Optimising series solution methods for flow over topography—Part 2

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Abstract

We consider the procedure used for computing series solutions to two dimensional fully non-linear flow over topography. Even though we use the simplest model for flow over topography there are many challenges when it comes to computing solutions. We discuss update methods that iterate an initial estimate of the free surface to its final position. Updates are performed at a discrete set of knot points. We show that using information about upstream knot point updates is beneficial for the update of downstream knot points. When we are careful about how updates are performed, an order of magnitude decrease in the total number of free surface updates occurs.

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1 Introduction

In Part 1 of this paper [2] we presented the background for this work. It is the update method, used to iterate the free surface from an initial estimate, to its final position, that we consider in this part of the paper. Computationally the free surface is updated at a set of knot points. The algorithm in Part 1 computed updates in isolation, that is the update of the jth knot point was calculated without regard for the update of any other point. We show that using information about the update of upstream knot points typically results in an order of magnitude decrease in the number of iterations required to achieve a specified accuracy.

In Section 2 a description of both the original and the improved update methods is presented. In Section 3 results are presented and summarised.

2 Update method

As discussed in Part 1 [2] of the paper the free surface $y = \eta(x)$ is determined at a set of knot points. At locations in between knot points, its value is determined from a Fourier sine series interpolant. The free surface position is then updated according to how well the transformed stream function ψ satisfies the cost function.

The y coordinate of each knot point is updated according to

$$\eta^{(i+1)}(x_j) = \eta^{(i)}(x_j) - c\,\delta\eta^{(i)}(x_j)\,, \tag{1}$$

where c is a relaxation parameter that can be altered as needed to ensure the convergence to a solution of the free boundary problem is optimised and the superscript (i) refers to the iteration count. The quantity $\delta\eta^{(i)}$ is determined by comparing two estimates of the velocity potential Φ at the point $(x_j, \eta^{(i)}(x_j))$. The first estimate is based on the estimate of the transformed stream function ψ_N given in equation (8) from Part 1 [2]:

$$\Phi_N(x,y) = \phi_N(x,y) + x + \frac{1}{4s}[(x+s)^2 - y^2]\left[\frac{1}{\eta_s} - 1\right],$$

where $\phi_N(x,y) = \sum_{n=1}^{N} a_n\bar{u}_n + b_n\bar{v}_n$. The coefficients a_n and b_n are the same as those in equation (8) from Part 1 [2] with $\bar{u}_n$ and $\bar{v}_n$ related to u_n and v_n in the same equation through the Cauchy–Riemann equations.

The second estimate of Φ on the free surface is determined by solving for u in equation (4) from Part 1 [2] and integrating along the free surface (a streamline) to obtain

$$\Phi_B[x_j,\eta(x)] = \int_{-s}^{x_j}\left(\left[\frac{2}{F^2}(1-\eta(x)) + 1\right]\left[1 + \left(\frac{d\eta}{dx}\right)^2\right]\right)^{1/2} dx\,. \tag{2}$$

Note that square brackets, $[x_j, \eta(x)]$, indicate that Φ_B is obtained from an integral along $\eta(x)$. Then the increment in equation (1) is

$$\delta\eta^{(i)}(x_j) = \Phi_B[x_j,\eta^{(i)}(x)] - \Phi_N(x_j,\eta^{(i)}(x_j))\,. \tag{3}$$

This update method performs reliably on subcritical flow problems. However, approximately 200 iterations are required when the obstacle height produces waves near the maximum wave height. The main criticism of this method is that each update is performed in "isolation". That is the increment $\delta\eta(x_j)$ is computed without regard for how nearby points are updated. Given the form of the potential Φ_B in equation (2), as upstream knots are updated then the value Φ_B at downstream knot points is altered. This observation forms the basis of a "feedback" update method.

The "feedback" update procedure works as follows. The first knot point $x_1 = -s$ is fixed at $\eta(x_1) = 1$ to satisfy the upstream conditions. The update for $\eta(x_2)$ is computed using equations (2) and (3), as previously. At the third knot point we are required to compute Φ_B using equation (2). At this point we use the fact that $\eta(x_2)$ has been updated. Φ_B is computed using a surface through the updated knot point. This procedure is continued at all knot points downstream of x_2. At the computation for the update of knot point x_{k+1} we use the fact that updates for knot points x_2 to x_k have already been computed. These updated η values are employed in the computation of Φ_B in equation (2).

