Some remarks on the inverse eigenvalue problem for real symmetric Toeplitz matrices

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(Received 8 October 2004, revised 17 November 2005)

Abstract

A theorem about the bounds of solutions of the Toeplitz Inverse Eigenvalue Problem is introduced and proved. It can be applied to make a better starting generator for iterative numerical methods. This application is tested through a short *Mathematica* program. Also an optimisation method for solving the Toeplitz Inverse Eigenvalue Problem with a global convergence property is presented. A global convergence theorem is proved.

Contents

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See http://anziamj.austms.org.au/V46/CTAC2004/Li for this article, © Austral. Mathematical Soc. 2006. Published January 6, 2006. ISSN 1446-8735

1 Introduction

The inverse Toeplitz eigenvalue problem (ToIEP) is to obtain a real vector $\mathbf{r} = [r_1, r_2, \ldots, r_n]^t$ so that the Toeplitz matrix

$$T(\mathbf{r}) = \begin{bmatrix} r_1 & r_2 & \cdots & r_{n-1} & r_n \\ r_2 & r_1 & \cdots & r_{n-2} & r_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ r_{n-1} & r_{n-2} & \cdots & r_1 & r_2 \\ r_n & r_{n-1} & \cdots & r_2 & r_1 \end{bmatrix} \qquad (1)$$

has a prescribed set of real numbers $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ as its spectrum.

Landau [7] proved that every set of n real numbers is the spectrum of an $n \times n$ real symmetric Toeplitz matrix. As the proof is nonconstructive, Newton-type iteration methods are still the main methods to build up such Toeplitz matrices.

The critical task for applying Newton's method is to choose a starting point or an initial approximation properly, otherwise the iterations either diverge or converge to a point which is not a solution. The issue for ToIEP is also mentioned by Laurie [8] and Trench [15]. Theorem 1 in Section 2 gives the bounds of each component of a solution $\mathbf{r}$. Therefore it provides guidance for choosing a starting point. A more reliable starting generator is thus produced. A short *Mathematica* program using this generator is given in Section 3.

There are two categories of iterative methods for solving ToIEP. One [2, 15] exploits the Toeplitz structure while the other [5, 6, 8] does not. The difference between the two categories is discussed in [1]. All these methods except Trench's do not possess a global convergence property. Trench's method appears to be globally convergent; however, this is not proved. In Section 4 the Levenberg–Marquardt (L-M) method [13, 14] with a global convergence feature is presented. The method itself does not need any knowledge of the Toeplitz structure, but its convergence does depend on it.

2 Bounds of solutions

Theorem 1 gives the bounds of each component of a solution $\mathbf{r}$.

Theorem 1 *If* $\mathbf{r} = [r_1, r_2, \ldots, r_n]^t$ *is a solution of the* ToIEP, *then*

$$r_1 = \sigma_1/n\,, \tag{2}$$

and

$$|r_i| \le \sqrt{\frac{n\sigma_2 - \sigma_1^2}{2n(n-i+1)}}\,, \quad i = 2, \ldots, n\,, \tag{3}$$

where $\sigma_k = \sum_{i=1}^n \lambda_i^k$.

Proof: Equation (2) is well known [15]. Moreover,

$$\sigma_2 - (\sigma_1^2/n) = \operatorname{trace}(T^2) - nr_1^2 = 2\sum_{i=1}^{n-1} i r_{n+1-i}^2\,, \tag{4}$$

which implies (3), since all terms on the right hand side of (4) are non-negative. See that for the problem with standardized eigenvalues ($\sigma_1 = 0$, $\sigma_2 = 1$) [15],

$$|r_i| \le \frac{1}{\sqrt{2(n-i+1)}}\,, \quad i = 2, \ldots, n\,. \tag{5}$$

♠

Theorem 1 gives a clear criterion for selecting an initial approximation when an iterative method is applied. The following well known theorem [9, e.g.] follows immediately from the fact that $T(\mathbf{r})$ has the same eigenvalues as the matrix $D^{-1}T(\mathbf{r})D$, where D is the diagonal matrix whose ith diagonal element is $(-1)^{i+1}$.

