Dispersed combat as mass action with finite search

A. H. Pincombe[1] B. M. Pincombe[2] C. E. M. Pearce[3]

(Received 9 January 2016; revised 6 November 2016)

Abstract

Improvements to models of battle attrition are necessary because current models cannot explain battle attrition. Agent based simulations indicate that calculated attrition is substantially different when agents are not assumed to have unlimited detection capabilities. However, agent based models are limited to small force sizes and there is no evidence that the changes in calculated attrition occur for large force sizes. We develop a probabilistic model, based on Bernoulli trials, to check if limited detection capabilities result in significant changes to calculated attrition when force sizes are large, as in battle datasets. Our model is a search model and we convert it to an attrition model via the same processes used in current models, and include the same assumptions for factors other than detection range. We find two series

DOI:10.21914/anziamj.v57i0.10447, Published November 10, 2016, as part of the Proceedings of the 12th Biennial Engineering Mathematics and Applications Conference. ISSN 1445-8810. (Print two pages per sheet of paper.)

solutions to the model, one for small force sizes, the other for large force sizes, and find numerically that the two solutions strongly overlap. The new model makes a difference to calculated attrition when force sizes are small, but not when they are large. However, the model makes a difference to calculated attrition for all force sizes if the battlefield area is increased to maintain a sparse force density. Our approach is mathematical, not requiring application knowledge, and several of the assumptions underlying mass action models are raised in our discussion.

Contents

1 Introduction

If complex applications are sufficiently regular and those regularities can be learnt, then applications can be managed intuitively by experienced practitioners [5, pp. 240]. We may be able to derive equations to explain the data in such applications. Over the past century, several applications were modelled by mass action and, although the modelling resulted in many publications, the models were not so successful in explaining application data, resulting in a call for better models in various applications, including infectious disease transmission [4]. Our interest is in finding better models of battle attrition.

Mass action models of battle attrition define connections between force sizes and the attrition suffered by each force, and are useful when force sizes, or population sizes, are the only important variables. When the predictions of mass action models are compared with battle data, there are large unexplained errors (with coefficient of determination $R^2 = 0.1$) [8, 9]. Problems due to temporal aggregation of the data [13] were reduced through the use of ratio models [14]. Large errors between models and data can indicate the need for one or more extra variables. Can we introduce new variables that explain the variance in battle attrition without making the models too complex for practical use? The identification of important variables from battle datasets has proven difficult, and although we recently showed [15] that combat support resources (such as artillery, tanks and aircraft) are important, what form the influence takes cannot currently be derived from battle data.

We use one of the current mass action models and all but one of its assumptions to create a more complex model with extra variables: a probabilistic model based on Bernoulli trials. The addition of extra complexity is justified if attrition estimates are significantly changed. We remove the assumption that force members have an unlimited detection range and replace it with finite detection ranges. This approach does not require application knowledge and can be used to identify a set of variables that make a significant difference to the calculated values of attrition. Finite detection ranges are chosen for this initial attempt because simulation results indicate that they make a difference to estimated attrition [6]. However, simulations are limited to small force sizes and results may not scale up to larger force sizes. To isolate the effect of detection range, the new model is based on the other simplifying assumptions used in mass action attrition models. We aim to determine if finite detection ranges predict attrition that is significantly different from the attrition predicted by standard mass action models. The other assumptions of mass action models will be tested separately in the future.

1.1 Homogeneous Lanchester models

Homogeneous mass action models of battle attrition were developed independently by Chase [1], Osipov [11] and Lanchester [7]. Chase was a naval officer whereas Lanchester was a motor and aviation engineer mostly interested in aerial dogfights. Climactic naval battles and dogfights allow assumptions about all combatants being able to see and hurt all others. Both Chase and Lanchester used a bottom up approach based on simplified military theory. Osipov, an Army surveyor, worked empirically from a database of historical battles.

In mass action models, all variables other than force populations are represented by constant coefficients. In a detailed sense, this is unrealistic: coefficients will only be constant for short periods over small areas [10]. In an averaged or aggregated sense, constant coefficients imply the assumption that other variables are unimportant.

We refer to the two forces fighting a battle as the X-force and the Y-force. The following descriptions of the Lanchester equations are based on the work of Taylor [16].

