Time delayed discounted Steiner trees to locate two or more discounted Steiner points

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Abstract

A discounted Steiner tree is a weighted Steiner tree in which the costs of constructing the edges and values at the nodes are discounted over time. Discounted Steiner points are located to maximise the sum of the discounted cash flows, known as the net present value, and an algorithm for doing this for a single Steiner point, known as the discounted Steiner point algorithm, was previously established. An application of this problem is underground mine planning. This article proposes an algorithm to optimally locate two junction points, given a surface portal and three ore resource points, for maximum net present value, which includes the value of the ore bodies and the construction costs. The discounted Steiner point algorithm is extended to locate two

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junction points where time delays may occur at a discounted Steiner point before constructing the adjacent edges. The optimal locations of the junction points are obtained for a range of discount rates. Numerical trials show that this algorithm works well. A generalisation of the algorithm to locate more discounted Steiner points is also discussed.

Contents

1 Introduction

The Steiner problem seeks a shortest tree that connects every node in a given network [1]. A layout is a configuration of terminal points and Steiner points where the topology is specified but the locations of the Steiner points are not [2]. Hwang et al. [2] developed algorithms to locate multiple Steiner points which minimise the total length of a network. However, the problem analysed here locates Steiner points to maximise the net present value (NPV), which is different from the problem discussed by Hwang et al. [2].

The NPV is the value of future cash flows projected to the present time. Cash flows due to construction costs and time delays are dependent on the length of the edges in the network. We use a given discount rate to discount these cash flows. If the discount rate is zero, then the problem reduces to the classical Steiner problem [3].

The NPV depends on where the junctions are placed in the network. Sirinanda et al. [3, 4] described an iterative approach for locating a single discounted Steiner point. The Discounted Steiner Point Algorithm (DSPA) discussed by Sirinanda et al. [3] located a junction point to access most efficiently two ore bodies from the surface, thus maximising the NPV. Sirinanda et al. showed that in the maximum NPV network, the paths from the junction point to the surface portal and the first resource point make equal angles with the path from the junction point to the second resource point. The algorithm provides higher NPV compared with placing the junction point at the location where the network has the minimum development length.

This article describes a way of locating two discounted Steiner points to maximise the NPV for an underground mine. The DSPA is extended to locate two discounted Steiner points in a given network layout. In Section 2, we formulate and describe the optimisation problem. Section 3 introduces the Extension of the Discounted Steiner Point Algorithm (EDSPA) which is proposed for the scenario of a network with two junction points. Section 4 presents the numerical trials.

2 Problem formulation

Let $p_0 = (x_0, y_0, z_0)$, $p_1 = (x_1, y_1, z_1)$, $p_2 = (x_2, y_2, z_2)$ and $p_3 = (x_3, y_3, z_3)$ be points in an underground mine, where p_0 is a surface portal (or breakout point from existing infrastructure), and p_1, p_2 and p_3 are the drawpoints for ore deposits with values V_1, V_2 and V_3, respectively, as shown in Figure 1. We assume that when determining these values, any time discounting arising from the time taken to extract and process the ore has been applied. The objective is to locate two discounted Steiner points (junction points) $s_1 = (x_{s_1}, y_{s_1}, z_{s_1})$ and $s_2 = (x_{s_2}, y_{s_2}, z_{s_2})$ to maximise the NPV.

Figure 1: Locating two discounted Steiner points, s_1 and s_2.

