A dynamical systems model for fireline growth with suppression

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(received 26 October 2005; revised 8 August 2006)

Abstract

An elementary dynamical systems model for fireline growth is presented. It includes the effect of suppression applied from a set time after the start of the fire. Criteria for the likely success of the containment activities are derived from the model.

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See http://anziamj.austms.org.au/V47EMAC2005/Weber for this article, © Austral. Mathematical Soc. 2006. Published December 15, 2006. ISSN 1446-8735

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1 Introduction

There has been a long history of wildland fire spread modelling, most recently reviewed by Pastor et al. [4] and including models that have attempted to include chemical kinetics in a fundamental way, such as the paper by Assensio and Ferragut [3]. A few of the models attempted to include the effects of suppression on fire growth and the eventual fire size; for example, Anderson [1]. We introduce a dynamical systems model as an alternative way of considering and modelling fireline growth when there is active suppression applied from a set time after the fire was initiated. The intention is to use the mathematical framework of dynamical systems to illustrate a new way of describing the effect of fire suppression activities on the fireline and to develop criteria for the likely outcome of containment activities.

It is anticipated that each component of the present model could be refined to account for more details of actual situations and that the result could then form a simulation module within fire incident management systems.

2 Fireline growth without suppression

Consider first the simpler example of fireline growth from a point ignition and without suppression.

Let the fireline at any time t have length $L(t)$. The requirement of point ignition means that $L(0) = 0$ and the fireline should increase in length for all $t > 0$. A useful first formulation for fireline growth is

$$\frac{dL}{dt} = \frac{\alpha}{L^{\epsilon}} \tag{1}$$

with α is a constant of proportionality and ϵ is an exponent which we need to determine. After some elementary integration and matching of the initial condition $L(0) = 0$, we see that the exponent ϵ needs to be greater than minus one for a sensible model which correctly reflects that the fire starts at a point and grows in time. The simplest case is when $\epsilon = 1$ and we shall use this as our basic model throughout the rest of this paper.

To summarise, we select the model

$$\frac{dL}{dt} = \frac{\alpha}{L}, \tag{2}$$

with α a constant of proportionality, initial condition $L(0) = 0$ and solution for fireline growth $L(t) = \sqrt{2\alpha t}$, valid for all $t > 0$.

Note that the area of the fire scales approximately as the square of the fireline, and that with $\epsilon = 1$ this results in a fire area that grows linearly with time.

Incidentally, there is a link with this and the many mean curvature models for the growth of assorted physical, chemical and biological fronts; see for example the many examples by Pelcé [5]. This is also relevant in the context of previous geometrical models for wildland fire growth such as the ellipse model of Anderson et al. [2].

3 Fireline growth with suppression

Building upon the previous model without suppression, we now present a formulation where the natural processes of fire growth and the application of fire control strategies interact. The simplest example of a dynamical system which incorporates the model from the previous section for natural fireline growth and the assumptions that

- suppression reduces fireline growth $(-\beta S)$,
- fixed resources are available for suppression (Q),
- suppression efficacy is diminished by fireline growth $(-\gamma L)$,

is

$$\begin{aligned} \frac{dL}{dt} &= \frac{\alpha}{L} - \beta S\,, \\ \frac{dS}{dt} &= Q(t) - \gamma L\,, \end{aligned} \tag{3}$$

Here $L(t)$ is the length of active fireline at a time t, $S(t)$ is the length of suppressed fireline at a time t, and α, β and γ are constants in the model. $Q(t)$ is a quantifiable expression for all of the suppression resources applied at time t.

