A Reynolds uniform scheme for singularly perturbed parabolic differential equation

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Abstract

Time dependent convection diffusion problems with large Reynolds number are considered. Such a problem has been considered by using Shishkin's scheme, which was uniformly convergent with respect to large Reynolds number in order $\mathcal{O}(N^{-1}\log^2 N + M^{-1})$, where N and M are number of intervals in x direction and t direction respectively. A three-transition points scheme, four piecewise-uniform mesh, is introduced. The mesh partition, the barrier function, the estimate of truncation error and the techniques of proof are different from others. The new scheme is non-equidistant. It is proved uniformly convergent with respect to large Reynolds number in order $\mathcal{O}(N^{-1} + M^{-1})$. Our work is better than Shishkin's traditional

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scheme, while the computational procedure is as simple as Shishkin's scheme. This novel method also has the same accurate result as Bakhvalov–Shishkin's scheme, while the computational procedure is simpler than Bakhvalov–Shishkin's scheme. Shishkin's scheme and Bakhvalov–Shishkin's scheme are compared with the new method. Finally, numerical results support the theoretical results.

Contents

1 Introduction

Problems with large Reynolds number are of common occurrence in many branches of applied mathematics [3, 6, 7]. The presence of large Reynolds number leads to boundary layer phenomena. Various methods are available in the literature in order to obtain numerical solution [1, 2, 4, 7].

In this article the parabolic initial-boundary value problem with large Reynolds number is considered. It is well known that a uniform mesh does not properly solve such problems [3]. The simplest class of the non-uniform mesh is fitted piecewise-uniform mesh [3, 6], which is uniformly convergent with respect to large Reynolds number in order $\mathcal{O}(N^{-1}\log^2 N + M^{-1})$, where N and M are number of intervals in x direction and t direction respectively.

In order to improve the convergence rate, Shishkin's scheme with Bakhvalov technique [1] has been presented [5], which has first-order convergence rate, but it was more difficult to determine the mesh partition.

In this article a multi-transition points difference scheme is presented. The new method is computationally efficient. It has the same accurate result as Bakhvalov–Shishkin's scheme, while the computational procedure is as simple as Shishkin's scheme.

There are new ideas in this article: the mesh partition, the barrier function, the estimate of truncation error and the techniques of proof are different from others.

Firstly, a choice of three-transition points is presented, which is based on the multi-log function of mesh partition number. Secondly, multi-segment discrete mesh functions as barrier functions for numerical solution of the singular component are constructed. The barrier functions are increased monotonically at the same rate in the same segment, while they change at different rate in every transition points. The barrier functions are different in different situation. Thirdly, the estimation of the truncation error, especially in transition points, is different from Shishkin's traditional scheme.

Throughout this article $C, C_1, C_2, \ldots, c_1, c_2, \ldots$ denote positive constants that may take different values in different formulas, but that are always independent of N, M and ε.

2 Partial differential equation

Consider the parabolic initial-boundary value problem P_ε

$$\begin{cases} L_\varepsilon u(x,t) = \varepsilon u_{xx} + a(x,t)u_x - d(x,t)u_t = f(x,t)\,, & (x,t)\in\Omega\,, \\ u(x,0) = s(x)\,, & (x,0)\in S_x\,, \\ u(0,t) = q_0(t)\,, & (0,t)\in S_0\,, \\ u(1,t) = q_1(t)\,, & (1,t)\in S_1\,, \end{cases} \tag{1}$$

where $\Omega = (0,1)\times(0,1]\,$, $\bar\Omega = [0,1]\times[0,1]\,$, $S_x = \{(x,0)\mid 0\le x\le 1\}\,$, $S_0 = \{(0,t)\mid 0\le t\le 1\}\,$, $S_1 = \{(1,t)\mid 0\le t\le 1\}\,$, $a(x,t), d(x,t), f(x,t)$ are sufficiently smooth functions and satisfy the compatibility conditions [7] and also the following conditions

$$a(x,t) > \alpha > 0\,, \quad d(x,t) > \beta > 0\,, \tag{2}$$

ε is a sufficiently small parameter, without loss of generality, we let

$$0 < \varepsilon \ll \frac{\alpha}{4}\,. \tag{3}$$

Similar to the discussion in [6, 7], we have the following lemma.

