A simple mathematical model of wool scouring

J. F. Caunce* S. I. Barry* G. N. Mercer*

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Abstract

The transport of contaminants in a wool scour bowl is modelled by advection diffusion equations. Averaging over the thin layers of wool and the water beneath gives two coupled differential equations clearly showing the contaminant interaction between the layers, and leading to simple asymptotic relationships.

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*School Physical, Environmental & Mathematical Sciences, University of New South Wales at ADFA, Canberra, Australia. mailto:s.barry@adfa.edu.au

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1 Introduction

Wool scouring is the process of washing shorn wool to remove contaminants such as grease and dirt. In many scour bowls the wool sits in a layer on top of a tank of water and is propelled along the bowl, and agitated, by a series of harrows. This agitation releases dirt which settles into a series of tanks which are periodically drained, with the water cleaned and recycled. It is the process of washing and settling which we consider here.

There have been only a few models of the traditional scour bowl, mostly concentrating on the overall balance of dirt and grease in and out of the bowl to find the average equilibrium concentration [3, 4]. A study group [1] considered a simple model of transport within the bowl by compartmentalising the bowl into eight regions and balancing the fluxes of water, dirt and grease in and out of these regions. However, a more complete model of the contaminant transport from the wool into the bowl is needed since different sized materials settle at different rates; hence the first settling tanks collect larger particles than the later tanks. A better understanding of how dirt moves out of the wool and into these tanks can be used to operate the bowls more efficiently. In particular we need to know how changes in wool layer depth, wool velocity, input dirt concentrations, and input water velocity effect the eventual accumulation of dirt, and the particle size distribution, in the tanks.

Figure 1 illustrates the portion of the scour bowl we model. The wool is shown as the top layer, $x \in [0, x_0]$, with the harrows agitating a layer of water of roughly the same depth, $x \in [x_0, 2x_0]$. Beneath this is a region where the dirt settles, with transport governed less by agitation and more by the downward settling velocity q and the cross flow fluid velocities v_i, $i = 1, 2, 3$.

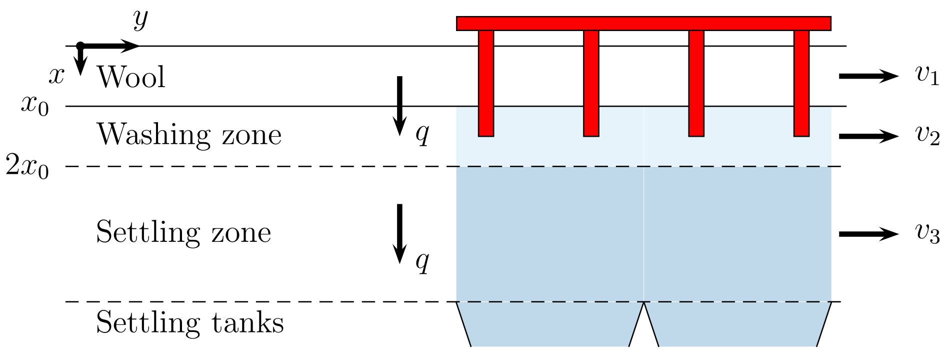

Figure 1: Schematic diagram of the wool mixing model, showing a thin layer of wool of thickness x_0 over a layer of water, being mixed by harrows.

Beneath this is shown the top of the pyramid settling tanks where no cross flow occurs.

In this report we outline the governing advection diffusion equations and non-dimensionalise them using typical parameter values. These equations are then averaged by integrating over the depth of the layers to produce a set of coupled ordinary differential equations for the averaged concentration in the wool and the water. Solving for the concentration, with various asymptotic relationships, illustrates the fundamental behaviour of the system.

2 Governing equations

The transport in the wool, mixing water and settling water layers are modelled by the advection diffusion equation within each layer, $i = 1, 2, 3$:

$$\frac{\partial c_i}{\partial t} = D_i \nabla^2 c_i - q \frac{\partial c_i}{\partial x} - v_i \frac{\partial c_i}{\partial y}\,, \tag{1}$$

where c_i is the concentration in each layer (wool, washing zone, settling zone), q is the settling velocity, v_i is the horizontal transport velocity, D_i is a diffusion-like term which represents the mixing action of the harrows, x is the coordinate down the tank, from the top of the wool, through the various layers, whereas y is the coordinate in the direction of wool motion. These partial differential equations are based on simple mass-flux balances.

