Monotone iterates for systems of nonlinear integro-elliptic equations

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Abstract

This paper deals with numerically solving systems of nonlinear integro-elliptic equations. We give a monotone iterative method, based on the method of upper and lower solutions. The construction of the initial upper and lower solution is discussed, and numerical experiments are presented.

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Keywords: integro-elliptic equations; monotone iterates; initial upper and lower solutions

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Contents

1 Introduction

Various reaction-diffusion-convection problems in the chemical, physical and engineering sciences are described by coupled systems of nonlinear integro-elliptic equations [2]. We give a numerical treatment for the system of two nonlinear integro-elliptic equations in the form

$$\begin{aligned} &-L_i u_i + f_i(x,u) + \int_\omega g_i^*(x,s,u(s))\,ds = 0\,, \quad x \in \omega\,, \\ &u_i(x) = \phi_i(x)\,, \quad x \in \partial\omega\,, \end{aligned} \tag{1}$$

where $u = (u_1, u_2)$, ω is a connected bounded domain in $\mathbb{R}^\kappa$ $(\kappa = 1, 2, \ldots)$ with boundary $\partial\omega$, and throughout, unless otherwise stated, $i = 1, 2$. The differential terms $L_i u_i$ are given by

$$L_i u_i = \sum_{\alpha=1}^{\kappa} \frac{\partial}{\partial x_\alpha}\left(D_i(x)\frac{\partial u_i}{\partial x_\alpha}\right) + \sum_{\alpha=1}^{\kappa} v_{i,\alpha}(x)\frac{\partial u_i}{\partial x_\alpha}\,,$$

where the coefficients of the differential operators are smooth and $D_i > 0$, in $\overline{\omega} = \omega \cup \partial\omega$. It is also assumed that the functions f_i, g_i^* and ϕ_i are smooth in their respective domains.

Pao [3] gave monotone iterative methods for numerical solutions of scalar nonlinear integro-elliptic boundary problems. Previously [1] I presented a monotone iterative method for solving systems of nonlinear integro-parabolic equations of Volterra type. In this paper, we apply and investigate the monotone approach of Boglaev [1] for solving the system of two nonlinear integro-elliptic equations (1). Our iterative scheme is based on the method of upper and lower solutions and associated monotone iterates. We formulate a nonlinear difference scheme for the numerical solution of (1) in Subsection 2.1. A monotone iterative method for the nonlinear difference scheme is given in Subsection 2.2. This requires the construction of initial upper and lower solutions, which are discussed in Subsection 2.3. Finally, the results of numerical experiments are presented in Section 3.

2 The monotone iterative method

2.1 The nonlinear difference scheme

For solving (1), we introduce a computational mesh $\bar{\omega}^h$ and consider the nonlinear difference scheme

$$\begin{aligned} &\mathcal{L}_i U_i(p) + f_i(p, U) + g_i(p, U) = 0\,, \quad p \in \omega^h\,, \\ &U_i(p) = \phi_i(p)\,, \quad p \in \partial\omega^h\,, \end{aligned} \tag{2}$$

where $U = (U_1, U_2)$ and $\partial\omega^h$ is the boundary of $\bar{\omega}^h = \omega^h \cup \partial\omega^h$. The difference operators $\mathcal{L}_i$ are defined by

$$\mathcal{L}_i U_i(p) = d_i(p) U_i(p) - \sum_{p' \in \sigma_i'(p)} a_i(p, p') U_i(p')\,,$$

where $\sigma_i'(p) = \sigma_i(p) \setminus \{p\}$, $\sigma_i(p)$ are stencils of the scheme at $p \in \omega^h$. We make the following assumptions on $\mathcal{L}_i$:

$$a_i(p, p') \geqslant 0, \quad d_i(p) \geqslant \sum_{p' \in \sigma_i'(p)} a_i(p, p')\,, \quad p \in \omega^h\,, \quad p' \in \sigma_i'(p)\,. \tag{3}$$

The integrals in (1) are approximated in (2) by the finite sums, based on a composite Newton–Cotes quadrature rule [5]

$$g_i(p, U) = \sum_{l=1}^{N} b_l g_i^*(p, p_l, U(p_l)), \qquad (4)$$

where b_l, $l = 1, \dots, N$, are nonnegative weights, and N is the number of mesh points $p \in \bar{\omega}^h$.