1.1.1 Direct fire model—the square law

In the direct fire model, $x(t)$ is both the number of shooters for the X-force and the number of targets for the Y-force, with a similar dual meaning for $y(t)$. The attrition rate for each force is proportional to the number of shooters on the opposing force,

$$\frac{dx}{dt} = -ay\,, \quad \frac{dy}{dt} = -cx\,, \tag{1}$$

where a and c are constant.

The direct fire model state equation is formed from equation (1):

$$\frac{dx}{dy} = \frac{ay}{cx}\,. \tag{2}$$

By integrating equation (2) we see that the quantity $cx^2 - ay^2$ is conserved. Because of this conserved quantity, equation (1) is often called the square law.

1.1.2 Empirical model—the log law

A second model is based on the results of tank trials [12]. Here the attrition rate for a force is proportional to its own size. Although we use the same pronumerals for the constants as in equation (1), they have different dimensions in the two cases:

$$\frac{dx}{dt} = -ax\,, \quad \frac{dy}{dt} = -cy\,. \tag{3}$$

The state equation,

$$\frac{dx}{dy} = \frac{ax}{cy}\,, \tag{4}$$

conserves $c\log x - a\log y$. This model is often called the log law.

1.1.3 Indirect fire model—the linear law

For the indirect fire model, the attrition rate depends on both the number of shooters and the number of targets:

$$\frac{dx}{dt} = -axy\,, \quad \frac{dy}{dt} = -cxy\,, \tag{5}$$

where a and c are constant. Again we use the same pronumerals as in the previous two models, but with different dimensions. Although Lanchester [7] stated that the indirect fire model represents the one-on-one conflicts of ancient battles, the formulation does not quite match one-on-one fighting.

This was observed in experiments on duelling behaviour in ants and resulted in the modified model [2]

$$\frac{dx}{dt} = -a\min(x, y)\,, \quad \frac{dy}{dt} = -c\min(x, y)\,. \tag{6}$$

The modified indirect fire model is actually a mixed model. If $y < x$, then y suffers attrition according to the log law equation (3) while x suffers attrition according to the square law equation (1), and the opposite for $x < y$.

The state equation for equation (5) is

$$\frac{dx}{dy} = \frac{a}{c}\,, \tag{7}$$

with conserved quantity $cx - ay$. This model is known as the linear law.

The initial conditions $x(0) = x_0$, $y(0) = y_0$ are the same for equations (1), (3) and (5).

2 The new model

We develop a new model, based on Bernoulli trials, where soldiers independently seek detections. The model links the size of the area of awareness for each soldier to the likelihood that each soldier will have a target.

Consider a particular Y-force soldier, surrounded by an area a_y which represents the limits of his awareness. If the total area of the battlefield is A, then the probability that an individual X-force soldier, is within a_y is a_y/A and the probability that the X-force soldier is not within a_y is $1 - a_y/A$. There are x X-force soldiers at time t, so the probability that none of them are in a_y is estimated by

$$p = \left(1 - \frac{a_y}{A}\right)^x. \tag{8}$$

The expected number of Y-force soldiers without potential targets is py, where y is the number of Y-force soldiers at time t. Thus the expected number of Y-force soldiers with at least one potential target is $(1-p)y = [1-(1-a_y/A)^x]y$.

In real war, what happens when a detection is made is a matter of tactics: equations (1), (3) and (5) assume that a soldier shoots at a target if he has one. We use assumptions identical to those for equations (1), (3) and (5), except we do not assume unlimited detection. We replace the number of Y-force soldiers (y) in the Lanchester square law differential equations with the expected number of Y-force soldiers who have at least one target, and make the analogous change for the X-force:

$$\frac{dx}{dt} = -k_y\left[1-\left(1-\frac{a_y}{A}\right)^x\right]y \quad \text{and} \quad \frac{dy}{dt} = -k_x\left[1-\left(1-\frac{a_x}{A}\right)^y\right]x\,. \tag{9}$$

For convenience, we write $(1-a_y/A)^x = e^{-\alpha x}$ and $e^{-\beta y} = (1-a_x/A)^y$, where

$$\alpha = -\log(1-a_y/A) \quad \text{and} \quad \beta = -\log(1-a_x/A)\,. \tag{10}$$