For simplicity we make several reasonable assumptions: the diffusivity terms D are constants with $D_1 = D_2$, since the wool and water mix together; the diffusivity in the settling zone $D_3 \approx 0$ since little mixing occurs; the horizontal velocities vary, $v_1 \neq v_2$ but $v_2 = v_3$; the system has reached a steady state so $\partial c/\partial t = 0$; the wool layer and the washing water layer have the same depth, x_0; and, the diffusion in the y direction is assumed zero since $y_0 \gg x_0$ and hence the $D\partial^2 c/\partial y^2$ terms are small.

The boundary conditions are of no flux through the top surface and matching fluxes between each layer. Thus

$$\left[D\frac{\partial c}{\partial x} - qc\right]_{x=0} = 0\,, \quad D^+\frac{\partial c^+}{\partial x} = D^-\frac{\partial c^-}{\partial x}\,, \quad c^+ = c^-\,, \tag{2}$$

where c^+, c^- denote the transition between each of the layers.

Equation (1) is non-dimensionalised with respect to the length of a single tank y_0 and the height of the wool layer x_0 to give in non-dimensional coordinates (x, y)

$$\begin{aligned}
\frac{\partial c_1}{\partial y} &= D^*\frac{\partial^2 c_1}{\partial x^2} - q^*\frac{\partial c_1}{\partial x}\,, \quad x \in (0,1), && (3)\\
\frac{\partial c_2}{\partial y} &= \alpha D^*\frac{\partial^2 c_2}{\partial x^2} - \alpha q^*\frac{\partial c_2}{\partial x}\,, \quad x \in (1,2), && (4)\\
\frac{\partial c_3}{\partial y} &= -\alpha q^*\frac{\partial c_3}{\partial x}\,, \quad x > 2\,, && (5)
\end{aligned}$$

where the parameters are

$$D^* = \frac{Dy_0}{v_1 x_0^2}\,, \quad q^* = \frac{qy_0}{v_1 x_0}\,, \quad \alpha = \frac{v_1}{v_2}\,. \tag{6}$$

We now drop the * notation and assume all variables are non-dimensional unless otherwise stated. Alternative scalings exist which reduce the number of parameters to α and D^*/q^*, but the above scaling makes the transport processes more obvious physically.

To simplify these further we integrate the equations over the depth of each of the layers with notation

$$\bar{c}_1 = \int_0^1 c_1\,dx\,, \quad \bar{c}_2 = \int_1^2 c_2\,dx\,,$$

noting that the layers have non-dimensional depth one. Thus

$$\begin{aligned}
\frac{\partial \bar{c}_1}{\partial y} &= D\left(\frac{\partial c_1}{\partial x}(x=1) - \frac{\partial c_1}{\partial x}(x=0)\right) - q(c_1(x=1) - c_1(x=0)), && (7)\\
\frac{\partial \bar{c}_2}{\partial y} &= \alpha D\left(\frac{\partial c_2}{\partial x}(x=2) - \frac{\partial c_2}{\partial x}(x=1)\right) - \alpha q(c_2(x=2) - c_2(x=1)). && (8)
\end{aligned}$$

The boundary conditions eliminate the terms at $x=0$. Similarly at $x=2$, the bottom of the water mixing layer, matching implies $\frac{\partial}{\partial x}c_2(x=2)=0$. We now replace the derivative terms by their central difference

$$\frac{\partial c_1}{\partial x}(x=1) = c_2\left(x=\frac{3}{2}\right) - c_1\left(x=\frac{1}{2}\right),$$

and the average $\bar{c}_1 \approx c_1(1/2) \equiv c_1$ and $\bar{c}_2 \approx c_1(3/2) \equiv c_2$. The concentration term $c_1(x=1) \approx c_1(x=1/2)$ and $c_2(x=2) \approx c_2(x=3/2)$. Hence

$$\begin{aligned}
\frac{dc_1}{dy} &= -(D+q)c_1 + Dc_2\,, && (9)\\
\frac{dc_2}{dy} &= \alpha(D+q)c_1 - \alpha(D+q)c_2\,. && (10)
\end{aligned}$$

These equations, despite the averaging, have much of the expected behaviour. The $-(D+q)c_1$ term represents transport of contaminant from the wool to the water. The Dc_2 represents the transport of dirt by mixing back to the wool. Similarly the $-(D+q)c_2$ term represents the loss of material from the water, both to the wool, Dc_2, and the water beneath, qc_2. The equation for c_3 gives solution

$$c_3(x,y) = c_2\left(y - \frac{x-2}{q\alpha}\right), \tag{11}$$

which simply represents a diagonal transport of the material with the velocities, q and v_3 away from the base of the washing water zone.