Lemma 1 *Under certain smoothness and compatibility conditions on the data $a(x,t), d(x,t), f(x,t)$, the solution of P_ε has the decomposition $u(x,t) = v(x,t) + w(x,t)\,$, where $v(x,t)$ is the smooth component and $w(x,t)$ is the singular component. For any integers i,j, satisfying $0\le i+j\le 3\,$, $v(x,t)$ and $w(x,t)$ satisfy*

$$\left|\frac{\partial^{i+j} v}{\partial x^i \partial t^j}(x,t)\right| \le C(1+\varepsilon^{2-i}), \tag{4}$$

$$\left|\frac{\partial^{i+j} w}{\partial x^i \partial t^j}(x,t)\right| \le C\varepsilon^{-i} e^{-\alpha x/\varepsilon}. \tag{5}$$

3 New techniques of mesh partition

We consider the non-equidistant mesh partition in x direction $0 = x_0 < x_1 < \cdots < x_N = 1$, where we set $h_i = x_i - x_{i-1}$, $1 \le i \le N$; $\bar{h}_i = \frac{h_i + h_{i+1}}{2}$, $1 \le i \le N-1$. We consider the equidistant mesh partition in t direction $0 = t_0 < t_1 < \cdots < t_M = 1$, where we set $\tau = 1/M$, $1 \le i \le M$. For all $N = 2^m \in [16, 3.7 \times 10^6]$, where $m \ge 4$, satisfy

$$\begin{cases} e^e < N < e^{e^e}\,, \\ e < \log N < e^e\,, \\ 1 < \log\log N < e\,, \\ 0 < \log\log\log N < 1\,. \end{cases} \tag{6}$$

Consider the following three-transition points in x direction:

$$\begin{cases} \tau_3 = \frac{\varepsilon \log\log\log N}{\alpha}\,, \\ \tau_2 = \tau_3 + \min\{\frac{\varepsilon \log\log N}{\alpha}, \frac{1}{4}\}\,, \\ \tau_1 = \tau_2 + \min\{\frac{\varepsilon \log N}{\alpha}, \frac{1}{4}\}\,. \end{cases} \tag{7}$$

Note: τ_3 is always a transition point under the condition (3).

The mesh partition is depended on the following three cases.

Case 1: $\frac{\varepsilon \log N}{\alpha} < \frac{1}{4}$.

There are three transition points: $\tau_3 = (\varepsilon \log\log\log N)/\alpha$, $\tau_2 = \tau_3 + (\varepsilon \log\log N)/\alpha$, $\tau_1 = \tau_2 + (\varepsilon \log N)/\alpha$ (see Figure 1). Therefore the domain $(0, 1]$ is subdivided into four subintervals: $I_1 = (0, \tau_3]$, $I_2 = (\tau_3, \tau_2]$, $I_3 = (\tau_2, \tau_1]$, $I_4 = (\tau_1, 1]$.

There are $N/4$ mesh points in four subintervals respectively. In each interval we use equidistant mesh (see Figure 2), so we let $h_1 = h_2 = \cdots = h_{N/4} = H_4$, $h_{N/4+1} = h_{N/4+2} = \cdots = h_{N/2} = H_3$, $h_{N/2+1} = h_{N/2+2} = \cdots = h_{3N/4} = H_2$, $h_{3N/4+1} = h_{3N/4+2} = \cdots = h_N = H_1$.