Theorem 2 *For a given set of real numbers* $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *, if*

$$\mathbf{r} = [r_1, r_2, \ldots, r_n]^t$$

is a solution of the ToIEP*, then*

$$\tilde{\mathbf{r}} = [r_1, -r_2, \ldots, (-1)^{n-1} r_n]^t$$

is also a solution of the ToIEP*.*

Theorem 2 shows that the solutions of the ToIEP exist in pairs. It is helpful when we try to locate all possible solutions of the problem.

3 A *Mathematica* program

The starting generator is usually a subtle issue when applying iterative methods. Trench, Laurie and other authors have mentioned the issue for solving ToIEP [9, 15]. Some generators make a unified starting value for $r_2, r_3, \ldots, r_n$, for example $1/2(n-1)$, ignoring the differences among these components. Theorem 1 shows that the bounds for r_2 and r_n differ by nearly $\sqrt{n}$ times. When n is large the ignorance will not be acceptable. The short *Mathematica* program in Algorithm 1 is designed for solving ToIEP which shows how the results of Theorem 1 are used to initiate the subroutine *FindRoot*. The ith component of a starting point $\mathbf{r}$ is chosen randomly between

$$\pm 0.5\sqrt{\frac{n\sigma_2-\sigma_1^2}{2n(n-i+1)}}$$

using *Random*[], which produces a random number between 0 and 1. The algorithm is quite simple: just solve the equations obtained by equating corresponding coefficients of the characteristic polynomial of $T(\mathbf{r})$ and $P(x) = (x-\lambda_1)\cdots(x-\lambda_n)$. We test the program on a problem with an extremely irregularly clustered spectral data $\{1000, 100, 99, 5, 1\}$ which was first presented by Laurie [8].

Algorithm 1:

```
λ[1]=1000; λ[2]=100; λ[3]=99; λ[4]=5; λ[5]=1;
s1=Sum[λ[i],{i,1,5}]; s2=Sum[λ[i]^2,{i,1,5}];
a=s1/5;
m={{a,b,c,d,e}, {b,a,b,c,d}, {c,b,a,b,c}, {d,c,b,a,b}, {e,d,c,b,a}};
P[x_]:=Product[(x−λ[i]),{i,1,5}]
eqs=Table[ Coefficient[Det[x*IdentityMatrix[5]−m],x,i]==
Coefficient[P[x],x,i],{i,0,3}];
start[k_]:= (Random[]−0.5)Sqrt[5 s2−s1^2)/(10*(6−k))];
For[i=1, i ≤ 100, i++,
Do[sol=
FindRoot[eqs,{b,start[2]},{c, start[3]},{d, start[4]},{e, start[5]}];
Print[sol]]]
```

After 100 tries, the following 12 sets of solutions $(r_2, r_3, r_4, r_5) = (b, c, d, e)$

with $r_1 = a = 241.000$ were obtained: {168.853, 212.453, 209.583, 165.547}, {-191.89, 218.846, -155.583, 159.154}, {-192.256, 218.631,-155.536, 158.369}, {168.986, 212.011, 210.26, 164.989}, {-211.225, 169.31, -166.489, 211.69}, {193.838, 217.022, 152.043, 163.978}, {210.868, 168.858, 167.156, 213.142}, {-185.502, 160.523, -224.893, 220.477}, {193.472, 217.237, 152.089, 164.763}, {186.977, 159.793, 224.821, 217.207}, {167.541, 210.216, 212.945, 170.784}, {167.399, 210.668, 212.278, 171.332}.

Actually, from Theorem 2, we have obtained 24 sets of solutions. By changing the sign of b and d of the above sets we get the other 12 sets of the solutions. I expect to obtain more solutions (possibly $5! = 120$ solutions, see [3, 6]) if we try more times.