We have thus reduced the coupled partial differential equations to an averaged set of coupled, linear, ordinary differential equations which have classic phase plane type solutions which we explore in the next section.

3 Solutions

The solutions to the system (10) is simply found to be

$$\begin{bmatrix} c_1(y) \\ c_2(y) \end{bmatrix} = a\mathbf{v}_1 e^{\lambda_1 y} + b\mathbf{v}_2 e^{\lambda_2 y} \tag{12}$$

where $\mathbf{v}_i$ are the eigenvectors and λ_i are the eigenvalues of the matrix

$$\begin{bmatrix} -(D+q) & D \\ \alpha(D+q) & -\alpha(D+q) \end{bmatrix}. \tag{13}$$

Both the eigenvalues are real, negative, and unique for physical parameter ranges. The constants a and b are found by application of the conditions at $y=0$. Evaluating this solution is trivial in MATLAB® when written in diagonalised form. In full the eigenvalues and eigenvectors are

$$\lambda = \frac{-(D+q)}{2}\left((1+\alpha) \pm \sqrt{(1+\alpha)^2 + \frac{4\alpha D}{D+q} - \alpha}\right), \tag{14}$$

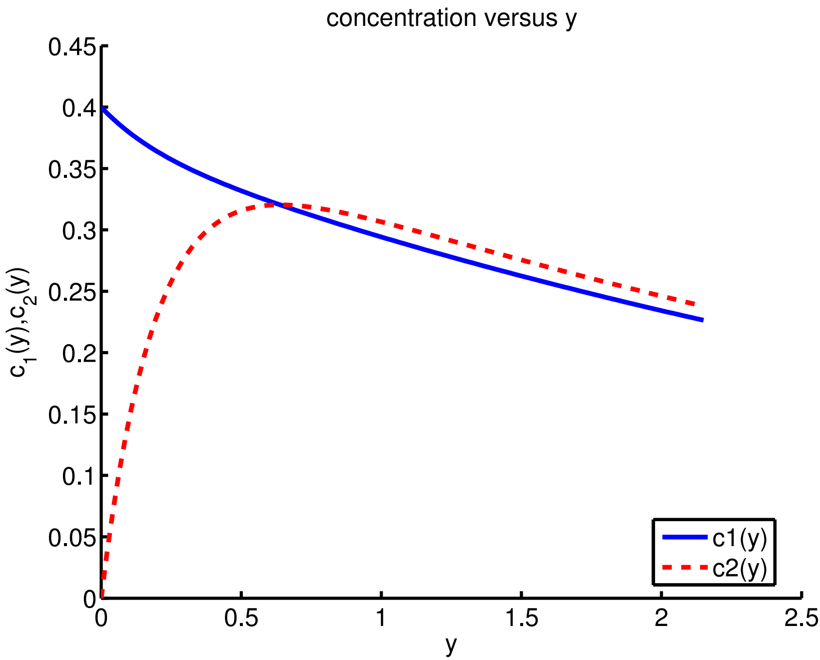

Figure 2: Concentration in the wool $c_1(y)$ and water $c_2(y)$ showing rapid and slow exponential responses.

and

$$\mathbf{v} = \begin{bmatrix} D \\ \lambda + (D+q) \end{bmatrix} . \tag{15}$$

A typical solution is illustrated in Figure 2 showing the concentration, $c(y)$ versus distance y, in both the wool and the water for physically typical values of $q^* = 0.3$, $D^* = 0.7$, $\alpha = 8$, $c_1(0) = 0.4$, and $c_2(0) = 0$. That is dirty wool entering from the left along with clean water. The solution shows the wool, c_1, getting cleaner while the water, c_2, initially gets dirtier (from the wool) before eventually becoming cleaner. The solution also shows the two exponential type behaviour, with both solutions having an initially fast transient phase followed by a matched slow exponential decay governed by