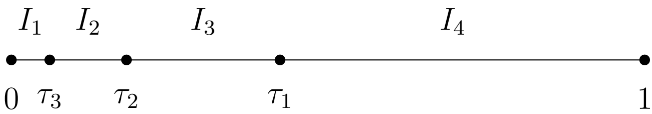

FIGURE 1: Three-transition points in Case 1

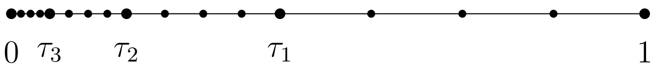

FIGURE 2: Piecewise Uniform Mesh Partition with $N = 16$ for Case 1

Case 2: $\frac{\varepsilon \log\log N}{\alpha} < \frac{1}{4}$ and $\frac{\varepsilon \log N}{\alpha} \geq \frac{1}{4}$.

Note that $\varepsilon^{-1} \leq (4\log N)/\alpha$. There are three transition points: $\tau_3 = (\varepsilon \log\log\log N)/\alpha$, $\tau_2 = \tau_3 + (\varepsilon \log\log N)/\alpha$, $\tau_1 = \tau_2 + \frac{1}{4}$. Therefore the domain $(0, 1]$ is subdivided into four subintervals: $I_1 = (0, \tau_3]$, $I_2 = (\tau_3, \tau_2]$, $I_3 = (\tau_2, \tau_1]$, $I_4 = (\tau_1, 1]$.

There are $N/4$ mesh points in four subintervals respectively. In each interval we use equidistant mesh, so we can let $h_1 = h_2 = \cdots = h_{N/4} = H_4$, $h_{N/4+1} = h_{N/4+2} = \cdots = h_{N/2} = H_3$, $h_{N/2+1} = h_{N/2+2} = \cdots = h_{3N/4} = H_2$, $h_{3N/4+1} = h_{3N/4+2} = \cdots = h_N = H_1$.

Case 3: $\frac{\varepsilon \log\log N}{\alpha} \geq \frac{1}{4}$.

Note that $\varepsilon^{-1} \leq (4\log\log N)/\alpha$. There are two transition points: $\tau_3 = (\varepsilon \log\log\log N)/\alpha, \tau_2 = \tau_3 + \frac{1}{4}$. Therefore the domain $(0, 1]$ is subdivided into three subintervals: $I_1 = (0, \tau_3]$, $I_2 = (\tau_3, \tau_2]$, $I_4 = (\tau_2, 1]$.

There are $N/2$ mesh points in I_4 and $N/4$ mesh points in I_1, I_2 respectively. In each interval we use equidistant mesh, so we can let $h_1 = h_2 = \cdots = h_{N/4} = H_4$, $h_{N/4+1} = h_{N/4+2} = \cdots = h_{N/2} = H_3$, $h_{N/2+1} = h_{N/2+2} = \cdots = h_N = H_1$.

4 Difference equation

Multi-transition points difference scheme is constructed as Problem $P_\varepsilon^{N,M}$

$$\begin{cases} L_\varepsilon^{N,M}U(x_i,t_j) &= \varepsilon\delta_x^2U(x_i,t_j)+a(x_i,t_j)D_x^+U(x_i,t_j)-d(x_i,t_j)D_t^-U(x_i,t_j) \\ &= f(x_i,t_j)\,, \quad (x_i,t_j)\in\Omega^{N,M}\,, \\ U(x_i,0) &= s(x_i)\,, \quad (x_i,0)\in\Gamma^N\,, \\ U(0,t_j) &= q_0(t_j)\,, \quad (0,t_j)\in\Gamma_0^M\,, \\ U(1,t_j) &= q_1(t_j)\,, \quad (1,t_j)\in\Gamma_1^M\,, \end{cases} \tag{8}$$