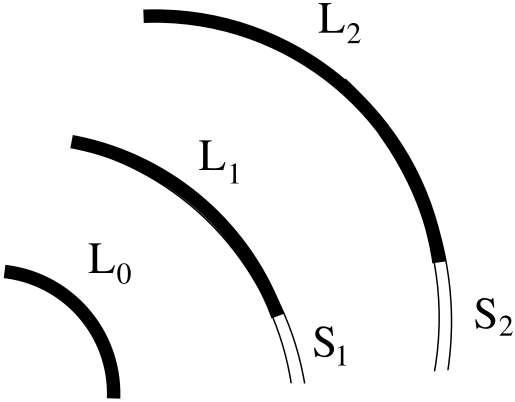

Figure 1: Schematic of a growing fireline at times $t = 0, 1$ and 2, showing active part, L, and suppressed part, S, at each time.

4 Initial conditions

We envisage that the starting situation is a point ignition at $t = 0$, and that there is no suppression activity until a time t_S. Hence, the fireline grows just as in the previous section for time t where $0 \leq t \leq t_S$. This means that the fireline will reach a finite, non-zero size before suppression activities commence. At this time, the length of suppressed fireline is zero (that is, $S = 0$ at $t = t_S$) and the model above, governing equation (3), is begun.

This is equivalent to assuming that the model (3) is started with $L = L_S$ and $S = 0$ at $t = t_S$. We then analyse the expected behaviour in our model and try to predict the success of the suppression activity. Figure 1 is a schematic of a growing fireline with suppression applied, showing increasing active and suppressed parts at successive times.

5 Behaviour in the phase plane

Henceforth we assume that the available suppression resources are constant; that is, Q is in dependent of time. There is only one critical point, found by setting

$$\frac{dL}{dt} = 0 \quad \text{and} \quad \frac{dS}{dt} = 0\,,$$

which gives the simultaneous equations

$$\begin{aligned} \frac{\alpha}{L} - \beta S &= 0\,, \\ Q - \gamma L &= 0\,. \end{aligned}$$

The critical point is then $(L, S) = (Q/\gamma, \alpha\gamma/(\beta Q))$.

The nature of this sole critical point is determined by examining the behaviour in a neighbourhood of the critical point; we find that it is a saddle. To illustrate the behaviour of the system we examine the numerically determined phase plane of a representative example shown in Figure 2. The saddle point is denoted by an open circle. The dashed bold arrows refer to the manifolds of the saddle points (the stable manifolds are denoted by the arrows pointing towards the saddle point, whereas the unstable manifold is denoted by the arrows pointing away from this critical point). The phase plane plot shows the path within the (L, S) plane of the fireline growth suppression model as it evolves in time according to the governing equations (3), for four initial conditions. The trajectories are labelled accordingly in the figure. As in the previous section the initial condition for this model is a nonzero value for the active fireline and a zero value for the suppressed fire line (that is, no suppression at the beginning). There are then three different types of behaviour depending on the values of these initial conditions, as determined from the phase plane in Figure 2.

Initial conditions between $(0, 0)$ and $(L_{\mathbf{cr}1}, 0)$ Examining the behaviour of the trajectory labelled (1) with initial values $(30, 0)$, we see that initially