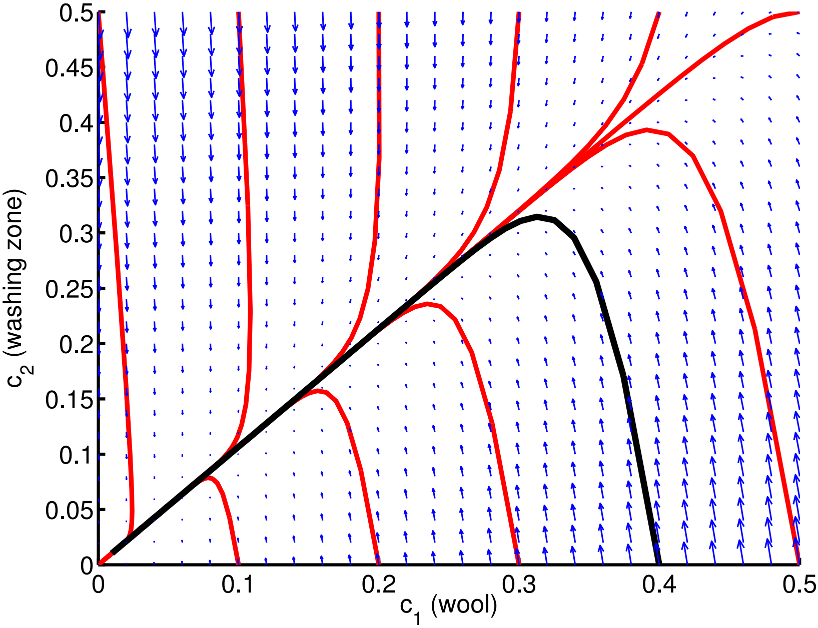

Figure 3: Phase plane of dirt concentration in wool, c_1 and water, c_2 as y changes.

the leading eigenvector/eigenvalue.

In Figure 3 the solutions $c_1(y)$ and $c_2(y)$ from equation (12) are plotted on a phase plane as c_2 versus c_1, with the same parameter set as Figure 1. For example, the bold trajectory through $c_1 = 0.4$, $c_2 = 0$ represents the solution when $c_1(0) = 0.4$, $c_2(0) = 0$. As y changes c_2 rapidly changes, as shown in Figure 2 until $y \approx 0.6$. This is the effect of the second exponential term in (12). After this point both c_1 and c_2 progress uniformly hence in Figure 3 all trajectories head along the same straight line to the origin as the slowly varying exponential term in (12) dominates. The slope of this latter transition is given by the leading eigenvector in equation (15).

One of our aims is to find simple solutions and to gain a better understanding of the system with this approximation. By making the assumption that $c_2(y) \approx c_2(0)$ it is not hard to show that

$$c_1(y) \approx c_1(0)e^{-(D+q)y} + c_2(0)\frac{D}{D+q}\left(1 - e^{-(D+q)y}\right) , \tag{16}$$

which is valid for small values of $(D+q)y$. This clearly indicates the dominant behaviour of the initial rapid decay exponential term $\exp[-(D+q)y]$ coming from the most negative eigenvalue.

By considering a Taylor series in y we also obtain the asymptotic results valid for small $(D+q)y$ of

$$\begin{aligned} c_1(y) &\approx c_1(0)(1-(D+q)y) + c_2(0)Dy\,, && (17)\\ c_2(y) &\approx c_2(0) + \alpha(D+q)c_1(0)y - \alpha(D+q)c_2(0)y\,. && (18) \end{aligned}$$

This clearly shows the interaction between the elements, with the $(D+q)y$ term reducing the concentration in c_1 and increasing it in c_2.

If the diffusivity term is small, $D \approx 0$, equivalent to no agitation of the wool, then the system reduces easily to

$$\begin{aligned} c_1(y) &= c_1(0)e^{-qy}\,, && (19)\\ c_2(y) &= c_1(0)\frac{\alpha}{\alpha-1}e^{-qy} + \left(c_2(0) - \frac{\alpha}{\alpha-1}c_1(0)\right)e^{-\alpha qy}\,, && (20) \end{aligned}$$

for $\alpha \neq 1$. Note that for $\alpha > 1$, the usual physical case, for large y

