Nonlinear stability in seismic waves

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Abstract

We analyse a passive system featuring a neutrally stable short-wavelength mode. The system is modelled by the Nikolaevskiy equation relevant to elastic waves, reaction-diffusion systems and convection. After quickly falling onto a centre manifold, the system then exhibits slow decay. Using the centre manifold technique, we deduce that the decay law is the inverse square root of time. The result is confirmed by direct computations of the system.

Contents

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1 Introduction

The Nikolaevskiy equation was originally derived for elastic and seismic waves [1] aiming, in particular, to explain the experimentally detected dominant frequencies. Subsequently, the equation was also linked to Rayleigh-Benard convection [2] and reaction-diffusion systems [3]–[5]. The initial focus within the equation was on the formation and stability of patterns such as stationary rolls, which emerge from an instability in a spatially uniform state [6]–[9]. Further attention was given to more complex dynamics, especially chaos [3, 10]. Generally, the Nikolaevskiy equation includes two groups of terms—the dispersion terms and dissipation/excitation terms, with the latter group being responsible for the growth or decay of the patterns. In this paper we focus on the effects of dissipation/excitation in seismic waves, so for simplicity we consider the nondimensional Nikolaevskiy equation

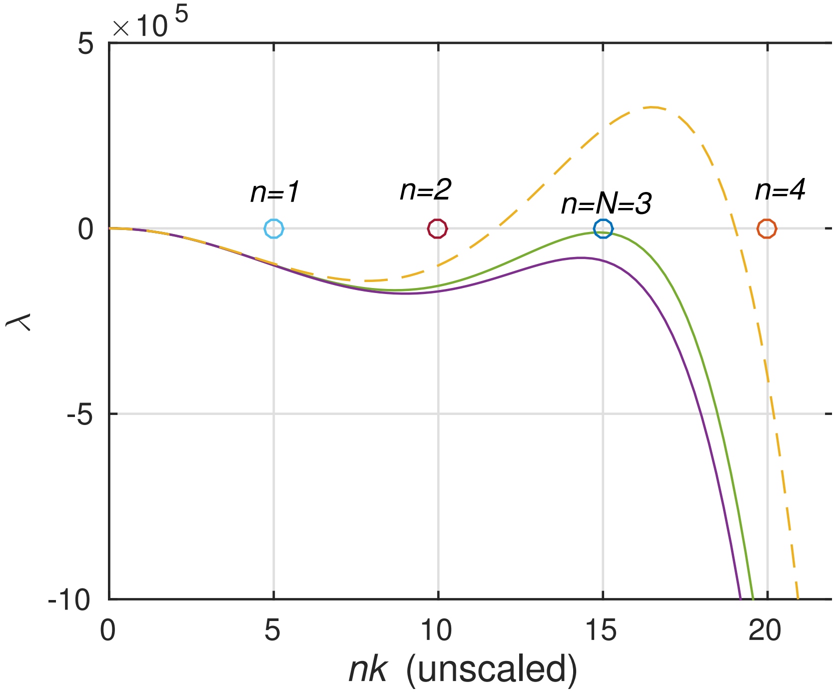
$$\partial_t v = A\partial_x^2 v + C\partial_x^4 v + F\partial_x^6 v + Gv\partial_x v\,, \tag{1}$$

where $A > 0$, $C > 0$, $F > 0$ and G are constants. For reaction-diffusion systems the dispersion terms are not part of the equation. We integrate (1) over x and rewrite it in terms of the derivative, $v = \partial_x u$. After rescaling $u = \alpha u_1$, $x = \beta x_1$ and $t = \gamma t_1$ and requiring that three coefficients in the new equation be units (this is achieved by appropriate selection of the three scaling factors α, β and γ) we obtain

$$\partial_t u = \partial_x^2 u + \alpha\partial_x^4 u + \partial_x^6 u + (\partial_x u)^2 + E\,, \tag{2}$$

where the subscripts '1' are omitted for simplicity, E is the constant of integration, and α is a free parameter. In the context of elastic waves, v represents the velocity in the reference frame moving with the wave. In the

Figure 1: The increment λ versus wave number k for an active system (dashed line) and passive system (as an example, the mode with $N = 3$ is shown as neutral).