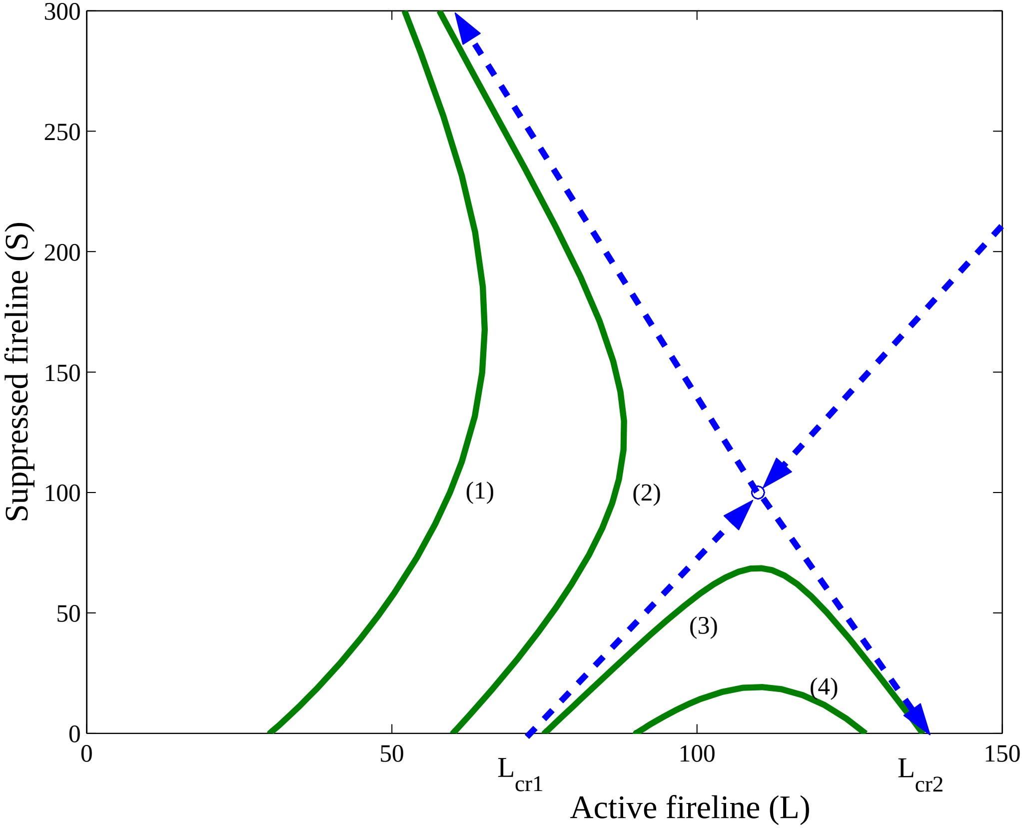

Figure 2: Phase-plane plot for the fireline growth and suppression model with $\alpha = 11$, $\beta = 0.001$, $\gamma = 0.01$ and $Q = 1.1$.