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \to \begin{bmatrix} \alpha - 1 \\ \alpha \end{bmatrix} \frac{c_1(0)}{\alpha - 1} e^{-qy}\,, \tag{21}$$

showing that the equivalent slope on the phase plane, Figure 3, is $\alpha/(\alpha-1)$ which is not affected by settling q. That is, the relative concentration of c_1 and c_2 are dominated by the velocities α. For $\alpha = 1$ the $c_2(y)$ equation is modified by

$$c_2(y) = (c_1(0)qy + c_2(0))e^{-qy}\,. \tag{22}$$

Importantly we calculate the maximum value of $c_2(y)$ since, as shown in Figure 2, for typical parameters, the concentration in the water rises to a maximum before dropping. This maximum is then transported down to the settling tanks by the solution to c_3, equation (11), effectively giving the tank which will clog the most. Thus differentiation of the c_2 solution in equation (22) gives a maximum at

$$y_{\max} = \frac{1}{1-\alpha q} \ln\left(\frac{c_1(0)}{\alpha c_1(0) + (1-\alpha)c_2(0)}\right) \tag{23}$$

for $\alpha \neq 1$, so long as the solution exists, and for $\alpha = 1$

$$y_{\max} = \frac{c_1(0) - c_2(0)}{c_1(0)q}\,, \quad c_1(0) \neq 0\,. \tag{24}$$

One can then use these equations to both predict which bowl will fill with settling contaminant first as well as allowing operators to vary input parameters (wool speed, cross flow velocities) to obtain a desired settling outcome. A similar, but more complicated, expression can be found for the full eigenvector solution in equation (12).

Other, reasonably physical, simplifications occur. For example if $v_2 = 0$, then the term $v_2 \partial c_2/\partial y = 0$ and so $c_1 = c_2$ giving

$$c_1(y) = c_1(0)e^{-qy}\,; \tag{25}$$

that is, the concentration in the wool is independent of the washing action as material diffuses out then back into the wool. If $q = 0$ which is appropriate for neutrally buoyant grease, rather than dirt, the solution simplifies to

$$\begin{bmatrix} c_1(y) \\ c_2(y) \end{bmatrix} = \frac{\alpha c_1(0) + c_2(0)}{1+\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{c_1(0) - c_2(0)}{1+\alpha} \begin{bmatrix} 1 \\ -\alpha \end{bmatrix} e^{-D(1+\alpha)y}\,. \tag{26}$$

showing that both the wool and water diffuse to the same value. Naturally, if $c_1(0) = c_2(0)$, then there is no y dependence. Assuming that the exponential term decays fast enough, this means that the grease concentrations in the

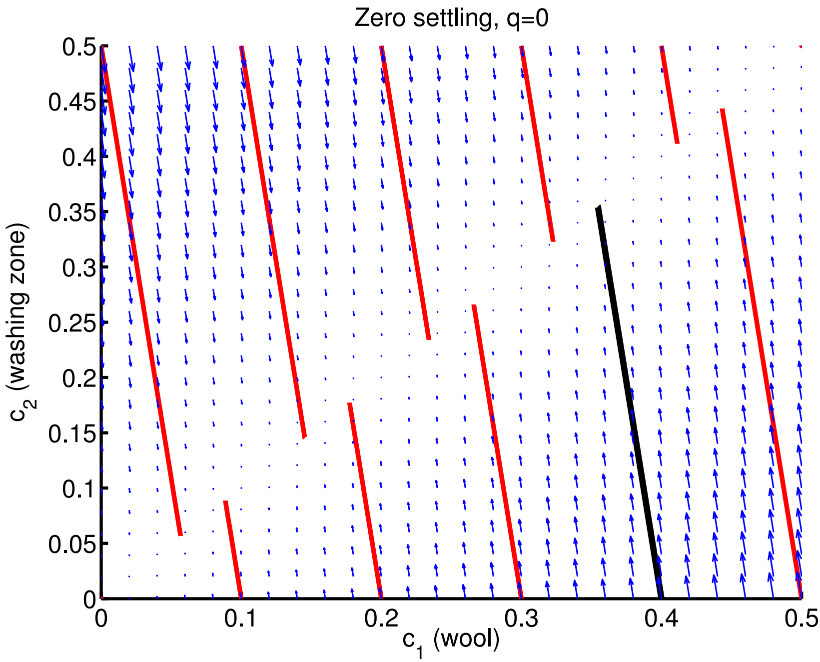

Figure 4: Phase plane of grease concentration in wool c_1 and water c_2 as y changes for $q = 0$.

water at the end of the bowl, where grease is removed to make lanolin, are solely given by

$$c = \frac{\alpha c_1(0) + c_2(0)}{1 + \alpha} \tag{27}$$

with little dependence on D or the dimensions of the tank. This latter result is shown in Figure 4 as a phase plane, with the same parameters as Figures 2 and 3 apart from the settling velocity being $q = 0$. The bold trajectory again shows concentrations starting from $c_1(0) = 0.4$, $c_2(0) = 0$. As y increases the concentrations tend linearly to the first terms in equation (26) as the second exponential term dies away.