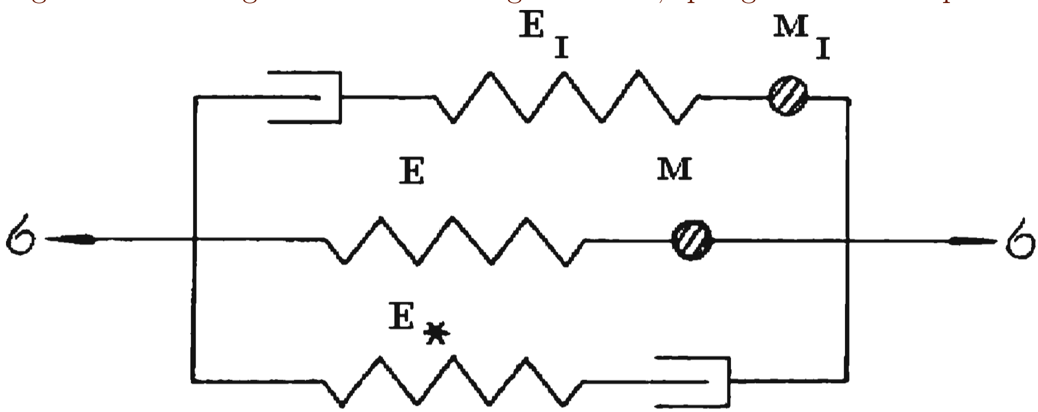


context of reaction-diffusion systems, u stands for the phase of oscillations of chemical concentration and $E = 0$ (the transition $u \to u + Et$ eliminates E).

When $\alpha > 2$, for a small perturbation $\sim e^{\lambda t + ikx}$ of the stationary state u =const, the curve representing the increment $\lambda(k)$ as a function of the wave number k is partly located above zero (see the dashed line in Figure 1). Nikolaevskiy [1] introduced the wave number k_* corresponding to the maximum of λ which translates into the dominant seismic frequency $\nu = ck_*$, where c is the average wave velocity.

As recently argued [11], for a passive system, such as an elastic wave, self-

Figure 2: The Voigt element consisting of masses, springs and friction pistons.



excitation is prohibited because of the absence of internal energy sources. In other words, rocks at rest (corresponding to the state $v(x, 0) = 0$ or $u(x, 0) = \text{const}$) cannot start moving on their own. Strunin [11] provided a more detailed discussion, but here observe Figure 2 which illustrates the absence of internal sources of energy in the Voigt element, used to derive the stress-strain relation [1, 6]. In applications to systems where an internal energy supply exists, self-excitation is possible; for example, in reaction-diffusion systems where energy is internally generated by reactions [4]. These arguments do not mean that the dominant frequency cannot be explained within the Nikolaevskiy equation, but, in our view, the interpretation of this frequency needs to be modified. The dominant frequency mode is the one that exhibits the *slowest decay* relative to the other modes, rather than *fastest growth*. Such a mode survives for longer periods of time compared to the other modes, and thus is the one recorded experimentally. Accordingly, the curve $\lambda(k)$ should lie entirely below zero, as shown by the solid lines in Figure 1). This leads to the question of how the decay progresses in time. Although the decaying dynamics may be less interesting compared to nontrivial pattern formation in systems with self-excitation (such as stationary or chaotic patterns in the excited Nikolaevskiy equation), they still deserve attention.

Upon adopting periodic boundary conditions, simulating a closed loop of elastic material, we expand into the Fourier series

$$u(x,t) = \sum_{n=-\infty}^{\infty} A_n(t)e^{inkx} . \tag{3}$$

Here $A_{-n} = A_n^*$ to ensure that u is real-valued.

For elastic waves, since the original equation is written in terms of the velocity v, not u (see (1)), the adopted condition of periodicity of u is a more strict condition than periodicity of v: generally, the function u is allowed to be non-periodic.

Substituting (3) into (2) we obtain

$$\frac{dA_n}{dt} = \left[-(kn)^2 + \alpha(kn)^4 - (kn)^6\right] A_n - k^2 \sum_{m=-\infty}^{\infty} A_{n-m}A_m m(n-m) . \tag{4}$$