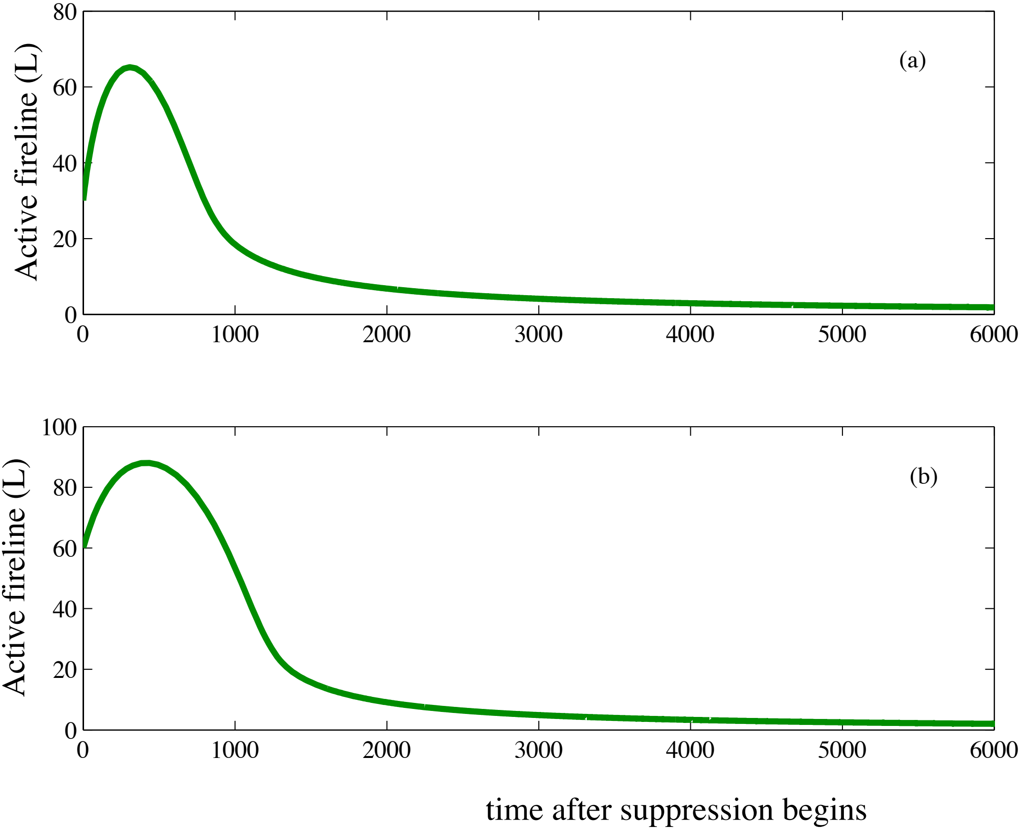

Figure 3: Shows the timeplots of the trajectories from Figure 2. (a) shows the time evolution of Trajectory (1) from Figure 2 whose initial condition is $(L, S) = (30, 0)$, and (b) shows the time evolution of Trajectory (2) from Figure 2 whose initial condition is $(L, S) = (60, 0)$. Both curves show that the active fireline reduces to zero in the long term. Similar behaviour is seen for any initial condition between $(L, S) = (0, 0)$ and $(L, S) = (L_{\text{cr1}}, 0)$.

the active fireline increases. However, in the long term, the active fireline reduces to zero. This can be seen clearly in the timeplots of Figure 3(a). Similar behaviour can be seen for Trajectory (2) in Figure 2; that is, for initial condition $(L, S) = (60, 0)$. All initial conditions in this region behaves in a manner similar to that described above; that is, the active fireline increases initially, but in the long term the active fireline reduces to zero. In other words, sufficient resources were available to successfully suppress the active fireline.

Initial conditions between $(L_{\mathbf{cr}1}, 0)$ and$(L_{\mathbf{cr}2}, 0)$ Trajectories (3) and (4) in the phase plane (Figure 2) illustrate the behaviour for initial conditions in this region. The corresponding timeplots are shown in Figure 4. See that initially the suppressed fireline increases, but later it goes to zero, which results in the entire fireline being active. Running the model beyond this time results in the suppressed fireline becoming negative, and the model (3) ceases to be physically meaningful. Hence for initial conditions in this region, the available resources to fight the fire only appear, at the beginning, to be sufficient to control the active fireline (with the increase in suppressed fireline) but soon the entire fireline becomes active.

Initial conditions beyond $(L_{\mathbf{cr}2}, 0)$ For initial conditions in this region, the resources available to fight the fire are not sufficient to cause any suppression whatsoever. From the unstable manifold, we clearly see that running the model will result in negative suppressed fireline and hence the model is not physically meaningful.