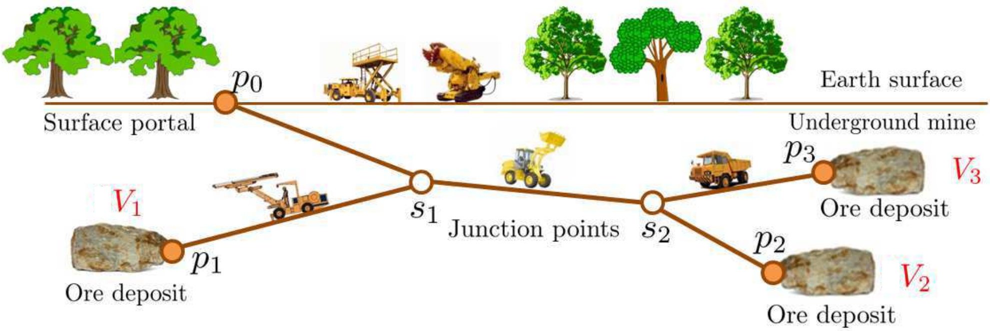
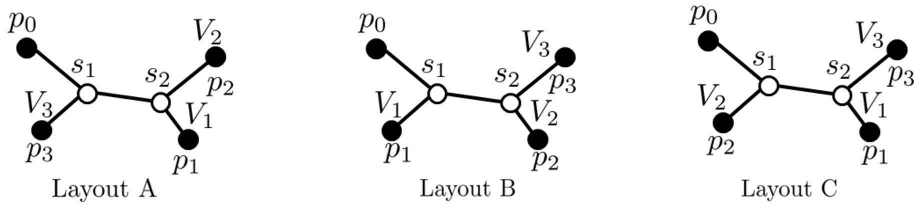


Figure 2: Basic layouts for a network with two discounted Steiner points.



The decline links p_0s_1, s_1p_1, s_1s_2, s_2p_2 and s_2p_3 are constructed one at a time. The order of the access construction process for the mine illustrated in Figure 1 is $p_0s_1 \to s_1p_1 \to s_1s_2 \to s_2p_2 \to s_2p_3$. The order that the access points on deposits p_i $(i = 0, 1, 2, 3)$ are accessed remain fixed throughout this article. A discounted Steiner network with four given points has three possible network layouts, as shown in Figure 2.

A new algorithm is needed to optimally locate two discounted Steiner points for the three layouts in Figure 2. The new algorithm, EDSPA, uses DSPA as a subroutine. For all three layouts the procedure is the same, except the parameters that are passed to DSPA change. The order of reaching access points p_1, p_2 and p_3 changes according to the layout, causing time delays in the construction of the corresponding decline links. Therefore, the DSPA is modified to take account of these time delays. Such a network is a *time delayed discounted Steiner network*. The objective function is formulated to account for these time delays.

Consider a network with a point p_0 where the network breaks out from existing infrastructure, two draw points p_1 and p_2 where ore is extracted, and a discounted Steiner point s. Before starting the process there is a time delay of t_{d_1}. The resource (ore) at the point p_1 is extracted before p_2 and there is a time delay t_{d_2} before reaching the point p_2. Let l_0, l_1 and l_2 denote the lengths of the decline links p_0s, sp_1 and sp_2, respectively. Let D be the decline development rate. Then the total time to reach p_1 is $t_{d_1} + t_0 + t_1$ and the total time to reach p_2 is $t_{d_1} + t_0 + t_1 + t_{d_2} + t_2$, where $t_0 = l_0/D$, $t_1 = l_1/D$ and $t_2 = l_2/D$. Let $r = 1 + d$ where d is the discount rate. The cash flow sum generated from the ore extraction is given by Sirinanda et al. [3, Lemma 2] and simplifies to

$$\text{NPV}^d_{\text{ext}} = V_1 r^{-t_{d_1}} r^{-(l_0+l_1)/D} + V_2 r^{-(t_{d_1}+t_{d_2})} r^{-(l_0+l_1+l_2)/D}.$$

Let the access construction costs $V_c = CD/\log r$ where C is the development cost rate. Sirinanda et al. [3, Lemma 1] calculated these costs and they simplify to

$$\text{NPV}^d_{\text{con}} = V_c r^{-t_{d_1}} [1 - r^{-(l_0+l_1)/D}(1 - r^{-t_{d_2}}) - r^{-t_{d_2}} r^{-(l_0+l_1+l_2)/D}] .$$

The total NPV is the combination of cash flows generated from the ore extraction and access construction costs. Hence,

$$\begin{aligned}
\text{NPV} &= \text{NPV}^d_{\text{ext}} - \text{NPV}^d_{\text{con}} , \\
&= [V_1 + V_c(1 - r^{-t_{d_2}})] r^{-t_{d_1}} r^{-(l_0+l_1)/D} \\
&\quad + (V_2 + V_c) r^{-(t_{d_1}+t_{d_2})} r^{-(l_0+l_1+l_2)/D} - V_c r^{-t_{d_1}} . \qquad (1)
\end{aligned}$$