For the linearised equation (4), the modes behave as $A_n \sim e^{\lambda_n t}$ with

$$\lambda_n = -(kn)^2 + \alpha(kn)^4 - (kn)^6 .$$

Consider the limiting case of $\alpha = 2$, when one of the modes is only neutrally stable. The increment curve $\lambda(k)$ touches zero when $nk = 1$. Therefore, if such a neutral mode is chosen to be the Nth Fourier mode, then the wave number is

$$k = 1/N . \tag{5}$$

Thus, under the linearised version of the model, we have one neutral mode and a discrete set of exponentially decaying modes. This situation is ideal for the centre manifold technique; it allows us to asymptotically describe the decaying dynamics at large times. Inserting $\alpha = 2$ and (5) into (4) gives

$$\frac{dA_n}{dt} = \left[-\left(\frac{n}{N}\right)^2 + 2\left(\frac{n}{N}\right)^4 - \left(\frac{n}{N}\right)^6\right] A_n - \left(\frac{1}{N}\right)^2 \sum_{m=-\infty}^{\infty} A_{n-m}A_m m(n-m) . \tag{6}$$

2 Decay on the centre manifold

We start with the case when the neutral mode is the first, $N = 1$. Using $N = 1$ in (6) and restricting attention to a few leading modes,

$$\begin{aligned} \frac{dA_1}{dt} &= 4A_2A_1^* + 12A_3A_2^* + \cdots\,, \\ \frac{dA_2}{dt} &= -36A_2 - A_1^2 + 6A_3A_1^* + \cdots\,, \\ \frac{dA_3}{dt} &= -576A_3 - 4A_1A_2 + \cdots\,. \end{aligned} \tag{7}$$

Centre manifold theory states that the modes which experience a stage of exponential decay caused by the linear terms (in (7) these modes are A_2 and A_3), drop onto a surface, or manifold, where they then evolve slowly [12]. On the manifold these fast modes become connected to the neutral mode by a stiff algebraic expression. As a consequence, they become dependent on time not independently but via the neutral mode. We seek the modes A_2 and A_3 in the form of power series in A_1 and A_1^*:

$$\begin{aligned} A_2 = {}& a_1A_1 + b_1A_1^* \\ &+ a_2A_1A_1^* + m_2A_1^2 + n_2A_1^{*2} \\ &+ a_3A_1^2A_1^* + b_3A_1^{*2}A_1 + g_3A_1^3 + h_3A_1^{*3} \\ &+ w_4A_1^4 + x_4A_1^3A_1^* + y_4A_1^2A_1^{*2} + z_4A_1A_1^{*3} + l_4A_1^{*4} + \cdots\,, \end{aligned} \tag{8}$$

$$\begin{aligned} A_3 = {}& p_1A_1 + q_1A_1^* \\ &+ p_2A_1A_1^* + f_2A_1^2 + k_2A_1^{*2} \\ &+ p_3A_1^2A_1^* + q_3A_1^{*2}A_1 + v_3A_1^3 + y_3A_1^{*3} + \cdots\,. \end{aligned} \tag{9}$$

We substitute (8) and (9) into the second and third equations of (7) while simultaneously replacing dA_1/dt with the first equation of (7). Collecting like powers of A_1, A_1^* and their products with the help of computer algebra (Maxima) we obtain equations for the leading non-zero coefficients of the series (8) and (9),

$$m_2 = -\frac{1}{36}\,, \quad \nu_3 = -\frac{4}{576}\,, \quad m_2 = \frac{1}{5184}\,,$$
$$x_4 = \frac{6\nu_3 - 8m_2^2}{36} = -\frac{13}{93312}\,. \tag{10}$$

The structure of the power series is (the coefficients and stars are omitted)

$$\begin{aligned} A_2 &\sim A_1^2 + A_1^4 + A_1^6 + \cdots\,, \\ A_3 &\sim A_1^3 + A_1^5 + A_1^7 + \cdots\,, \\ A_4 &\sim A_1^4 + A_1^6 + A_1^8 + \cdots\,, \end{aligned}$$

and similarly for higher orders. Based on (10) and (7), the slow evolution on the manifold, up to the fifth order, is

$$\begin{aligned} \frac{dA_1}{dt} &= 4(m_2 A_1^2 + x_4 A_1^3 A_1^* + \cdots)A_1^* + 12(\nu_3 A_1^3 + \cdots)(m_2 A_1^{*2} + \cdots) \\ &= -\frac{1}{9}A_1^2 A_1^* - \frac{29}{46656}A_1^3\,(A_1^*)^2 + \cdots\,. \end{aligned} \tag{11}$$