We also assume that the mesh $\bar{\omega}^h$ is connected. This means that for two interior mesh points $\tilde{p}$ and $\hat{p}$, there exists a finite set of interior mesh points $\{p_1, p_2, \dots, p_s\}$ such that

$$p_1 \in \sigma'(\tilde{p}), \ p_2 \in \sigma'(p_1), \dots, \ p_s \in \sigma'(p_{s-1}), \ \hat{p} \in \sigma'(p_s). \qquad (5)$$

We now formulate the discrete maximum principle.

Lemma 1. *Let the coefficients of the difference operators $\mathcal{L}_i$ satisfy (3) and the mesh $\bar{\omega}^h$ be connected as defined by (5). If mesh functions $W_i(p)$ satisfy the conditions*

$$(\mathcal{L}_i + \bar{c}_i)W_i(p) \geqslant 0, \quad p \in \omega^h, \quad W_i(p) \geqslant 0, \quad p \in \partial\omega^h,$$

where $\bar{c}_i(p) \geqslant 0$, then $W_i(p) \geqslant 0$ in $\bar{\omega}^h$.

Samarskii [4] proved this lemma. Since the problems are linear, the same results apply for $-W_i(p)$.

2.2 The iterative method

Two vector mesh functions $\widetilde{U}(p) = (\widetilde{U}_1(p), \widetilde{U}_2(p))$ and $\widehat{U}(p) = (\widehat{U}_1(p), \widehat{U}_2(p))$ are called ordered upper and lower solutions of (2), if they satisfy the relation

$\widetilde{U}(p) \geqslant \widehat{U}(p)$, $p \in \bar{\omega}^h$ and the inequalities

$$
\begin{aligned}
&\mathcal{L}_i \widehat{U}_i(p) + f_i(p, \widehat{U}) + g_i(p, \widehat{U}) \leqslant 0 \\
&\qquad\qquad \leqslant \mathcal{L}_i \widetilde{U}_i(p) + f_i(p, \widetilde{U}) + g_i(p, \widetilde{U}) \,, \quad p \in \omega^h \,, \qquad (6) \\
&\widehat{U}_i(p) \leqslant \phi_i(p) \leqslant \widetilde{U}_i(p) \,, \quad p \in \partial\omega^h \,.
\end{aligned}
$$

For a given pair of ordered upper and lower solutions $\widetilde{U}$ and $\widehat{U}$, we define the sector $\langle \widehat{U}, \widetilde{U} \rangle = \{U(p) : \widehat{U}(p) \leqslant U(p) \leqslant \widetilde{U}(p) \,, \quad p \in \bar{\omega}^h\}$. We assume that on $\langle \widehat{U}, \widetilde{U} \rangle$, f_i and g_i^* satisfy the constraints

$$
\begin{cases}
\dfrac{\partial f_i}{\partial u_i}(p, U) \leqslant c_i(p) \,, \quad \dfrac{\partial g_i^*}{\partial u_i}(p, p_l, U(p_l)) \leqslant 0 \,, \quad 1 \leqslant l \leqslant N \,, \\
\dfrac{\partial f_i}{\partial u_{i'}}(p, U) \leqslant 0 \,, \quad \dfrac{\partial g_i^*}{\partial u_{i'}}(p, p_l, U(p_l)) \leqslant 0 \,, \quad 1 \leqslant l \leqslant N \,, \quad i \neq i' \,,
\end{cases} \qquad (7)
$$

where $c_i(p)$ are nonnegative bounded functions in $\bar{\omega}^h$.

We now construct an iterative method for solving (2) in the following way:

$$
\begin{cases}
(\mathcal{L}_i + c_i) Z_i^{(n)}(p) = -\mathcal{R}_i(p, U^{(n-1)}) \,, \quad p \in \omega^h \,, \\
Z_i^{(n)}(p) = U_i^{(n)}(p) - U_i^{(n-1)}(p) \,, \quad p \in \bar{\omega}^h \,, \\
Z_i^{(1)}(p) = \phi_i(p) - U_i^{(0)}(p) \,, \quad Z_i^{(n)}(p) = 0 \,, \quad n \geqslant 2 \,, \quad p \in \partial\omega^h \,,
\end{cases} \qquad (8)
$$

where $c_i(p)$ are defined in (7), and

$$
\mathcal{R}_i(p, U^{(n-1)}) := \mathcal{L}_i U_i^{(n-1)}(p) + f_i(p, U^{(n-1)}) + g_i(p, U^{(n-1)}) \,, \qquad (9)
$$

are the residuals of the difference scheme (2) on $U^{(n-1)}$.