where

$$\begin{aligned}
D_x^+U(x_i,t_j) &= \frac{U(x_{i+1},t_j)-U(x_i)}{h_{i+1}}\,,\\
D_x^-U(x_i,t_j) &= \frac{U(x_i,t_j)-U(x_{i-1},t_j)}{h_i}\,,\\
\delta_x^2U(x_i,t_j) &= \frac{D_x^+U(x_i,t_j)-D_x^-U(x_i,,t_j)}{\bar h_i}\,,\\
D_t^-U(x_i,t_j) &= \frac{U(x_i,t_j)-U(x_i,t_{j-1})}{\tau}\,,\\
\bar\Omega^N &= \{x_i\}_{i=0}^N\,,\\
\Omega^N &= \{x_i\}_{i=1}^{N-1}\,,\\
\bar\Omega^M &= \{t_j\}_{j=0}^M\,,\\
\Omega^M &= \{t_j\}_{j=1}^M\,,\\
\bar\Omega^{N,M} &= \bar\Omega^N\times\bar\Omega^M\,,\\
\Omega^{N,M} &= \Omega^N\times\Omega^M\,,\\
\Gamma^N &= \{(x_i,0)\mid x_i\in\bar\Omega^N\}\,,\\
\Gamma_0^M &= \{(0,t_j)\mid t_j\in\Omega^M\}\,,\\
\Gamma_1^M &= \{(1,t_j)\mid t_j\in\Omega^M\}\,,\\
\Gamma^{N,M} &= \Gamma^N\cup\Gamma_0^M\cup\Gamma_1^M\,.
\end{aligned}$$

Analogous to that argument in [6, 7], it is easy to prove the following discrete minimum principle and uniform stability result.

Lemma 2 *Suppose that a mesh function $U(x_i,t_j)$ satisfy*

$$\begin{cases} L_\varepsilon^{N,M}U(x_i,t_j) & \leq 0\,, \quad (x_i,t_j)\in\Omega^{N,M}\,, \\ U(x_i,t_j) & \geq 0\,, \quad (x_i,t_j)\in\Gamma^{N,M}\,, \end{cases} \tag{9}$$

then $U(x_i,t_j)\geq 0$ holds for all $(x_i,t_j)\in\bar{\Omega}^{N,M}$.

Lemma 3 *The solution $U(x_i,t_j)$ of difference equation $P_\varepsilon^{N,M}$ for all $(x_i,t_j)\in\bar{\Omega}^{N,M}$ satisfies*

$$|U(x_i,t_j)| \leq C_1 \max_{\Omega^{N,M}} \left|L_\varepsilon^{N,M}U(x_i,t_j)\right| + C_2 \max_{\Gamma^{N,M}} |U(x_i,t_j)|\,. \tag{10}$$

The solution of difference equation $P_\varepsilon^{N,M}$ has the decomposition

$$U(x_i,t_j) = V(x_i,t_j) + W(x_i,t_j)\,, \tag{11}$$

where $V(x_i,t_j)$ and $W(x_i,t_j)$ satisfy

$$\begin{cases} L_\varepsilon^{N,M}V(x_i,t_j) & = L_\varepsilon v(x_i,t_j)\,, & (x_i,t_j)\in\Omega^{N,M}\,, \\ V(x_i,0) & = v(x_i,0)\,, & (x_i,0)\in\Gamma^{N}\,, \\ V(0,t_j) & = v(0,t_j)\,, & (0,t_j)\in\Gamma_0^{M}\,, \\ V(1,t_j) & = v(1,t_j)\,, & (1,t_j)\in\Gamma_1^{M}\,, \end{cases} \tag{12}$$

and

$$\begin{cases} L_\varepsilon^{N,M}W(x_i,t_j) & = L_\varepsilon w(x_i,t_j)\,, & (x_i,t_j)\in\Omega^{N,M}\,, \\ W(x_i,0) & = w(x_i,0)\,, & (x_i,0)\in\Gamma^{N}\,, \\ W(0,t_j) & = w(0,t_j)\,, & (0,t_j)\in\Gamma_0^{M}\,, \\ W(1,t_j) & = w(1,t_j)\,, & (1,t_j)\in\Gamma_1^{M}\,. \end{cases} \tag{13}$$

5 ε-uniform convergence

Applying Lemma 3, it is easy to prove the following lemma.