Note that cubic splines satisfying the not-a-knot condition are used to approximate the partially updated surface in the intermediate calculations required for the feedback update method. Linked to this is the issue of how the surface is treated downstream of the most downstream updated knot point. Suppose $\eta(x_k)$ is the most downstream updated knot point. There are two possible approaches for the computation of the integral in equation (2) in order to obtain an update for $\eta(x_{k+1})$. The first method uses a spline to interpolate the updated $\eta(x_k)$ with the old values for $\eta(x_l)$ where $l > k$. The second updates $\eta(x_l)$ for $l > k$ by the same quantity that $\eta(x_k)$ has been updated and places a spline through these points. The second method usually results in a smoother function in the integral (2) near the point, x_{k+1} where the current update is being computed and is the approach we adopt for our feedback computations presented in Section 3.

3 Results

In this section we compare the performance of the series solution procedure using the two update methods described in the previous section. We choose a cosine shaped obstacle with maximum height h and base width 4:

$$f^b(x) = \begin{cases} \frac{h}{2}\left(1+\cos\left[\frac{\pi x}{2}\right]\right) , & -2 \le x \le 2 , \\ 0 , & \text{otherwise.} \end{cases}$$

The Froude number is chosen to be $F = 0.5$. In every example the initial estimate of the free surface is $\eta(x) = 1$, the interval on which a solution is sought is given by $s = 7$, the number of knot points used is $M = 100$ and the number of terms in the series is $N = 100$.

Figure 1 shows a comparison after the first iteration of the two update schemes. The obstacle has height $h = 0.1$. Without feedback the relaxation parameter has value $c = 0.2$ and the parameter used in the knot spacing is $p = 10$. When feedback is used $c = 0.5$ and $p = 10$. The feedback enhances the profile of the free surface considerably as the downstream waves appear with crests and troughs in favourable positions (on comparison with the final solution). Without feedback the first iteration provides a "hydraulic fall" profile for the free surface.

Figure 2 shows a comparison of the free surface profiles for the same obstacle as in Figure 1 after 200 iterations. Also shown are plots of the r.m.s. errors in each of the boundary conditions on the upper and lower surfaces. Without feedback convergence to the final solution is a slow process, but with feedback roughly twenty iterations provides a solution for which errors are almost optimal.

Figure 3 displays the behaviour of the two update methods in a region of the parameter space where the obstacle produces extreme waves. The obstacle has height $h = 0.153$. Without feedback the relaxation parameter has value $c = 0.1$ and the parameter used in the knot spacing is $p = 0.8$. When