A simple approximation is derived when we retain only the leading term in (11),

$$\frac{dA_1}{dt} = -\frac{1}{9}A_1^2 A_1^*\,. \tag{12}$$

For the real and imaginary parts defined by $A_1 = Z + iY$ we get the system

$$\begin{aligned} \frac{dZ}{dt} &= -\frac{1}{9}\left(Z^3 + ZY^2\right)\,, \\ \frac{dY}{dt} &= -\frac{1}{9}\left(Z^2 Y + Y^3\right)\,, \end{aligned} \tag{13}$$

which has the solution

$$Z = \frac{Z_0}{\sqrt{t}}\,, \quad Y = \frac{Y_0}{\sqrt{t}}\,, \tag{14}$$

for constant Z_0 and Y_0. Further, inserting (14) into (13) we establish that Z_0 and Y_0 are connected by

$$Z_0^2 + Y_0^2 = \frac{9}{2}\,. \tag{15}$$

The individual values of Z_0 and Y_0 depend on a specific trajectory governed by (7).

Figure 3 is an example of a numerical solution of the three-component system (7), obtained using the solver of Roberts [13]. On the vertical axis we plot the imaginary and real parts multiplied by $\sqrt{t}$. This way we show the asymptotic stage ($t \to \infty$) more vividly as the curve becomes horizontal over a longer range relative to the early stage of the dynamics. In contrast, the traditional log-log plot would give a much shorter horizontal stretch of the asymptotic stage, athough this is the stage of interest. Recall that the $1/\sqrt{t}$ regime is asymptotic, therefore the early stage in Figure 3 is to be ignored.

The settling of the curves in Figure 3 to constants proves that Re A_1 and Im A_1 are eventually proportional to $1/\sqrt{t}$. An inspection of the settled levels confirms the prediction (15).

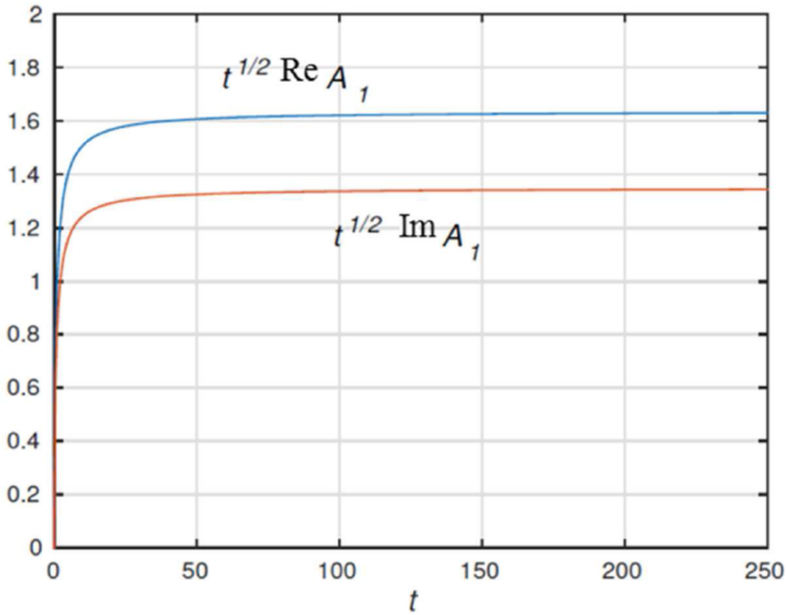
Now consider the case when the neutral mode is the second node, $N = 2$. Using $N = 2$ and $n = 1, 2, 3, 4$ in (6) we obtain

$$\begin{aligned}
\frac{dA_1}{dt} &= -\frac{9}{64}A_1 + 6A_4A_3^* + 3A_3A_2^* + A_2A_1^* + \cdots\,,\\
\frac{dA_2}{dt} &= 4A_4A_2^* + \frac{3}{2}A_3A_1^* - \frac{1}{4}A_1^2 + \cdots\,,\\
\frac{dA_3}{dt} &= -\frac{255}{64}A_3 + 2A_4A_1^* - A_1A_2 + \cdots\,,\\
\frac{dA_4}{dt} &= -36A_4 - \frac{3}{2}A_1A_3 - A_2^2 + \cdots\,.
\end{aligned} \tag{16}$$

The modes A_1, A_3 and A_4 are sought in the form of a power series in A_2 and A_2^*, leading to the centre manifolds

$$A_1 = 0\,, \quad A_3 = 0\,, \quad A_4 = -\frac{1}{36}A_2^2 - \frac{1}{5832}A_2^3A_2^* + \cdots\,. \tag{17}$$

Figure 3: Settling of the inverse-square-root law for the neutral mode $N = 1$ from (7); the initial condition is $A_1 = A_2 = A_3 = 1 + i$.