We introduce the notation

$$
F_i(p, U) = c_i(p) U_i(p) - f_i(p, U) - g_i(p, U) \,, \qquad (10)
$$

and prove a monotone property of F_i.

Lemma 2. *Let (7) hold, and let U, V be any two functions in $\langle \widehat{U}, \widetilde{U} \rangle$ such that $U(p) \geqslant V(p)$. Then*

$$
F_i(p, U) \geqslant F_i(p, V) \,, \quad p \in \bar{\omega}^h \,. \qquad (11)
$$

Proof: From (10), we have

$$\begin{aligned} F_1(p,U) - F_1(p,V) &\\ = c_1(p)[U_1(p) - V_1(p)] &\\ - [f_1(p,U_1,U_2) - f_1(p,V_1,U_2)] - [f_1(p,V_1,U_2) - f_1(p,V_1,V_2)] &\\ - [g_1(p,U_1,U_2) - g_1(p,V_1,U_2)] - [g_1(p,V_1,U_2) - g_1(p,V_1,V_2)]\,. \end{aligned}$$

By applying the mean-value theorem to the last four terms and utilizing the assumptions of the lemma, we conclude (11) for $i = 1$. Similarly, we can prove (11) for $i = 2$. ♠

The following theorem establishes the monotone property of the iterative method (8).

Theorem 3. *Assume that the coefficients of the difference operators $\mathcal{L}_i$ in (2) satisfy (3) and the computational mesh $\bar{\omega}^h$ is connected (5). Let $f_i(p,U)$ and $g_i^*(p,U)$ satisfy (7), where $\widetilde{U}$ and $\widehat{U}$ are ordered upper and lower solutions of (2) which satisfy (6). Then the sequences $\{\overline{U}^{(n)}\}$, $\{\underline{U}^{(n)}\}$ generated by (8) with, respectively, $\overline{U}^{(0)} = \widetilde{U}$ and $\underline{U}^{(0)} = \widehat{U}$, are ordered upper and lower solutions to (2) and converge monotonically to their respective solutions $\overline{U}$ and $\underline{U}$,*

$$\underline{U}^{(n-1)}(p) \leqslant \underline{U}^{(n)}(p) \leqslant \underline{U}(p) \leqslant \overline{U}(p) \leqslant \overline{U}^{(n)}(p) \leqslant \overline{U}^{(n-1)}(p)\,, \quad p \in \bar{\omega}^h\,. \tag{12}$$

If U^* is any other solution in $\langle \widehat{U}, \widetilde{U} \rangle$, then

$$\underline{U}(p) \leqslant U^*(p) \leqslant \overline{U}(p)\,, \quad p \in \bar{\omega}^h\,. \tag{13}$$

Proof: Since $\overline{U}^{(0)} = \widetilde{U}$ is an upper solution, then from (6) and (8) we conclude that

$$(\mathcal{L}_i + c_i)\overline{Z}_i^{(1)}(p) \leqslant 0\,, \quad p \in \omega^h\,, \quad \overline{Z}_i^{(1)}(p) \leqslant 0\,, \quad p \in \partial\omega^h\,.$$

From Lemma 1, it follows that

$$\overline{Z}_i^{(1)}(p) \leqslant 0\,, \quad p \in \bar{\omega}^h\,. \tag{14}$$

Similarly, for a lower solution $\underline{U}^{(0)} = \widehat{U}$, we conclude that

$$\underline{Z}_i^{(1)}(p) \geqslant 0\,, \quad p \in \bar{\omega}^h\,. \tag{15}$$