The objective function (1) is rewritten as

$$\text{NPV} = \bar{V}_1 r^{-(l_0+l_1)/D} + (\bar{V}_2 + \bar{V}_c) r^{-(l_0+l_1+l_2)/D} - \bar{V}_c r^{t_{d_2}} , \qquad (2)$$

where $\bar{V}_1 = [V_1 + V_c(1 - r^{-t_{d_2}})]r^{-t_{d_1}}$, $\bar{V}_2 = V_2 r^{-(t_{d_1}+t_{d_2})}$ and $\bar{V}_c = V_c r^{-(t_{d_1}+t_{d_2})}$.

The objective function (2) is similar to the objective function analysed by Sirinanda et al. [3] but with modified parameter values. The only differences are the constants V_1, V_2 and V_c, but they do not affect the optimisation. Therefore, the DSPA described by Sirinanda et al. [3] is used to locate the discounted Steiner point in a time delayed discounted Steiner network.

In some cases, before applying the DSPA, the total value of the cash flows generated from the ore production and access construction costs at one discounted Steiner point needs to be calculated. This value is defined as the aggregated value V_s. The aggregated value at a discounted Steiner point is written in terms of the discounted values at the adjacent nodes in the network. For the discounted Steiner point s connected to two vertices p_2 and p_3 with values V_2 and V_3, respectively, the aggregated value is

$$\begin{aligned} V_s &= V_2 r^{-l_2/D} + V_3 r^{-(l_2+l_3)/D} - \int_0^{l_2+l_3} C r^{-x/D}\,dx \\ &= V_2 r^{-l_2/D} + (V_3 + V_c) r^{-(l_2+l_3)/D} - V_c\,, \end{aligned} \tag{3}$$

where the distances l_2 and l_3 are the Euclidean distances from s to p_2 and s to p_3, respectively. The point p_2 is accessed before p_3.

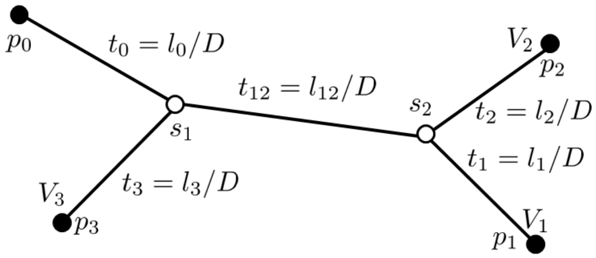
3 Extension of the Discounted Steiner Point Algorithm

In this section, the Extension of the Discounted Steiner Point Algorithm (EDSPA) is proposed to locate two discounted Steiner points for Layout A shown in Figure 2. Below we describe the NPV for Layout A.

In Figure 3, for Layout A, the cash flow sum generated from the ore extraction is

$$\mathrm{NPV}^A_{\mathrm{ext}} = V_1 r^{-(t_0+t_{12}+t_1)} + V_2 r^{-(t_0+t_{12}+t_1+t_2)} + V_3 r^{-(t_0+t_{12}+t_1+t_2+t_3)}\,. \tag{4}$$

Figure 3: The NPV calculations for Layout A.



By substituting $t_0 = l_0/D$, $t_{12} = l_{12}/D$, $t_1 = l_1/D$, $t_2 = l_2/D$, and $t_3 = l_3/D$ into equation (4),

$$\mathrm{NPV}^A_{\mathrm{ext}} = V_1 r^{-(l_0+l_{12}+l_1)/D} + V_2 r^{-(l_0+l_{12}+l_1+l_2)/D} + V_3 r^{-(l_0+l_{12}+l_1+l_2+l_3)/D} .$$

The decline links p_0s_1, s_1s_2, s_2p_1, s_2p_2 and s_1p_3 need to be constructed sequentially. Sirinanda et al. [3, Lemma 1] calculated