Lemma 4 *At each mesh point $(x_i, t_j) \in \bar{\Omega}^{N,M}$, the error in the smooth component satisfy the ε-uniform error bound*

$$|V(x_i, t_j) - v(x_i, t_j)| \le C_1 N^{-1} + C_2 M^{-1}\,, \tag{14}$$

where $V(x_i, t_j)$ is the solution of (12) and $v(x, t)$ is the smooth component in Lemma 1.

Lemma 5 *At each mesh point $(x_i, t_j) \in \bar{\Omega}^{N,M}$, the error in the singular component satisfy the ε-uniform error bound*

$$|W(x_i, t_j) - w(x_i, t_j)| \le C(N^{-1} + M^{-1})\,, \tag{15}$$

where $W(x_i, t_j)$ is the solution of (13) and $w(x, t)$ is the singular component in Lemma 1

Proof: The discussion is separately into the following three cases.

Case 1: $\frac{\varepsilon \log N}{\alpha} < \frac{1}{4}$.

Firstly we consider the region $\{(x_i, t_j) \mid x_i \in [\tau_1, 1], t_j \in \bar{\Omega}^M\}$. Introducing the mesh function in one dimension $P(x_i)$, which is the solution on the constant coefficient finite difference equation

$$\begin{cases} \varepsilon\delta^2 P(x_i) + \alpha D^+ P(x_i) = 0, x_i \in \Omega^N\,, \\ P(0) = 1\,, \quad P(1) = 0\,, \end{cases} \tag{16}$$

where

$$D^+P(x_i) = \frac{P(x_{i+1}) - P(x_i)}{h_{i+1}}\,,$$
$$D^-P(x_i) = \frac{P(x_i) - P(x_{i-1})}{h_i}\,,$$
$$\delta^2 P(x_i) = \frac{D^+P(x_i) - D^-P(x_i)}{\bar{h}_i}\,.$$

It can be proved as in [2, 3] that

$$D^+P(x_i) \le 0\,, \quad 0 \le i < N\,, \tag{17}$$
$$0 \le P(\tau_1) \le CN^{-1}\,. \tag{18}$$

Consider the barrier function

$$\Psi^{\pm}(x_i, t_j) = C_1 P(x_i) + C_2 t_j M^{-1} \pm W(x_i, t_j)\,. \tag{19}$$

Choosing C_1, C_2 large enough, applying Lemma 3, leads to

$$|W(x_i, t_j)| \le C(N^{-1} + M^{-1}) \quad \text{in } \{(x_i, t_j) \mid x_i \in [\tau_1, 1], t_j \in \bar{\Omega}^M\}. \tag{20}$$

Lemma 1 leads to

$$|w(x_i, t_j)| \le C(N^{-1} + M^{-1}) \quad \text{in } \{(x_i, t_j) \mid x_i \in [\tau_1, 1], t_j \in \bar{\Omega}^M\}. \tag{21}$$

Therefore in $\{(x_i, t_j) \mid x_i \in [\tau_1, 1], t_j \in \bar{\Omega}^M\}$

$$|W(x_i, t_j) - w(x_i, t_j)| \le C(N^{-1} + M^{-1}),. \tag{22}$$

Secondly, consider the result in $\{(x_i, t_j) \mid x_i \in [0, \tau_1], t_j \in \bar{\Omega}^M\}$. Now we introduce the mesh function in one dimension

$$Q(x_i) = \begin{cases} c_4 \frac{\tau_3}{N\varepsilon} \varphi_4(x_i) + Q(\tau_3)\,, & 0 \le i \le \frac{N}{4}\,, \\ c_3 \frac{\tau_2 - \tau_3}{N\varepsilon} e^{-\alpha\tau_3/\varepsilon} \varphi_3(x_i) + Q(\tau_2)\,, & \frac{N}{4} \le i \le \frac{N}{2}\,, \\ c_2 \frac{\tau_1 - \tau_2}{N\varepsilon} e^{-\alpha\tau_2/\varepsilon} \varphi_2(x_i)\,, & \frac{N}{2} \le i \le \frac{3N}{4}, \end{cases} \tag{23}$$