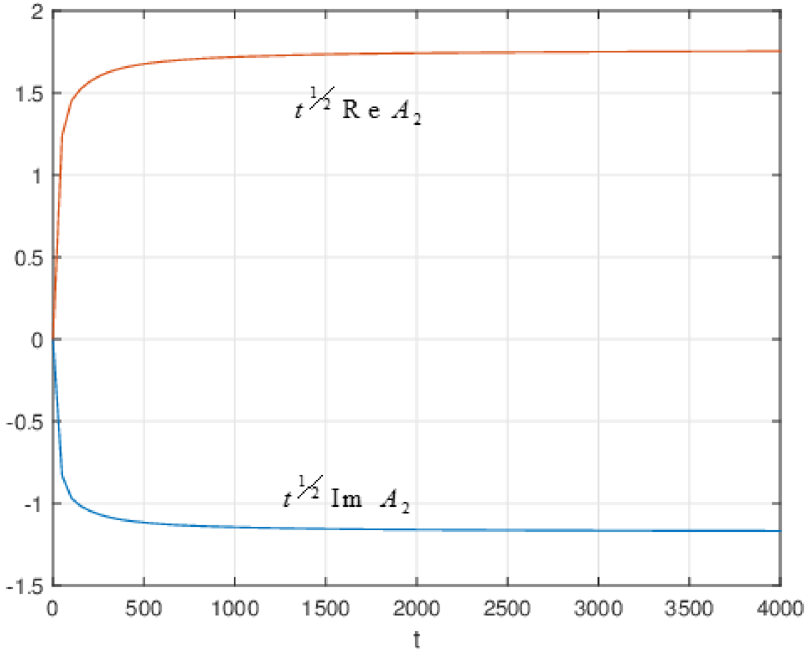
This results in the slow motion on the manifold according to

$$\frac{dA_2}{dt} = -\frac{1}{9}A_2^2A_2^* - \frac{1}{1458}A_2^3A_2^{*2} + \cdots . \tag{18}$$

To leading order, equation (18) has the same form as (12), so that the neutral mode decays as the inverse square root of time. This is confirmed by the numerical solution of (16), see Figure 4. Figure 5 shows the exponential decay of A_1 and A_3 towards their respective centre manifolds $A_1 = 0$ and $A_3 = 0$.

For the case of the neutral mode with $N = 3$, equation (6) gives the system

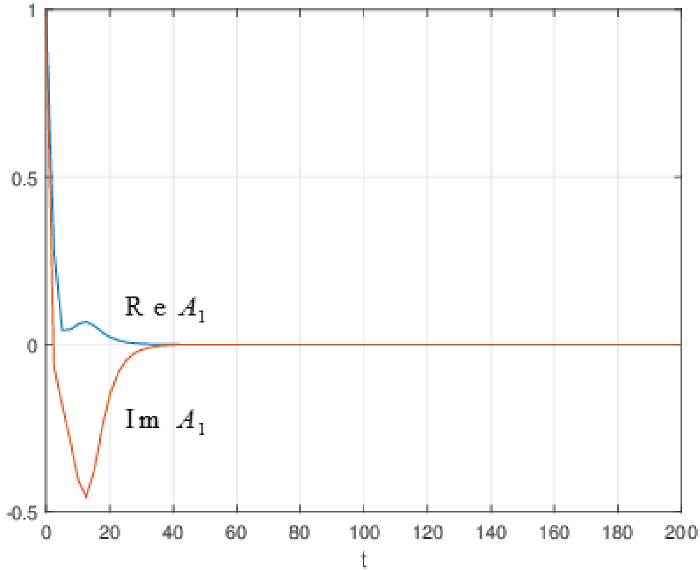
Figure 4: Settling of the inverse-square-root law for the neutral mode $N = 2$ from (16); the initial condition is $A_1 = A_2 = A_3 = A_4 = 1 + i$.