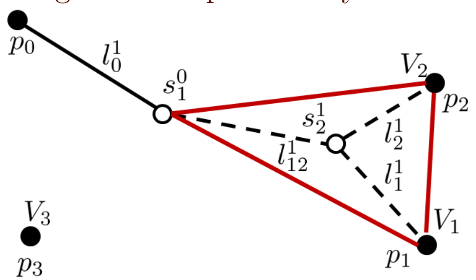
$$\begin{aligned}\mathrm{NPV}^A_{\mathrm{con}} &= \int_0^{l_0} Cr^{-x/D}\,dx + r^{-l_0/D}\int_0^{l_{12}} Cr^{-x/D}\,dx + r^{-(l_0+l_{12})/D}\int_0^{l_1} Cr^{-x/D}\,dx \\ &\quad + r^{-(l_0+l_{12}+l_1)/D}\int_0^{l_2} Cr^{-x/D}\,dx + r^{-(l_0+l_{12}+l_1+l_2)/D}\int_0^{l_3} Cr^{-x/D}\,dx ,\\ \mathrm{NPV}^A_{\mathrm{con}} &= V_c[1 - r^{-(l_0+l_{12}+l_1+l_2+l_3)/D}] .\end{aligned}$$

Since the construction is a cost, cash flows generated from access construction have a negative value. The total NPV is the combination of cash flows generated from ore production and access construction costs. Hence,

$$\begin{aligned}\mathrm{NPV}^A &= \mathrm{NPV}^A_{\mathrm{ext}} - \mathrm{NPV}^A_{\mathrm{con}} \\ &= V_1 r^{-(l_0+l_{12}+l_1)/D} + V_2 r^{-(l_0+l_{12}+l_1+l_2)/D} \\ &\quad + (V_3 + V_c) r^{-(l_0+l_{12}+l_1+l_2+l_3)/D} - V_c . \qquad (5)\end{aligned}$$

We now describe an algorithm for optimally locating two discounted Steiner points.

Figure 4: Step 1 for Layout A.



In the algorithm, $p_0 = (x_0, y_0, z_0)$, $p_1 = (x_1, y_1, z_1)$ and $p_2 = (x_2, y_2, z_2)$ are the surface portal (or breakout point from existing infrastructure) and the access points for ore deposits with values $\bar{V}_1$ and $\bar{V}_2$, respectively, where $\bar{V}_1$ and $\bar{V}_2$ are as in equation (2). The notation DSPA$(p_0, p_1, p_2, \bar{V}_1, \bar{V}_2)$, or similar, is used when the DSPA is called from within the current algorithm. In the first iteration of the DSPA, the discounted Steiner point is initialised at the classical Steiner point.

A superscript i denotes the value of the variable at the ith iteration. The initialisation step and Steps 1 and 2 for the first iteration of the algorithm are described below.

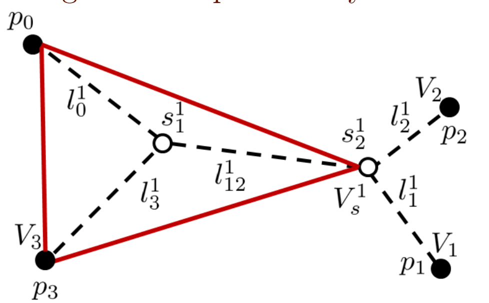
Initialisation For Layout A in Figure 3, the first discounted Steiner point s_1 is initialised at the mid point of the line p_0p_3 and named s_1^0, the second discounted Steiner point s_2 is initialised at the mid point of the line p_1p_2 and named s_2^0, and the initial NPV, NPV(0), is determined.

Step 1: iteration $(i = 1)$ Calculate the distance (see Figure 3)

$$l_0^1 = \sqrt{(x_{s_1}^0 - x_0)^2 + (y_{s_1}^0 - y_0)^2 + (z_{s_1}^0 - z_0)^2}\,.$$

Before reaching the point s_1^0 there is a time delay t_{d_1} in the network where $t_{d_1} = l_0^1/D$. The path from s_2^1 to p_2 is constructed immediately after reach-

Figure 5: Step 2 for Layout A.



ing p_1, so no time delay is applied for reaching p_2 and therefore $t_{d_2} = 0$. It follows from equation (2) that this step is written compactly as $\text{DSPA}(s_1, p_1, p_2, V_1 r^{-l_0^1/D}, V_2 r^{-l_0^1/D})$.