Thus equation (9) becomes

$$\frac{dx}{dt} = -k_y(1-e^{-\alpha x})y \quad \text{and} \quad \frac{dy}{dt} = -k_x(1-e^{-\beta y})x\,. \tag{11}$$

In the limit $x, y \to \infty$, equation (11) reverts to the form of the square law (1)

$$\frac{dx}{dt} = -k_y y \quad \text{and} \quad \frac{dy}{dt} = -k_x x\,. \tag{12}$$

When αx and βy approach zero, equation (11) approaches the form of the linear law (5)

$$\frac{dx}{dt} = -k_y \alpha xy \quad \text{and} \quad \frac{dy}{dt} = -k_x \beta xy\,. \tag{13}$$

For sufficiently large force sizes, our model is almost identical to the square law, and our model is almost identical to the linear law for sufficiently small force sizes.

2.1 Scaling, generalisation and solution of equations

For convenience, we introduce the changes of variables $X = \alpha x$ and $Y = \beta y$ to equation (11) to obtain

$$\frac{dX}{dt} = -\frac{k_y \alpha}{\beta}\left(1 - e^{-X}\right) Y \quad \text{and} \quad \frac{dY}{dt} = -\frac{k_x \beta}{\alpha}\left(1 - e^{-Y}\right) X . \tag{14}$$

The two parts of equation (14) are combined to obtain

$$\frac{dY}{dX} = \frac{\beta^2 k_x}{\alpha^2 k_y} \frac{(1 - e^{-Y})X}{(1 - e^{-X})Y}, \tag{15}$$

which is separable and, when integrated from (X, Y) to (X_0, Y_0), involves integrals of the form

$$F(Z_0) - F(Z) = \int_Z^{Z_0} \frac{s}{1 - e^{-s}}\, ds\,, \tag{16}$$

for $Z = X, Y$ and $Z_0 = X_0, Y_0$. Since

$$\int_Z^{Z_0} \frac{s}{1 - e^{-s}}\, ds = \int_0^{Z_0} \frac{s}{1 - e^{-s}}\, ds - \int_0^{Z} \frac{s}{1 - e^{-s}}\, ds\,, \tag{17}$$

we have

$$F(Z) = \int_0^{Z} \frac{s}{1 - e^{-s}}\, ds\,. \tag{18}$$

Equation (18) is evaluated numerically. Symbolic logic programs may claim an analytical solution in terms of the dilogarithm function, but the dilogarithm is a simple transformation of the integral in equation (18) and also must be evaluated numerically. To allow comparison with the square law, we opt for approximate solutions in terms of elementary functions, and we develop two asymptotic solutions, $F_1(Z)$ and $F_2(Z)$. We begin with the absolutely convergent series

$$\frac{e^{xs}}{e^s - 1} = \sum_{i=0}^{\infty} B_i(x) \frac{s^{i-1}}{i!}\,, \quad 0 < |s| < 2\pi\,, \tag{19}$$

which is the generating function of the Bernoulli polynomials $B_i(x)$ [3]. These are related to the Bernoulli numbers B_i through $B_i(1) = B_i$ for $i \neq 1$ and $B_1(1) = -B_1 = 1/2 = 1 + B_1$. Thus setting $x = 1$ in (19) and multiplying by s provides

$$\frac{s}{1 - e^{-s}} = s + \sum_{i=0}^{\infty} B_i \frac{s^i}{i!}, \tag{20}$$

with the series absolutely convergent for $0 < |s| < 2\pi$. The series in equation (20) is also absolutely convergent for $|s| \leqslant r$ if $0 < r < 2\pi$. Hence it is absolutely convergent for $|s| < 2\pi$ and we integrate termwise to get

$$F_1(Z) = \frac{Z^2}{2} + \sum_{i=0}^{\infty} B_i \frac{Z^{i+1}}{(i+1)!} \quad \text{for } |Z| < 2\pi. \tag{21}$$