4 An optimisation method

In the above program the ToIEP is converted to the system of polynomial equations,

$$f_i(r_2, \ldots, r_n) = c_i(r_2, \ldots, r_n) - p_i = 0\,, \quad i = 2, \ldots, n\,. \tag{6}$$

where c_i and p_i are coefficients of the λ^{n-i} term of the characteristic polynomial of $T(r)$ with $r_1 = \sigma_1/n$ and the polynomial $P(x) = (x-\lambda_1)\cdots(x-\lambda_n)$, respectively. We now apply the least squares method to find the solution of the equations. The objective function to be minimised here is

$$F(r_2, \ldots, r_n) = \frac{1}{2}\sum_{i=2}^{n} f_i^2(r_2, \ldots, r_n)\,.$$

If at a stage in the minimisation process $F(\bar{r_2}, \ldots, \bar{r_n}) = 0$, then $\mathbf{r} = [r_1, \bar{r_2}, \ldots, \bar{r_n}]^t$ is a solution of the ToIEP. The Levenberg–Marquardt (L-M) method solves this minimisation problem. The L-M method is widely recognized as one of the most reliable methods for nonlinear least squares problems. It works extremely well for functions without a high degree of nonlinearity [10, 11, 12]. A hybrid version of the L-M method was developed by Powell [14]. When the elements of the Jacobian of the system of equations are exact, the method has a global convergence property under some conditions. Note that a minimisation program with global convergence property means for *any* starting point it always converges to either a local minimum or a global minimum, but not always to a global minimum [4]. We state this Powell's result as Theorem 3.

Theorem 3 (Powell) *If the functions f_i have continuous, bounded first derivatives then the L-M method will finish after a finite number of iterations, due to*

$$F(\mathbf{x}) < E$$

or

$$F(\mathbf{x}^{(k)}) \geq M\|\mathbf{g}^{(k)}\|_2 \,,$$

where E and M are assigned fixed positive values before the iterations begin and $\mathbf{g}^{(k)}$ is the gradient vector of $F(x)$ at the kth iterate $\mathbf{x} = \mathbf{x}^{(k)}$.

See that if the iteration terminates due to $F(\mathbf{x}) < E$ (E is a very small number) then $\mathbf{x}$ is approximately a global minimum of the $F(\mathbf{x})$ and is also a solution of $f_i = 0$; if the iteration stops due to $F(\mathbf{x}^{(k)}) \geq M\|\mathbf{g}^{(k)}\|_2$ (M is a very large number) $\mathbf{x}^{(k)}$ is approximately a local minimum of $F(\mathbf{x})$. Interestingly, the functions f_i of a ToIEP satisfy all conditions of Theorem 3. Thus we have the following theorem:

Theorem 4 *Powell's version of L-M method for solving* ToIEP *has a global convergence property.*

Proof: Let $\mathbf{x} = (r_2, \ldots, r_n)$ and $\mathbf{x}^{(0)}$ be an initial approximation to the problem. Then the method restricts all iterates $\mathbf{x}^{(k)}$ to the set

$$S = \{\mathbf{x} : F(\mathbf{x}) \leq F(\mathbf{x}^{(0)})\} \,.$$

We claim that S is a compact set. As F is a continuous function S must be closed. Hence we only need to show that S is bounded. It can be shown that

$$f_2 = (n-1)r_2^2 + (n-2)r_3^2 + \cdots + r_n^2 - (n\sigma_2 - \sigma_1^2)/2n\,. \tag{7}$$

Let $c = \sqrt{2F(\mathbf{x}^{(0)})}$, then the inequality $|f_2| \leq c$ gives

$$|r_i| \leq \sqrt{\frac{n\sigma_2 - \sigma_1^2 + 2nc}{2n(n-i+1)}}\,, \quad i = 2, \ldots, n\,. \tag{8}$$

Thus S is bounded. Because all the derivatives f_i' are polynomials on the compact set S, they must be continuous and bounded. ♠

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