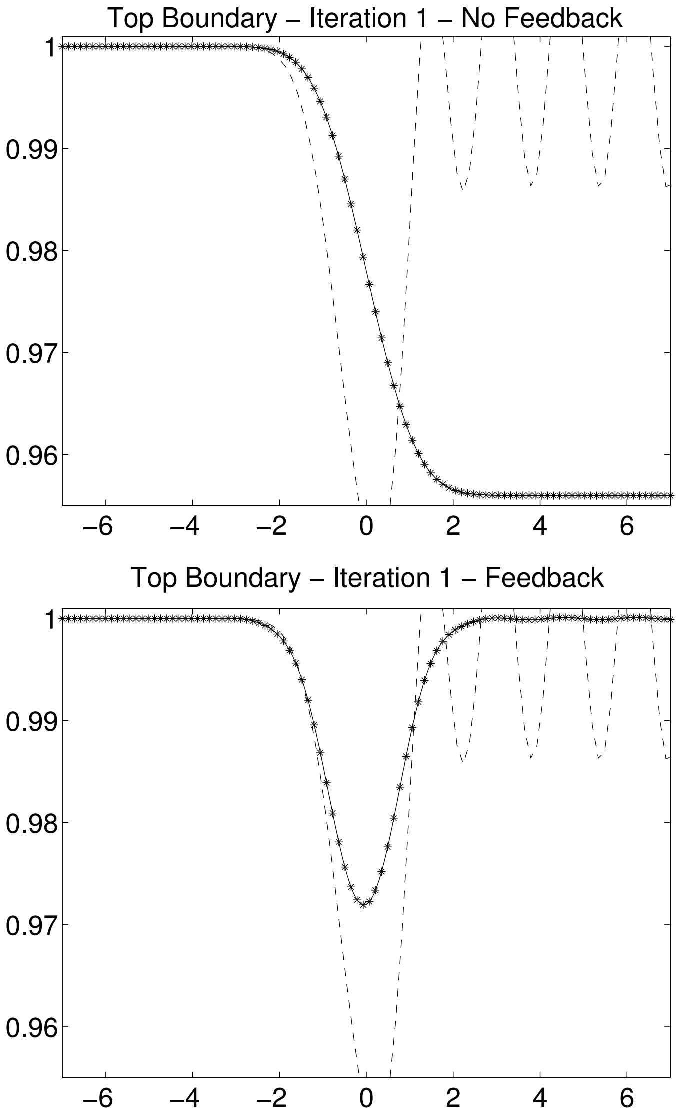

Figure 1: A comparison of the two update schemes after one iteration, obstacle height $h = 0.1$. In each case the final solution is plotted as a dashed line.

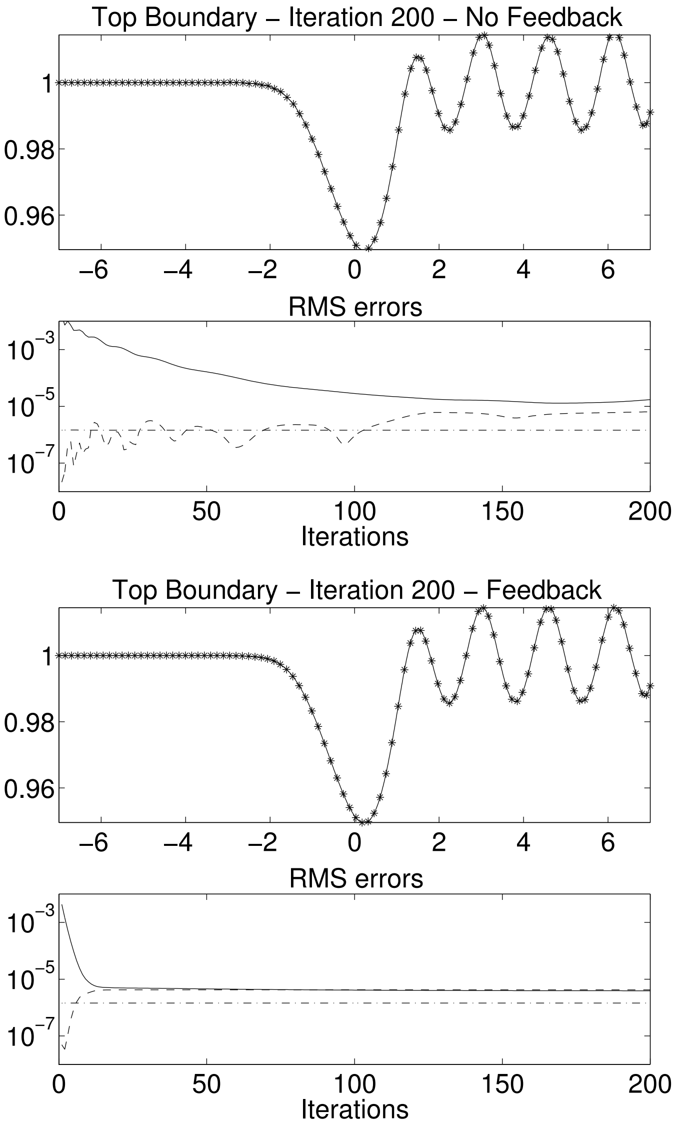

Figure 2: A comparison of the two update schemes after 200 iterations, for an obstacle height $h = 0.1$. Also plotted are the errors in the upper (dashed line) and lower (dash-dot) boundary conditions, and the error in the integrated Bernoulli function (solid line).