A second asymptotic expansion, suitable for large values of Z, is derived for the integral in equation (18) using the relationship $(1-t)^{-1} = \sum_{i=0}^{\infty} t^i$, where, in this case, $t = e^{-s}$. Termwise integration of the resulting expansion gives

$$F_2(Z) = \frac{Z^2}{2} - \sum_{i=1}^{\infty} \left(\frac{Ze^{-iZ}}{i} + \frac{e^{-iZ}}{i^2} - \frac{1}{i^2} \right), \tag{22}$$

which is simplified using

$$Z \log(1 - e^{-Z}) = -\sum_{i=1}^{\infty} \frac{Ze^{-iZ}}{i}, \tag{23}$$

and

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}, \tag{24}$$

to obtain

$$F_2(Z) = \frac{Z^2}{2} + \frac{\pi^2}{6} + Z \log(1 - e^{-Z}) - \sum_{i=1}^{\infty} \frac{e^{-iZ}}{i^2}. \tag{25}$$

Integration of equation (15) defines a constant valued ratio

$$\frac{F(Y_0) - F(Y)}{F(X_0) - F(X)} = \frac{\beta^2 k_x}{\alpha^2 k_y} \,. \tag{26}$$

Equations of the same form as equation (26) also apply to the Lanchester square law, when $F(Z) = Z^2$, and to the linear law, when $F(Z) = Z$, and the same form of relation was found for a wide class of Lanchester style equations [16]. In terms of the actual force sizes, equation (26) gives

$$\frac{F(\beta y_0) - F(\beta y)}{F(\alpha x_0) - F(\alpha x)} = \frac{\beta^2 k_x}{\alpha^2 k_y} \,. \tag{27}$$

3 Discussion

We use equation (11), coupled with the definitions of α and β (equation (10)), to develop approximate expressions linking x and a_y/A for two cases of which deviate from the square law: the first by 10%, representing a significant deviation; the second by 1%, representing an insignificant deviation. For a 10% deviation we have $e^{\alpha x} < 0.1$, leading to

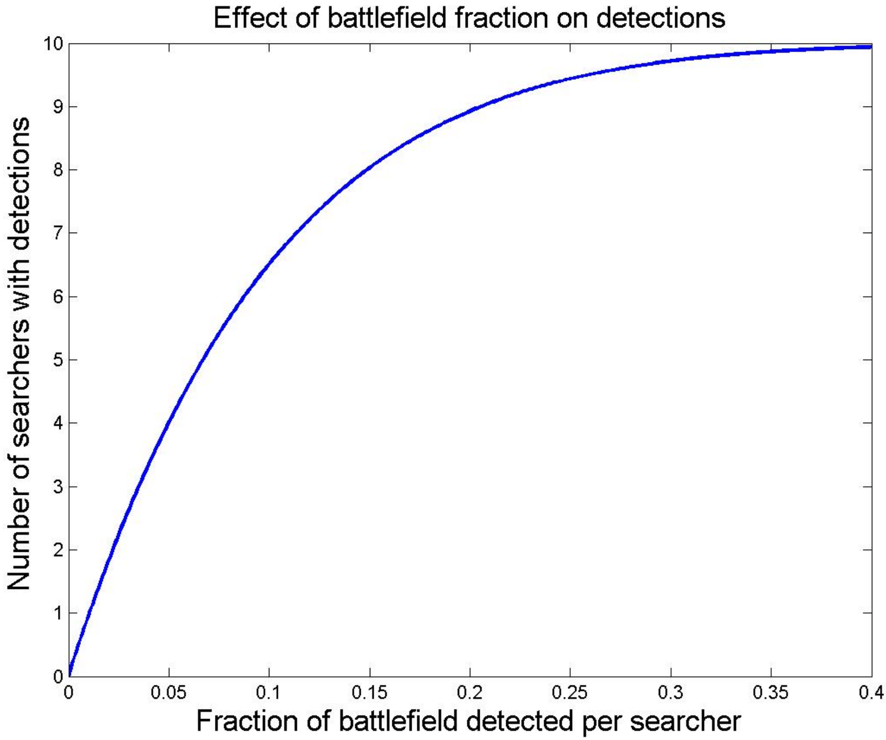
$$\frac{a_y}{A} \approx \frac{2.3}{x} \,, \tag{28}$$