Step 2: iteration $(i = 1)$ Use the DSPA to determine the new location of the first discounted Steiner point. The distances l_1^1 and l_2^1 were found in Step 1. From equation (3), the aggregated value at the point s_2 is

$$V_s^1 = V_1 r^{-l_1^1/D} + (V_2 + V_c) r^{-(l_1^1 + l_2^1)/D} - V_c\,.$$

There is no initial delay in the network so $t_{d_1} = 0$. However, the decline link $s_1^1 p_3$ is constructed after reaching the points p_1 and p_2. Therefore a time delay $t_{d_2} = (l_1^1 + l_2^1)/D$ is required. According to equation (2), this step is $\text{DSPA}(p_0, s_2^1, p_3, V_s^1 + V_c[1 - r^{-(l_1^1+l_2^1)/D}], V_3 r^{-(l_1^1+l_2^1)/D})$.

The Steps 1 and 2 are repeated until $|\text{NPV}(i) - \text{NPV}(i-1)| < \epsilon$, that is, the difference in the NPV at the ith and the $(i-1)$th iteration is smaller than some specified number ϵ. This NPV is calculated using equation (5).

The steps described above are shown in Algorithm 1. The EDSPA is used to locate two discounted Steiner points for Layout A.

For other layouts the algorithm procedure is the same as Algorithm 1. However, in Steps 1 and 2, the DSPA is applied to different sets of three points and

Algorithm 1: Extension of the Discounted Steiner Point Algorithm

Input: V_1, V_2, V_3, discount rate, development and cost rate of the declines, and locations of p_0, p_1, p_2, p_3 and ϵ.

Output: Optimal locations of two discounted Steiner points and the maximum NPV.

Initialisation: s_1^0 at the mid point of p_0 and p_3, s_2^0 at the mid point of p_1 and p_2, $\text{NPV}(0)$.

$i = 1$

repeat

Step 1

Calculate:

$$l_0^i = \sqrt{(x_{s_1}^{i-1} - x_0)^2 + (y_{s_1}^{i-1} - y_0)^2 + (z_{s_1}^{i-1} - z_0)^2}$$

Locate s_2^i by applying the $\text{DSPA}(s_1^{i-1}, p_1, p_2, V_1 r^{-l_0^i/D}, V_2 r^{-l_0^i/D})$.

Step 2

Update the aggregated value:

$$V_s^i = V_1 r^{-l_1^i/D} + (V_2 + V_c) r^{-(l_1^i + l_2^i)/D} - V_c$$

Locate s_1^i by applying $\text{DSPA}(p_0, s_2^i, p_3, V_s^i + V_c(1 - r^{-(l_1^i + l_2^i)/D}), V_3 r^{-(l_1^i + l_2^i)/D})$.

Calculate $\text{NPV}(i)$ using equation (5).

$i = i + 1$

until $|\text{NPV}(i) - \text{NPV}(i-1)| < \epsilon$

Outputs are the optimal locations of the discounted Steiner points $s_2^* = s_2^i$, $s_1^* = s_1^i$ and $\text{NPV}^* = \text{NPV}(i)$

their corresponding values V are different.

4 Numerical trials

In our numerical trials we use parameter values

- $V_1 = \$60$M, $V_2 = \$20$M, $V_3 = \$5$M, $C = \$6000\,\mathrm{m}^{-1}$, $D = 3640\,\mathrm{m/yr}$;
- $p_0 = (0, 1000, 0)$, $p_1 = (1000, 0, 0)$, $p_2 = (1000, 750, 0)$, $p_3 = (0, 0, 0)$ in metres;
- and discount rates $d = 0, 5, 10, \infty$, measured in %/yr.