where the coefficient c_2, c_3 and c_4 are large constant, $\varphi_j(x_i)$, $2 \le j \le 4$, are defined by

$$\begin{cases} \varphi_4(x_0) = 1\,, & \varphi_4(\tau_3) = 0\,, & \varphi_4(x_i) = \frac{\lambda_4^{N/4-i}-1}{\lambda_4^{-N/4}-1}\,, & 0 \le i \le \frac{N}{4}\,, \\ \varphi_3(\tau_3) = 1\,, & \varphi_3(\tau_2) = 0\,, & \varphi_3(x_i) = \frac{\lambda_3^{N/2-i}-1}{\lambda_3^{N/4}-1}\,, & \frac{N}{4} \le i \le \frac{N}{2}\,, \\ \varphi_2(\tau_2) = 1\,, & \varphi_2(\tau_1) = 0\,. & \varphi_2(x_i) = \frac{\lambda_2^{3N/4-i}-1}{\lambda_2^{N/4}-1}\,, & \frac{N}{2} \le i \le \frac{3N}{4}\,, \end{cases} \tag{24}$$

where

$$\begin{cases} \lambda_4 &= 1 + \frac{\alpha H_4}{\varepsilon}\,, \\ \lambda_3 &= 1 + \frac{\alpha H_3}{\varepsilon}\,, \\ \lambda_2 &= 1 + \frac{\alpha H_2}{\varepsilon}\,, \\ \lambda_1 &= 1 + \frac{\alpha H_1}{\varepsilon}\,. \end{cases} \tag{25}$$

It is easy to prove

$$|Q(x_i)| \le CN^{-1}. \tag{26}$$

At mesh points $\{(x_i, t_j) \mid x_i \in [0, \tau_1], t_j \in \bar{\Omega}^M\}$, consider the barrier function

$$\Phi^{\pm}(x_i, t_j) \;=\; C_1 Q(x_i) + C_2 N^{-1} + C_3 t_j M^{-1} \pm (W(x_i, t_j) - w(x_i, t_j))\,, \tag{27}$$

where C_i, $i \in [1, 3]$, are large constant. Choosing C_i, $i \in [1, 3]$, large enough, applying Lemma 3, we have

$$|W(x_i, t_j) - w(x_i, t_j)| \le C(N^{-1} + M^{-1}), \tag{28}$$

holds for $\{(x_i, t_j) \mid x_i \in [0, \tau_1], t_j \in \bar{\Omega}^M\}$.

Combining (22) and (28) yields

$$|W(x_i, t_j) - w(x_i, t_j)| \le C(N^{-1} + M^{-1})\,, \quad (x_i, t_j) \in \bar{\Omega}^{N,M} \tag{29}$$

Case 2: $\frac{\varepsilon \log\log N}{\alpha} < \frac{1}{4}$ and $\frac{\varepsilon \log N}{\alpha} \geq \frac{1}{4}$.

Introduce the mesh function in one dimension $R(x_i)$,

$$R(x_i) = \begin{cases} c_4 \frac{\tau_3}{N\varepsilon}\psi_4(x_i) + R(\tau_3)\,, & 0 \leq i \leq \frac{N}{4}\,, \\ c_3 \frac{\tau_2-\tau_3}{N\varepsilon} e^{-\alpha\tau_3/\varepsilon}\psi_3(x_i) + R(\tau_2)\,, & \frac{N}{4} \leq i \leq \frac{N}{2}\,, \\ c_2 \frac{\log N}{N} e^{-\alpha\tau_2/\varepsilon}\psi_2(x_i) + R(\tau_1)\,, & \frac{N}{2} \leq i \leq \frac{3N}{4}\,, \\ c_1 \frac{1}{N}\psi_1(x_i)\,, & \frac{3N}{4} \leq i \leq N\,, \end{cases} \tag{30}$$

where c_j, $j \in [1,4]$, are large constant; $\psi_j(x_i)$, $1 \leq j \leq 4$, are defined by

