/10.1007/BF00128764 C1273

[2] S. R. Belward, P. J. Higgins, and W. W. Read. Optimising series solutions methods for flow over topography—Part 2. *ANZIAM J.*, 46(E):C1286–C1295, 2005. http://anziamj.austms.org.au/V46/CTAC2004/Belw C1274

[3] S. R. Belward, W. W. Read, and P. J. Higgins. Efficient series solutions for non-linear flow over topography. *ANZIAM J.*, 44(E):C96–C113,

2003. http://anziamj.austms.org.au/V44/CTAC2001/Belw C1273, C1277, C1282

[4] L. K. Forbes and L. W. Schwartz. Free-surface flow over a semicircular obstruction. *J. Fluid Mech.*, 114:299–314, 1982. C1273

[5] P. J. Higgins, W. W. Read, and S. R. Belward. A series solution method for free boundary problems arising from flow over topography. Technical report, School of Mathematical and Physical Sciences, James Cook University, 2004. C1274, C1284

[6] W. W. Read. Series solutions for Laplace's equation with nonhomogeneous mixed boundary conditions and irregular boundaries. *Math. Comput. Modelling*, 17(12):9–19, 1993. http://dx.doi.org/10.1016/0895-7177(93)90023-R C1273, C1277

[7] W. W. Read, S. R. Belward, and P. J. Higgins. An efficient iterative scheme for series solutions to Laplacian free boundary problems. *ANZIAM J.*, 44(E):C644–C663, 2003. http://anziamj.austms.org.au/V44/CTAC2001/Read C1273, C1282

[8] L. N. Trefethen. *Spectral Methods in MATLAB*. SIAM, Philadelphia, 2000. C1279