The EDSPA is applied to Layout A (Figure 3) and the optimal locations of the two discounted Steiner points are obtained for a range of discount rates. As shown in Figure 6, for higher discount rates, p_1 and p_2 are accessed sooner while the distances from the discounted Steiner point s_2 to points p_1 and p_2 increase. Table 1 shows the improvement of the NPV compared with the network where the two Steiner points are located at the classical Steiner locations. The discounted Steiner point algorithm gives an improvement for all finite discount rates and greater improvements as the discount rate increases.

For the same set of inputs, Layout B is infeasible and Layout C produces solutions with slightly lower values for the NPV.

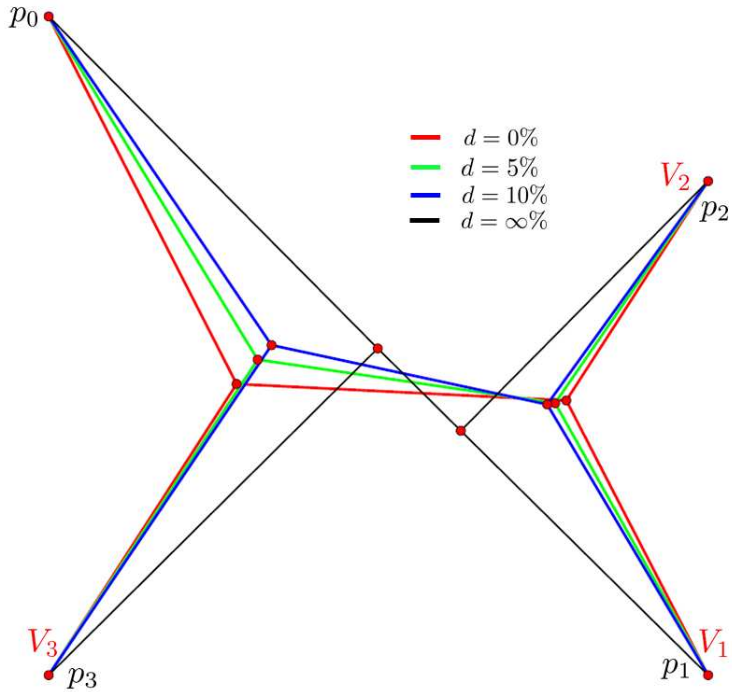
5 Conclusion

The discounted Steiner point algorithm is extended to locate two discounted Steiner points. The extension of the discounted Steiner point algorithm is applied to a hypothetical mine data-set and the performance is evaluated. The algorithm provides higher NPV compared with the placement of the

Table 1: Improvement of the NPV for different discount rates. The NPV is measured in \$M and the improvement is measured in \$k.

Discount (%/yr)	Optimal location of the points s_1, s_2	NPV with EDSPA	NPV without EDSPA	Improvement
0	$(286, 443, 0)$, $(786, 418, 0)$	69.888	69.888	0
5	$(317, 481, 0)$, $(769, 415, 0)$	68.209	68.177	32
10	$(338, 503, 0)$, $(757, 414, 0)$	66.683	66.584	99
∞	$(500, 500, 0)$, $(625, 375, 0)$	0.000	0.000	0

Figure 6: The optimal locations of the discounted Steiner points.



discounted Steiner points at the classical positions. The numerical trials suggest that the EDSPA converges rapidly. However, the convergence is hard to show mathematically. We plan to consider this in future work.

For any given set of four input points, at most two of the three layouts (Figure 2) are feasible. If there are two feasible layouts, then the locations of the discounted Steiner points are substantially different in the two layouts, but based on the limited numerical trials performed so far, it appears that there is typically only a small difference in the NPV.

In the method described here, the iterations alternate between the two Steiner points. In principle, the method could be extended to networks with three or more Steiner points (i.e., four or more draw points) by iterating cyclically through all of the Steiner points. If the method were to be extended in this way, then it could potentially be used to optimise the layout of mines with four or more draw points by applying the method to each possible layout (topology). However, there are two issues that make this process more complicated. One issue is the rapid (superexponential) increase in the number of layouts that need to be considered as the number of Steiner points increases. The other issue is that for any given topology, various orders of cycling through the Steiner points are possible, and it is unclear what effect the choice of order has on the convergence of the process. These issues are matters for further research.

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