From (8), in the notation $W^{(n)} = \overline{U}^{(n)} - \underline{U}^{(n)}$, we have

$$(\mathcal{L}_i + c_i)W_i^{(1)}(p) = F_i(p, \overline{U}^{(0)}) - F_i(p, \underline{U}^{(0)})\,,$$

where F_i are defined in (10). Since $\overline{U}^{(0)}(p) \geqslant \underline{U}^{(0)}(p)$, by Lemma 2, we conclude that the right hand sides in the difference equations are nonnegative. The positivity property in Lemma 1 implies $W_i^{(1)}(p) \geqslant 0$, and this leads to (12) with $n = 1$. We now prove that $\overline{U}_i^{(1)}(p)$ and $\underline{U}_i^{(1)}(p)$ are upper and lower solutions (6), respectively. By the mean-value theorem, we obtain

$$g_i(p, \overline{U}^{(1)}) - g_i(p, \overline{U}^{(0)}) = \sum_{l=1}^{N} b_l \frac{\partial g_i^*}{\partial u_i}(p_l)\overline{Z}_i^{(1)}(p_l) + \sum_{l=1}^{N} b_l \frac{\partial g_i^*}{\partial u_{i'}}(p_l)\overline{Z}_{i'}^{(1)}(p_l)\,,$$

where $i' \neq i$, and partial derivatives are calculated at intermediate points which lie between $\overline{U}^{(1)}(p_l)$ and $\overline{U}^{(0)}(p_l)$. From (9), by the mean-value theorem for $f_i(p, \overline{U}^{(1)})$, we obtain

$$\begin{aligned}\mathcal{R}_i(p, \overline{U}^{(1)}) = &-\left(c_i - \frac{\partial f_i}{\partial u_i}\right)\overline{Z}_i^{(1)}(p) + \frac{\partial f_i}{\partial u_{i'}}\overline{Z}_{i'}^{(1)}(p) \\ &+ \sum_{l=1}^{N} b_l \left(\frac{\partial g_i^*}{\partial u_i}(p_l)\overline{Z}_i^{(1)}(p_l) + \frac{\partial g_i^*}{\partial u_{i'}}(p_l)\overline{Z}_{i'}^{(1)}(p_l)\right)\,,\end{aligned} \tag{16}$$

where $i' \neq i$, and the partial derivatives are calculated at intermediate points which lie between $\overline{U}^{(1)}(p_l)$ and $\overline{U}^{(0)}(p_l)$. From here, (12) with $n = 1$, (14) and (15), it follows that the partial derivatives satisfy (7). From (7), (14) and (16), we conclude that

$$\mathcal{R}_i(p,\overline{U}^{(1)}) \geqslant 0\,, \quad p \in \omega^h\,, \quad \overline{U}_i^{(1)}(p) = \phi_i(p)\,, \quad p \in \partial\omega^h\,.$$

Thus, $\overline{U}^{(1)}(p)$ is an upper solution. Using a similar proof, $\underline{U}^{(1)}(p)$ is a lower solution. By induction on n, then $\{\overline{U}^{(n)}(p)\}$ is a monotonically decreasing sequence of upper solutions and $\{\underline{U}^{(n)}(p)\}$ is a monotonically increasing sequence of lower solutions, which satisfy (12).

For each $p \in \omega^h$, one can conclude from (12) that the monotonically decreasing sequence $\{\overline{U}^{(n)}\}$ is bounded below by any lower solution $\underline{U}^{(n)}$, $n \geqslant 0$. Therefore, the sequence is convergent, and from (8), we conclude that $\lim \overline{Z}_i^{(n)}(p) = 0$, $p \in \bar{\omega}^h$, as $n \to \infty$. Now by linearity of the operators $\mathcal{L}_i$ and the continuity of f_i and g_i, we have also from (8) that the pair of the mesh functions $\overline{U}_i$ defined by $\overline{U}_i(p) = \lim \overline{U}^{(n)}(p)$, $p \in \bar{\omega}^h$, as $n \to \infty$, is an exact solution to (2). With a similar argument, the monotonically increasing sequence $\{\underline{U}^{(n)}\}$ converges to an exact solution $\underline{U}_i(p)$, $p \in \bar{\omega}^h$ to (2).

To show (13), we consider $\widetilde{U}$ and U^* as ordered upper and lower solutions. Since the sequence $\{\underline{U}^{(n)}\} = \{U^*\}$ consists of the single element U^* for all n, then from (12), we conclude that $\overline{U}(p) \geqslant U^*(p)$, $p \in \bar{\omega}^h$. Similarly, we can prove the left inequality in (13). Thus, we prove the theorem. ♠

2.3 Construction of initial upper and lower solutions

Here, we give some conditions on functions f_i and g_i^* to guarantee the existence of upper $\widetilde{U}$ and lower $\widehat{U}$ solutions, which are used as the initial iterations in the monotone iterative method (8).