of equations

$$\begin{aligned}
\frac{dA_1}{dt} &= -\frac{64}{729}A_1 + \frac{8}{3}A_4A_3^* + \frac{4}{3}A_3A_2^* + \frac{4}{9}A_2A_1^* + \frac{40}{9}A_5A_4^* + \frac{60}{9}A_6A_5^* + \cdots\,, \\
\frac{dA_2}{dt} &= -\frac{100}{729}A_2 + \frac{16}{9}A_4A_2^* + \frac{3}{2}A_3A_1^* - \frac{1}{9}A_1^2 + \frac{10}{3}A_5A_3^* + \frac{48}{9}A_6A_4^* + \cdots\,, \\
\frac{dA_3}{dt} &= \frac{8}{9}A_4A_1^* - \frac{4}{9}A_1A_2 + \frac{20}{9}A_5A_2^* + 4A_6A_3^* + \cdots\,, \\
\frac{dA_4}{dt} &= -\frac{784}{729}A_4 - \frac{3}{2}A_1A_3 - \frac{4}{9}A_2^2 + \frac{10}{3}A_5A_1^* + \frac{24}{9}A_6A_2^* + \cdots\,, \\
\frac{dA_5}{dt} &= -\frac{6400}{729}A_5 - \frac{8}{9}A_1A_4 - \frac{4}{3}A_2A_3 + \frac{12}{9}A_6A_1^* + \cdots\,, \\
\frac{dA_6}{dt} &= -36A_6 - \frac{10}{9}A_1A_5 - \frac{16}{9}A_2A_4 - A_3^2 + \cdots\,. \qquad (19)
\end{aligned}$$

Figure 5: The exponential decay (asymptotically) of A_1 and A_3 for the case $N = 2$.



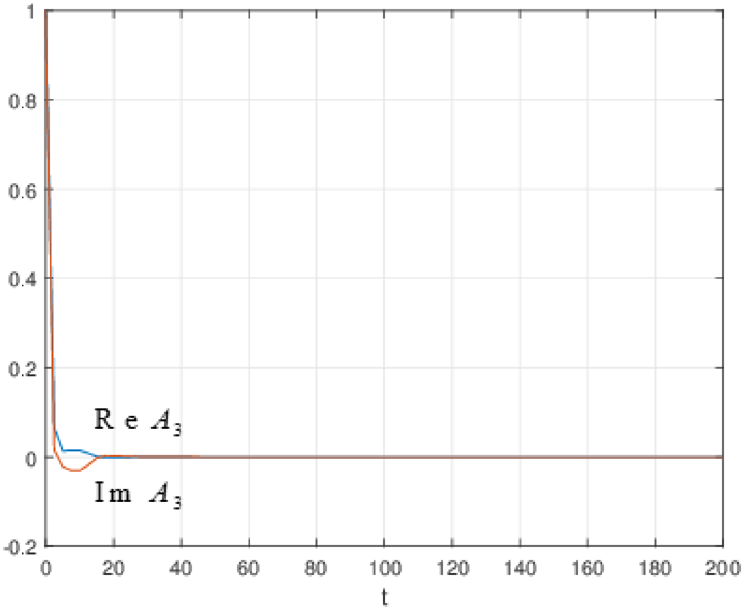
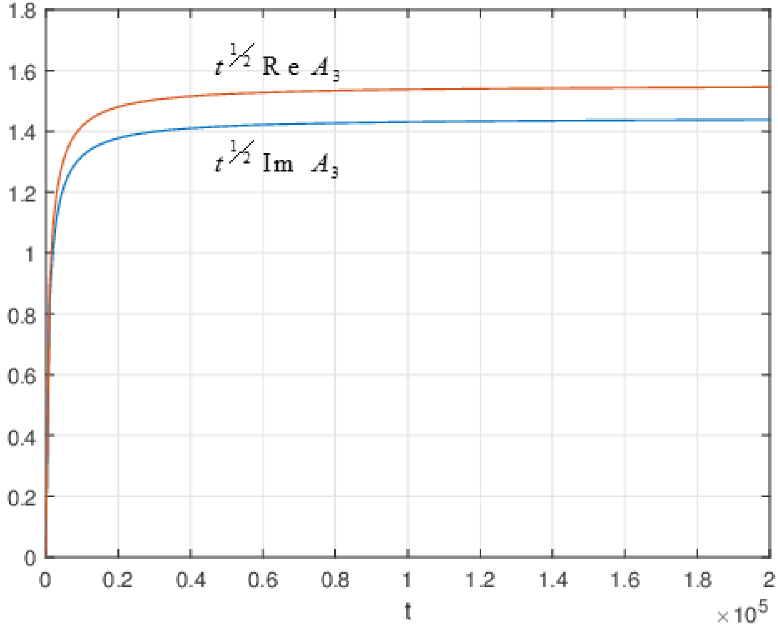


Figure 6: Settling of the inverse-square-root law for the neutral mode $N = 3$ from (19); the initial condition is $A_1 = A_2 = A_3 = A_4 = A_5 = A_6 = 1 + i$.