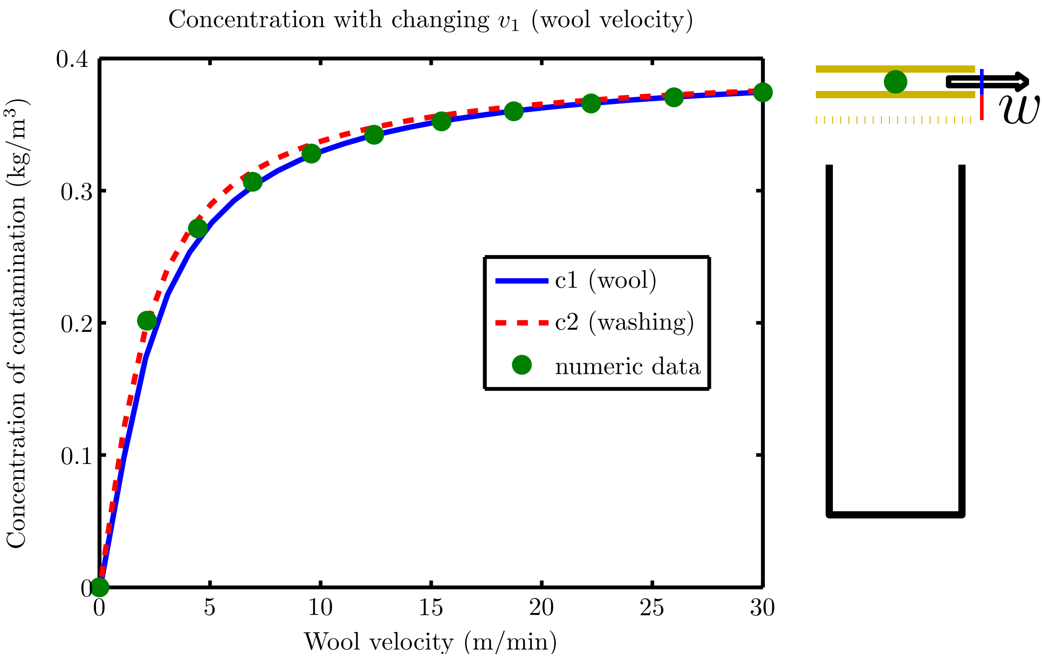

Figure 5: Concentration in the wool and water at a fixed y with varying wool velocity.

In many of the expressions above a decay term like $\exp[-(D+q)y]$ appears. In dimensional coordinates this is

$$\exp\left(\frac{-(D+qx_0)y}{v_1x_0^2}\right) \tag{28}$$

showing that the decay behaviour is faster (in y) if the depth of the wool x_0 is small or the velocity v_1 is small. If the depth of wool x_0 gets too large, then the system does not reach its final asymptotic state before reaching the end of the bowl. However, in this limit of large x_0 the averaging over the width of the layer becomes less valid and an x dependent model needs to be considered. Figure 5 shows this behaviour for changing v_1, that is illustrating $c_1(y=0.5)$, $c_2(y=0.5)$ from equation (12) in dimensional form as wool velocity v_1 changes. This figure uses the same parameters as early models with $y=0.5$ (corresponding to $y=1\,\mathrm{m}$ physically). This figure also shows results from a numerical simulation [2] which uses finite differences to solve the full time-dependent equations for an entire scour bowl of four tanks, including settling and drainage down the full length of the settling tanks. In contrast, the simplified model here only studies the region near the wool. As $v\to 0$ the concentration drops to zero, since this is equivalent to stationary wool with all the dirt settles out for small y hence being clean at $y=0.5$. As $v\to\infty$ the wool is moving so fast that at the point $y=0.5$ the wool has not had time for material to settle out and so the concentration approaches the initial condition of $c_1(0)=0.4$.

4 Conclusion

The washing action of a wool in the top of a wool scour bowl can be modelled well by using a simple integral averaging of the governing advection diffusion equations. This leads to a simple set of coupled ordinary differential equations which model the concentration moving between the wool and the water, with an initial exponential phase, mostly governed by an exponential decay $\exp[-(D+q)y]$ term, allowing the wool and water to come to a pseudo equilibrium state which then slowly decays exponentially with both the wool and water decaying in phase.

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