Bounded functions Let functions f_i, g_i^* and ϕ_i from (1) satisfy the following conditions:

$$\begin{cases} f_i(x,0) \leqslant 0\,, & g_i^*(x,s,0) \leqslant 0\,, \quad \phi_i(x) \geqslant 0\,, \\ f_i(x,u) \geqslant -\chi_i\,, & g_i^*(x,s,u) \geqslant -\nu_i\,, \quad u_i \geqslant 0\,, \end{cases} \tag{17}$$

where χ_i and ν_i are positive constants.

From here and (6), it follows that the functions

$$\widehat{U}_i(p) = 0\,, \quad p \in \bar{\omega}^h\,, \tag{18}$$

are lower solutions of (2).

We introduce the linear problems

$$\mathcal{L}_i \widetilde{U}_i(p) = \chi_i + \nu_i \|\omega\|\,, \quad p \in \omega^h\,, \quad \widetilde{U}_i(p) = \phi_i(p)\,, \quad p \in \partial\omega^h\,, \tag{19}$$

where $\|\omega\|$ is the volume of the domain ω.

Lemma 4. *Let the conditions in (17) be satisfied. Then $\widehat{U}$ and $\widetilde{U}$ from, respectively, (18) and (19), are ordered lower and upper solutions to (2), such that*

$$0 \leqslant \widehat{U}_i(p) \leqslant \widetilde{U}_i(p)\,, \quad p \in \bar{\omega}^h\,. \tag{20}$$

Proof: From (17–19), by the maximum principle in Lemma 1, we conclude (20). We now show that $\widetilde{U}$ is an upper solution (6) to (2). From (6), (8), (17) and (19), we have

$$\mathcal{R}_i(p,\widetilde{U}) = [\chi_i + f_i(p,\widetilde{U})] + [\nu_i\|\omega\| + g_i(p,\widetilde{U})]\,,$$

where $p \in \omega^h$. Taking into account that the weights b_l, $1 \leqslant l \leqslant N$, in (4) satisfy $\sum_1^N b_l = \|\omega\|$ (Stroud [5] gave details), from (17), we conclude that $\mathcal{R}_i(p,\widetilde{U}) \geqslant 0$. From here and (20), we conclude that $\widehat{U}$ and $\widetilde{U}$ from, respectively, (18) and (19), are ordered lower and upper solutions to (2). ♠

Constant upper and lower solutions Let the functions f_i, g_i^* and ϕ_i from (1) satisfy the following conditions:

$$f_i(x,0) \leqslant 0\,, \quad g_i^*(x,s,0) \leqslant 0\,, \quad \phi_i(x) \geqslant 0\,. \tag{21}$$

It is clear that the functions from (18) are lower solutions of (2). We assume that there exist positive constants M_i such that

$$f_i(p,M) + \sum_{l=1}^{N} b_l g_i^*(p,p_l,M) \geqslant 0\,, \quad M = (M_1,M_2), \quad p \in \overline{\omega}^h\,, \tag{22}$$
$$\phi_i(p) \leqslant M_i\,, \quad p \in \partial\omega^h\,,$$

and introduce the functions

$$\widetilde{U}_i(p) = M_i\,, \quad p \in \bar{\omega}^h\,. \tag{23}$$

Lemma 5. *Let conditions (21) and (22) be satisfied. Then $\widehat{U}$ and $\widetilde{U}$ from, respectively, (18) and (23), are ordered lower and upper solutions to (2) and satisfy (20).*

Proof: The proof of the lemma repeats the proof of Lemma 4 with the following modification:

$$\mathcal{R}_i(p,\widetilde{U}) \geqslant f_i(p,M) + g_i(p,M) \geqslant 0\,, \quad p \in \omega^h\,.$$