$$\begin{cases} \psi_4(x_0) = 1\,, \quad \psi_4(\tau_3) = 0\,, \quad \psi_4(x_i) = \frac{\lambda_4^{N/4-i}-1}{\lambda_4^{N/4}-1}\,, & 0 \leq i \leq \frac{N}{4}\,, \\ \psi_3(\tau_3) = 1\,, \quad \psi_3(\tau_2) = 0\,, \quad \psi_3(x_i) = \frac{\lambda_3^{N/2-i}-1}{\lambda_3^{N/4}-1}\,, & \frac{N}{4} \leq i \leq \frac{N}{2}\,, \\ \psi_2(\tau_2) = 1\,, \quad \psi_2(\tau_1) = 0\,, \quad \psi_2(x_i) = \frac{\lambda_2^{3N/4-i}-1}{\lambda_2^{N/4}-1}\,, & \frac{N}{2} \leq i \leq \frac{3N}{4}\,, \\ \qquad\qquad\qquad\qquad\qquad\quad \psi_1(x_i) = 1 - x_i\,, & \frac{3N}{4} \leq i \leq N\,. \end{cases} \tag{31}$$

It is easy to prove

$$|R(x_i)| \leq CN^{-1}. \tag{32}$$

At every mesh points $(x_i, t_j) \in \bar{\Omega}^{N,M}$, construct the barrier function

$$\Theta^{\pm}(x_i, t_j) \;=\; C_1 Q(x_i) + C_2 t_j M^{-1} \pm (W(x_i, t_j) - w(x_i, t_j))\,, \tag{33}$$

where C_1, C_2 are large constant. Choosing $C_j, j \in [1, 2]$, large enough, applying Lemma 3, we have

$$|W(x_i, t_j) - w(x_i, t_j)| \leq C(N^{-1} + M^{-1})\,, \quad (x_i, t_j) \in \bar{\Omega}^{N,M}. \tag{34}$$

Case 3: $\frac{\varepsilon \log\log N}{\alpha} \geq \frac{1}{4}$.

Similar to the discussion in Case 1 and Case 2, thus we have the same results as Case 1 and Case 2. Lemma 5 is proved. ♠

Combining Lemma 4 and Lemma 5, we obtain the main theorem.

Theorem 6 *Let $u(x,t)$ be the solution of P_ε and $U(x_i,t_j)$ be the solution of $(P_\varepsilon^{N,M})$, then*

$$|W(x_i,t_j)-w(x_i,t_j)|\leq CN^{-1}+CM^{-1}, \tag{35}$$

holds at each mesh points $(x_i,t_j)\in\bar{\Omega}^{N,M}$.

6 Numerical results

Considering the particular problem from the problem class P_ε

$$\begin{cases} L_\varepsilon u(x,t)=\varepsilon u_{xx}+2u_x-u_t=e^{-t}\,, & (x,t)\in\Omega\,,\\ u(x,0)=\frac{e^{-2x/\varepsilon}-e^{-2/\varepsilon}}{1-e^{-2/\varepsilon}}\,, & (x,0)\in S_x\,,\\ u(0,t)=1+e^{-t}\,, & (0,t)\in S_0\,,\\ u(1,t)=e^{-t}\,, & (1,t)\in S_1\,. \end{cases} \tag{36}$$

The exact solution is

$$u(x,t)=\frac{e^{-2x/\varepsilon}-e^{-2/\varepsilon}}{1-e^{-2/\varepsilon}}+e^{-t}\,.$$

The maximum pointwise errors are defined by

$$E_\varepsilon^{N,M}=\max_{\bar{\Omega}_\varepsilon^{N,M}}|u(x_i,t_j)-U(x_i,t_j)|\,,\quad E^{N,M}=\max_{\varepsilon\in[1,2^{-36}]}E_\varepsilon^{N,M}. \tag{37}$$

We solve this problem using Shishkin's scheme, Bakhvalov–Shishkin's scheme (BS) and Multi-transition points scheme (MTP) respectively. The maximum pointwise errors are shown in Table 1 which shows that all three methods converge to the exact solution, but Multi-transition points scheme is better than Shishkin's scheme. Our method is a computationally efficient method. It is more useful in applications.