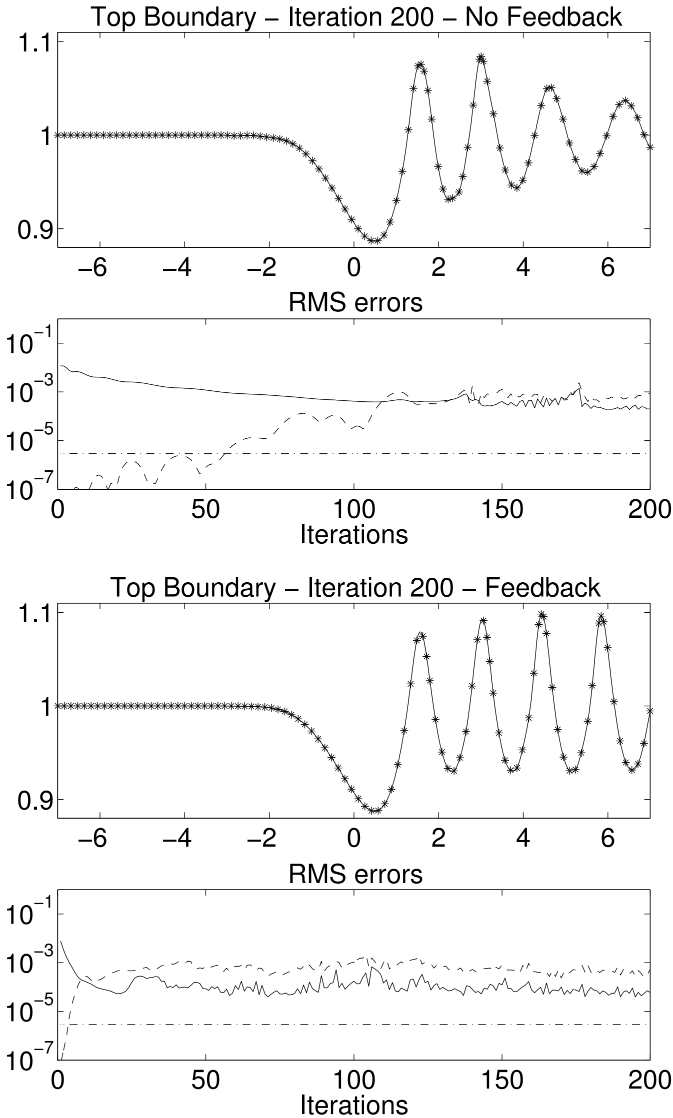

Figure 3: A comparison of the two update schemes after 200 iterations, for an obstacle height $h = 0.153$. Also plotted are the errors in the upper (dashed line) and lower (dash-dot) boundary conditions, and the error in the integrated Bernoulli function (solid line).

feedback is used $c = 0.45$ and $p = 0.4$. Note the surface $\eta(x)$ has points on it where η nears $\eta_{\max} = \frac{1}{2}F^2 + 1 = 1.125$ – see equation (4) from Part 1 [2]. Solutions in these regions are notoriously difficult to obtain [1]. The figure shows that without feedback the procedure is struggling and that after 200 iterations the free boundary problem is yet to be solved. By comparison, when feedback is included the free surface profile has the form we expect for highly nonlinear waves, with sharp crests and broad troughs. The plots of the r.m.s. errors also show the error in the integrated Bernoulli function settles to an almost constant value after approximately 20 itrerations. The oscillations in the error (as opposed to the constant r.m.s. errors shown in the feedback graph in Figure 2) highlight the sensitivity of the solution process in determining extreme waves. These oscillations in the errors may be eliminated by reducing the value of the constant c in equation (1) as iterations progress.

In summary, we have discussed the update procedure for computing series solutions to fully nonlinear two dimensional flow over topography. When downstream knot points are updated using information regarding the update of knot points upstream, an order of magnitude decrease in the number of free surface iterations for convergence to a solution occurs. This increase in efficiency will be extremely beneficial if similar gains can be made in the computation of three dimensional flow over topography.

References

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