♠

3 Numerical experiments

We consider a reaction-diffusion system with an unknown exact solution in $\bar{\omega} = \{0 \leqslant x_1 \leqslant 1, 0 \leqslant x_2 \leqslant 1\}$:

$$\begin{aligned}
&- D_1 \Delta u_1 + \zeta_1 u_1 \left(1 + e^{-u_2}\right) - \int_\omega u_2(s)\, ds = 0\,, \quad x \in \omega\,,\\
&- D_2 \Delta u_2 + \zeta_2 u_2 \left(1 + \frac{1}{1+u_1}\right) - \int_\omega u_1(s)\, ds = 0\,, \quad x \in \omega\,,\\
&\phi_i = 1 - \sin(\pi x_1)\sin(\pi x_2)\,, \quad x \in \partial\omega\,,
\end{aligned}$$

where $\Delta u = u_{x_1 x_1} + u_{x_2 x_2}$ and D_i, ζ_i are positive constants. For this test problem, we have in (1)

$$f_1 = \zeta_1 u_1 \left(1 + e^{-u_2}\right)\,, \quad f_2 = \zeta_2 u_2 \left(1 + \frac{1}{1+u_1}\right)\,, \quad g_i^* = -u_{i'}\,, \quad i \neq i'\,,$$

where for $u_i \geqslant 0$,

$$\begin{aligned}
&0 \leqslant \frac{\partial f_1}{\partial u_1} = \zeta_1 \left(1 + e^{-u_2}\right) \leqslant 2\zeta_1\,, \quad \frac{\partial f_1}{\partial u_2} = -\zeta_1 u_1 e^{-u_2} \leqslant 0\,,\\
&0 \leqslant \frac{\partial f_2}{\partial u_2} = \zeta_2 \left(1 + \frac{1}{1+u_1}\right) \leqslant 2\zeta_2\,, \quad \frac{\partial f_2}{\partial u_1} = -\frac{\zeta_2 u_2}{(1+u_1)^2} \leqslant 0\,,\\
&\frac{\partial g_i^*}{\partial u_i} = 0\,, \quad \frac{\partial g_i^*}{\partial u_{i'}} = -1\,, \quad i \neq i'\,.
\end{aligned}$$

From this, we choose $c_i = 2\zeta_i$ in the monotone iterative method (8). The conditions in (7) hold true without any extra restrictions. To guarantee (22), we assume that $\zeta_i \geqslant 1$ and choose $M_1 = M_2 = 1$. By Lemma 5, we conclude that $\widehat{U}$ and $\widetilde{U}$ from, respectively, (18) and (23), are ordered lower and upper solutions and satisfy (20).

We discretize the differential problem by the finite difference approximation on an uniform space mesh with the step size $h_1 = h_2 = h$ $(N = 1/h)$. The

Table 1: Numerical results for the test problem (see text for details).

N	32	64	128	256	512
		$D_1 = 1,\ D_2 = 10,\ \zeta_i = 10$			
error	2.9e-4	7.1e-5	1.8e-5	4.2e-6	8.4e-7
order	2.00	2.02	2.07	2.32	
# of iterations	11	11	11	11	11
		$D_1 = 1,\ D_2 = 0.1,\ \zeta_i = 10$			
error	4.8e-3	1.2e-3	3.0e-4	7.2e-5	1.4e-5
order	1.97	2.01	2.07	2.32	
# of iterations	13	13	13	13	13

stopping test for the monotone iterative method is chosen in the form

$$\max_{i=1,2}\left[\max_{p\in\bar{\omega}^h}\left|U_i^{(n)}(p)-U_i^{(n-1)}(p)\right|\right]\leqslant\delta\,,\quad \delta=10^{-8}\,.$$

We define the numerical error and order of the numerical error

$$\mathrm{error}(h)=\max_i\left[\max_{p\in\bar{\omega}^h}\left|\widetilde{U}_i(p)-\widetilde{U}_i^{\mathrm{ref}}(p)\right|\right],\quad \mathrm{order}(h)=\log_2\left(\frac{\mathrm{error}(h)}{\mathrm{error}(h/2)}\right),$$

where $\widetilde{U}_i^{\mathrm{ref}}(p)$ are reference solutions with $N = 1024$.

In Table 1, for different values of N and for the two sets of parameters $D_1 = 1$, $D_2 = 10$, $\zeta_i = 1$ and $D_1 = 1$, $D_2 = 0.1$, $\zeta_i = 1$, we present the numerical error, the order of the numerical error and numbers of monotone iterations. The data in the table indicate that numerical solutions converge to reference solutions with second-order accuracy, and the numbers of iterations is independent of N. The numerical experiments show that if N increases in reference solutions, then the order of the numerical error tends to second-order.

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