while for a 1% deviation

$$\frac{a_y}{A} \approx \frac{4.6}{x} \,. \tag{29}$$

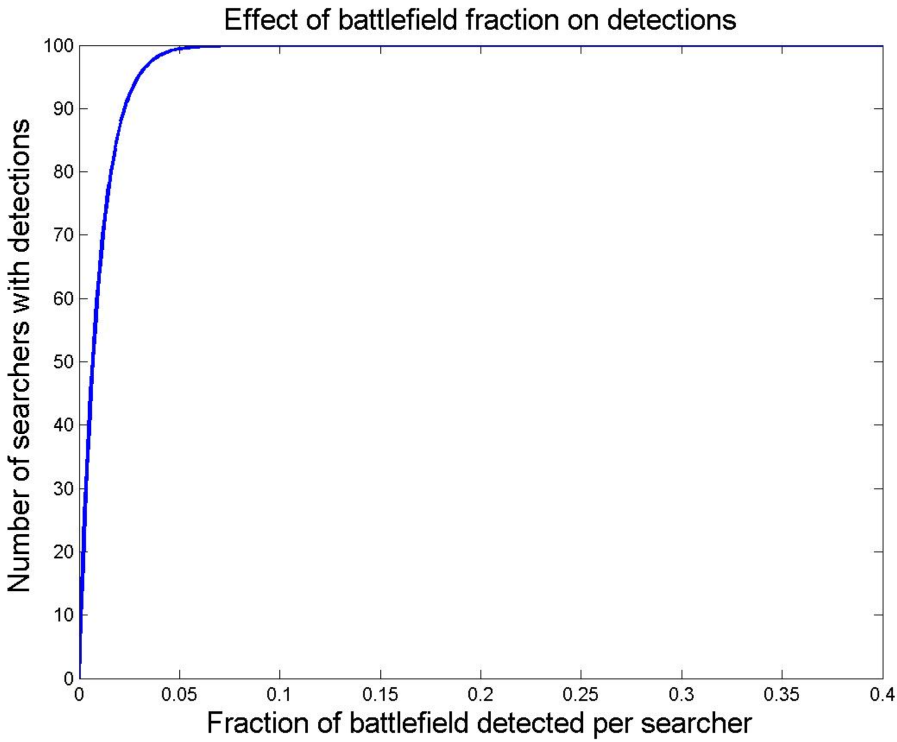
In Figures 1 and 2, the force sizes are ten soldiers or one hundred soldiers, respectively, the horizontal axes are the values of a_y/A, and the vertical axes the number of searchers with detections at $t = 0$ (given by the right hand side of the first equation in equation (9)). The two figures demonstrate how the number of force members with detections is related to the awareness areas and force sizes. For the square law, all soldiers always have detections. For Figure 1

Figure 1: Number of searchers with detections at $t = 0$ for ten searchers in each force.



there are ten soldiers in each force and attrition is approximately equivalent to the square law when $a/A = 0.4$, as expected from equation (29). For Figure 2 there are one hundred soldiers in each force and attrition is approximately the square law when $a/A > 0.05$, as expected from equation (29). The proportion of the battlefield area visible to searchers depends on both the value of a/A and the force size. When we increase the area A by the same factor as the increase in force sizes and look at values of a/A up to 0.04, then the results are almost the same as in Figure 1, indicating that attrition can be significantly altered for any force size as long as the population density is

Figure 2: Number of searchers with detections at $t = 0$ for 100 searchers in each force.



sparse.

Square law assumptions that are maintained include constant search cross sections, uniform distribution of fire, independence of individual shooters and homogeneity of force members (e.g., in weapons). So, like models based purely on mass action, the new model is unable to support analysis of the effectiveness of force members taking cover when under fire, or of concentrating their fire, or of individuals cooperating, or of heterogeneity between individuals. Uniform distribution of fire is a way of ignoring the difference between the number of searchers with detections and the number of targets that have been detected.

The extra variables in our model make a difference when force sizes are small, as in simulations, but not when force sizes are large, showing that the results do not scale up. Since the results from our model are not consistent over all force sizes, the new variables are unlikely to explain the evidence from battle datasets. However, our approach can be used to remove other assumptions and create other models.