Table 1: Comparison of the maximum pointwise errors $E^{N,M}$.

Scheme	Number of Intervals $N = M$			
	32	64	128	256
Shishkin	0.06516	0.03591	0.01965	0.01071
BS	0.06339	0.03241	0.01624	0.00810
MTP	0.05748	0.02847	0.01434	0.00727

7 Conclusion: compare the three difference schemes

In this section, Shishkin's scheme and Bakhvalov–Shishkin's scheme are compared with the new method.

Shishkin [6] proposed a simple transition point in solving large Reynolds number problem. Assume that we have an exponential boundary layer at $x = 0$, so the boundary layer function is $e^{-\alpha x/\varepsilon}$, for some fixed α. Shishkin's idea is to choose transition point $\tau = \min\{(\varepsilon \log N)/\alpha, \frac{1}{2}\}$. The piecewise-uniform meshes are

$$\bar{\Omega}_i = \left\{ x_i \mid x_i = \frac{2i\tau}{N}, i \leq \frac{N}{2};\ x_i = x_{i-1} + \frac{2(1-\tau)}{N}, \frac{N}{2} < i \leq N \right\}$$

Shishkin's scheme becomes uniform mesh when $\varepsilon \log(N)/\alpha \geq 1/2$. This piecewise-uniform mesh is only slightly more complex than a uniform mesh, because it is simply two uniform meshes glued together at a carefully chosen transition point. But the same problems as in [2] arise, and thus it cannot achieve a convergence rate of order $\mathcal{O}(N^{-1})$.

In order to improve the convergence rate, Bakhvalov–Shishkin's scheme is presented [5]. The original Bakhvalov mesh requires the solution of a nonlinear equation to determine the transition point where the mesh switches from coarse to fine. Instead Bakhvalov–Shishkin's scheme fix the transition

Table 2: Comparison of three difference schemes.

Scheme	Condition	Equidistant	Compute x_i	Convergence Rate
Shishkin	NO	$\frac{\varepsilon \log N}{\alpha} \geq \frac{1}{2}$	Simple	$\mathcal{O}(N^{-1}\log^2 N + M^{-1})$
BS	$\varepsilon \leq N^{-1}$	NO	Complicated	$\mathcal{O}(N^{-1} + M^{-1})$
MTP	NO	NO	Simple	$\mathcal{O}(N^{-1} + M^{-1})$

points as in the Shishkin's mesh. Bakhvalov–Shishkin's scheme introduces $\tau = \min\{(\varepsilon \log N)/\alpha, \frac{1}{2}\}$. So assume that

$$\varepsilon \leq N^{-1}. \tag{38}$$

Now the interval $[\tau, 1]$ is uniformly dissected into $N/2$ subintervals, while $[0, \tau]$ is partitioned into the same number of mesh intervals by inverting the function $e^{-\alpha x/\varepsilon}$. It specifies the x_i, $0 \leq i \leq \frac{N}{2}$ so that $e^{-\alpha x/\varepsilon}$ is a linear function in i, this leads to

$$x_i = \begin{cases} -\frac{2\varepsilon}{\alpha}\log\left(1 - \frac{2(N-1)i}{N^2}\right), & 0 \leq i \leq \frac{N}{2}, \\ \tau + \frac{2(1-\tau)i}{N}, & \frac{N}{2} \leq i \leq N, \end{cases} \tag{39}$$

Note: Bakhvalov–Shishkin's scheme is first-order convergent uniformly with respect to large Reynolds number, which is more accurate than Shishkin's original piecewise uniform scheme and is the same accuracy as our new method. However, it is more difficult to determine the mesh partition (39). The computational procedure is more complicated than our scheme. Another default is that it assumes condition (38). Our method has two advantages than Bakhvalov–Shishkin's scheme.

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