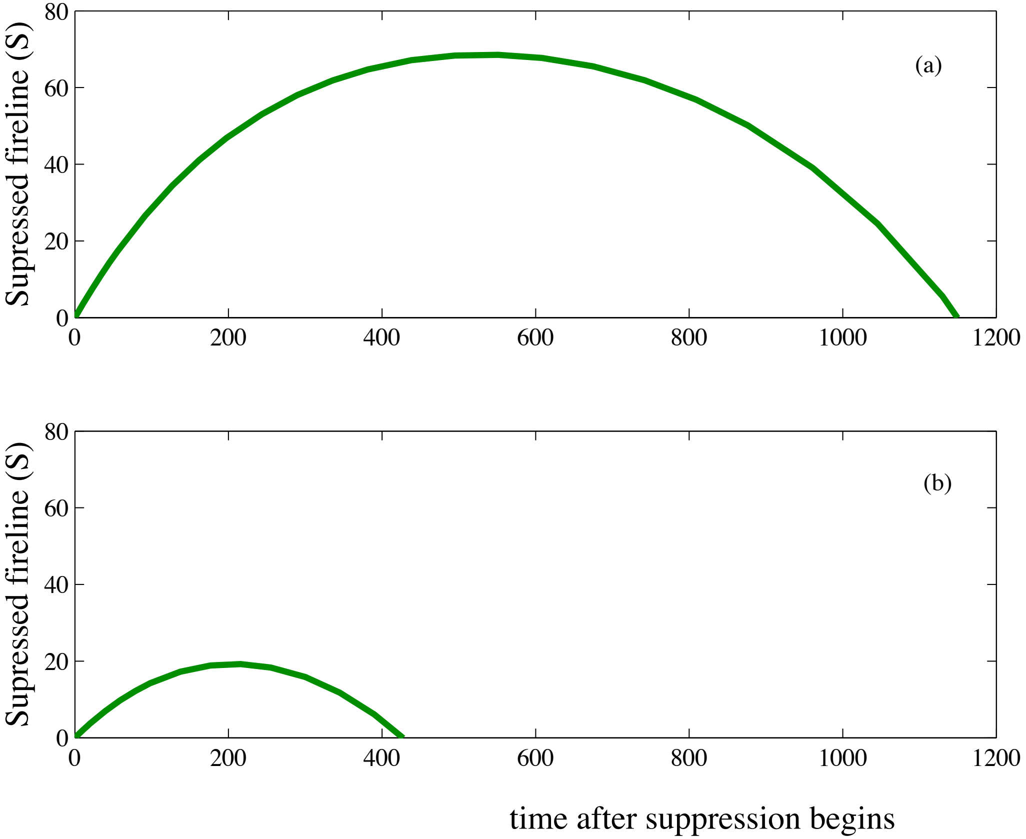

Figure 4: Shows the timeplots of the trajectories from Figure 2. (a) shows the time evolution of Trajectory (3) whose initial condition is $(L, S) = (75, 0)$, whereas (b) shows the time evolution of Trajectory (4) whose initial condition is $(L, S) = (90, 0)$. Both curves show that the suppressed fireline reduces to zero. Similar behaviour is seen for any initial condition between $(L, S) = (L_{\rm cr1}, 0)$ and $(L, S) = (L_{\rm cr2}, 0)$.

6 Success criteria

For initial active fireline L_0, see from the equation

$$\frac{dS}{dt} = Q - \gamma L \,,$$

that the relationship $Q = \gamma L_0$ is critical and hence we have the crude estimates that

- $L_0 > Q/\gamma$ means that the initial active fireline is beyond the capability of the suppression resources, whereas
- $L_0 < Q/\gamma$ means that suppression could be effective and the model needs to be run to ascertain the likely outcome.

From the phase plane portrait, L_{cr1} is a far superior value to use as an indicator of likely success. A linearised analysis could be used to calculate the eigenvectors and provide an approximate formula for L_{cr1}; however, in practice it is just as easy to numerically solve the model for any given set of parameters and determine L_{cr1} precisely.

7 Discussion

It is quite clear from the phase plane and associated time histories depicted in Figures 2–4 that there is a dynamical interplay between the intrinsic growth of the fireline and the applied suppression activities. The precise determination of the critical threshold for the success of the suppression activities is not possible analytically, but needs to be determined numerically as in our representative example. Interestingly, note that the model provides a method for quantifying the length of time for which suppression must be applied in order for it to result in a successful outcome. With a more precise calibration of the model this could be tested in the field in conjunction with land management agencies in the hope of assisting in the improved planning of suppression activities.

8 Conclusion

We have shown that it is possible to formulate a dynamical systems model which encapsulates the basic elements of fire growth prior to suppression and the subsequent effects of applied suppression. The application to actual fire situations will be the topic of a further investigation and will enable a determination of parameter values that accurately represent the rate of fire-line growth, the suppression resources, and the coefficients describing the interaction between fire size and suppression efficacy. This may result in further revision and extension of the model to include hitherto ignored aspects, such as the result of long-time ongoing fire suppression activities and the introduction of aerial support.

Acknowledgements We thank Ross Bradstock, Geoff Cary, Malcolm Gill and Karen King for helpful discussions during the course of this work.

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