Consider the case where $a_x < a_y$, giving the Y-force an advantage in detection range. If neither force makes any attempt to avoid detection, then the Y-force will always be able to surprise the X-force. If the Y-force also has an advantage in weapon range, then they may be able to destroy the X-force before the X-force can close the gap and begin retaliation. Normally, a force with a disadvantage in detection and weapon ranges will use natural terrain or engineered earthworks to provide cover and protection. By taking cover, the force reduces its detection cross section, thus reducing the detection range of its opponents. By using protection, a force may also be able to compensate for a disadvantage in weapon effectiveness.

Acknowledgements Professor Charles Pearce was tragically killed in a road accident after this model was developed.

References

[1] J. V. Chase. *Sea fights: A mathematical investigation of the effect of superiority of force in combats upon the sea*. Naval War College Archives, RG 8, Box 109, XTAV (1902), 1902. C308

[2] N. R. Franks and L. W. Partridge. Lanchester battles and the evolution of combat in ants. *Anim. Behav.*, 45:197–199, 1993. doi:10.1006/anbe.1993.1021 C310

[3] G. S. Gradschtein and I. M. Ryzhik. *Tables of Series, Products and Integrals*. Deutcher Verlag der Wissenschaften, 1996. C313

[4] N. C. Grassly and C. Fraser. Mathematical models of infectious disease transmission. *Nat. Rev. Microbiol.*, 6:477–487, 2008. doi:10.1038/nrmicro1845 C307

[5] D. Kahneman. *Thinking Fast and Slow*. Penguin, London, 2013. C306

[6] L. R. Kosowski, A. Pincombe and B. Pincombe. Irrelevance of the fractal dimension term in the modified fractal attrition equation. *ANZIAM J.*, 52:C988–C1011, 2011. doi:10.21914/anziamj.v52i0.3963 C307

[7] F. W. Lanchester. *Aircraft in warfare: The dawn of the fourth arm.* Constable, London, 1916. http://edc448uri.wikispaces.com/file/view/Lanchester+-+Aircraft+in+Warfare.pdf C308, C309

[8] T. W. Lucas and T. Türkeş. Fitting Lanchester equations to the Battles of Kursk and Ardennes. *Nav. Res. Log.*, 51:95–116, 2004. doi:10.1002/nav.10101 C307

[9] T. W. Lucas and J. A. Dinges. The effect of battle circumstances on fitting Lanchester equations to the Battle of Kursk. *Mil. Oper. Res.*, 9:17–30, 2004. http://www.mors.org/Publications/MOR-Journal/Online-Issues C307

[10] P. M. Morse and G. E. Kimball. *Methods of Operations Research*. Wiley, 1951. C308

[11] M. Osipov. The influence of the numerical strength of engaged forces in their casualties. Translated by R. L. Helmbold and A. S. Rehm. *Nav. Res. Log.*, 42:435–490, 1995. doi:10.1002/1520-6750(199504)42:3<435::AID-NAV3220420308>3.0.CO;2-2 C308

[12] R. Peterson. On the logarithmic law of combat and its application to tank combat. *Oper. Res.*, 15:557–558, 1967. doi:10.1287/opre.15.3.557 C309

[13] A. H. Pincombe, B. M. Pincombe and C. E. M. Pearce, Putting the art before the force. *ANZIAM J.*, 51:C482–C496, 2010. doi:10.0000/anziamj.v51i0.2584. C307

[14] A. H. Pincombe, B. M. Pincombe and C. E. M. Pearce. A simple battle model with explanatory power. *ANZIAM J.*, 51:C497–C511, 2010. doi:10.21914/anziamj.v51i0.2585. C307

[15] B. M. Pincombe and A. H. Pincombe. Mass action models of Falklands War battles. *ANZIAM J.*, 57:C235–C252, 2016. doi:10.21914/anziamj.v57i0.10450 C307

[16] J. G. Taylor. Lanchester models of warfare. Operations Research Society of America, Arlington, 1983. C308, C314

Author addresses

1. **A. H. Pincombe**, Pincombe Consulting, Australia.
   mailto:adrianpincombe@gmail.com

2. **B. M. Pincombe**, Defence Science and Technology Group
   mailto:Brandon.Pincombe@dsto.defence.gov.au

3. **C. E. M. Pearce**, School of Mathematical Sciences, University of Adelaide