Analysis of (19) results in the centre manifolds

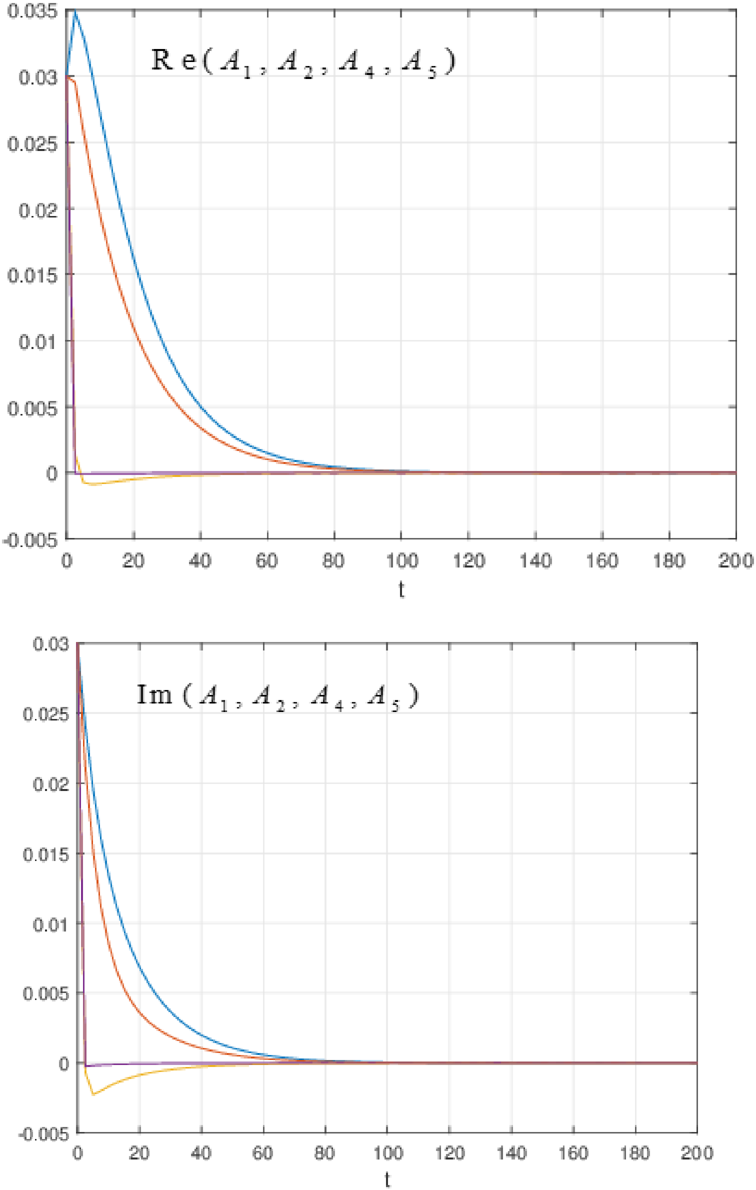
$$A_1 = 0\,, \quad A_2 = 0\,, \quad A_4 = 0\,, \quad A_5 = 0\,, \quad A_6 = -\frac{1}{36}A_3^2 + \cdots\,. \tag{20}$$

Then, the slow motion of the neutral mode on the centre manifold is governed by

$$\frac{dA_3}{dt} = -\frac{1}{9}A_3^2 A_3^* + \cdots\,. \tag{21}$$

The numerical solutions of (19), shown in Figure 6 and Figure 7, confirm the centre manifolds (20) and the law (21).

Figure 7: The exponential decay (asymptotically) of A_1, A_2, A_4 and A_5 for the case $\mathrm{N} = 3$.



3 Conclusions

Elastic waves and similar systems are characterised by the spectrum with dominant frequency/wave number. We interpret this dominant frequency as the one belonging to the mode with slowest decay (not fastest growth as usually adopted). We investigated the critical case when this mode is neutrally stable and, therefore, the decay occurs on a centre manifold. Asymptotic inverse-square-root laws for the decay are derived and confirmed numerically.

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