Solving approximate cloaking problems using finite element methods

Q. T. Le Gia[1] H. Gimperlein[2] M. Maischak[3]
E. P. Stephan[4]

Received 23 January 2017; revised 4 December 2017

Abstract

Motivated by the approximate cloaking problem, we consider a variable coefficient Helmholtz equation with a fixed wave number. We use finite element methods to discretize the equation. Numerical results are shown to exhibit cloaking behaviour.

Contents

DOI:10.21914/anziamj.v58i0.11729, Published 2017-12-05, as part of the Proceedings of the 18th Biennial Computational Techniques and Applications Conference. ISSN 1445-8810. (Print two pages per sheet of paper.)

1 Introduction

In 2005 and 2006 serious theoretical proposals [1, 6] and a widely reported experiment by Schurig et al. [9] were put forward for cloaking devices—structures that would not only render an object invisible but also undetectable to electromagnetic waves. The mathematical foundations of optical cloaking have developed significantly since then, see, for example, the excellent article by Greenleaf et al. [3].

The transformation optics approach to cloaking which uses a singular change of coordinates to blow up a point to the region being cloaked [4, 8] is difficult to analyse theoretically due to this singularity. Hence a rigorous numerical simulation should shed light on the problem.

In this paper, we will review the theoretical background of the approximate cloaking problem and propose a finite element method to numerically solve the problem. While there have been other papers, e.g., the paper by Cai et al. [2], describing cloaking experiments using the commercial finite-element COMSOL Multiphysics package, proper mathematical explanations were not available there. Here, we offer a summary of the theory, as well as a finite element solution using an open source package.

2 Mathematical problem

Let Ω be a bounded domain in $\mathbb{R}^n$ for $n = 2, 3$. Light waves passing through the domain Ω can be described by the wave equation

$$q(x)U_{tt} - \nabla \cdot (A(x)\nabla U) = 0.$$

Here, U is the displacement of the wave and $q(x)$ and $A(x)$ are functions that model the anisotropy of wave propagation through the domain Ω.

For a harmonic solutions $U = ue^{-ikt}$ we obtain the scalar Helmholtz equation

$$\nabla \cdot (A(x)\nabla u) + k^2 q(x)u = 0 \in \Omega. \tag{1}$$

The solution to the Helmholtz equation (1) is uniquely defined if either the Dirichlet condition

$$u = g \text{ on } \partial\Omega, \tag{2}$$

or the Neumann condition

$$\frac{\partial u}{\partial n} = \psi \text{ on } \partial\Omega \tag{3}$$

is given.

Let $H^1(\Omega)$ be the Sobolev space which consists of functions having first derivative in $L^2(\Omega)$. We define the Sobolev space $H^{1/2}(\partial\Omega)$ as follows: a function ϕ belongs to $H^{1/2}(\partial\Omega)$ if and only if ϕ is the restriction to $\partial\Omega$ of some function in $H^1(\Omega)$. Let $H^{-1/2}(\Omega)$ be the dual space of $H^{1/2}(\Omega)$. These are the natural spaces for Dirichlet and Neumann data of finite energy solutions. With respect to the Helmholtz equation (1), we define the map $\Lambda_{A,q} : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ as the solution of

$$\begin{cases} \Lambda_{A,q}(\psi) = u|_{\partial\Omega}, \\ u \text{ solves (1) with } \sum A_{ij}\dfrac{\partial u}{\partial x_j}\nu_i = \psi \text{ on } \partial\Omega. \end{cases} \tag{4}$$

Let B_r be the open ball of radius r, that is, $B_r = \{x \in \mathbb{R}^n : |x| < r\}$. Suppose the given domain Ω contains B_2. A specific structure $A_c(x), q_c(x)$ defined on the shell $B_2 \setminus B_1$ is said to cloak the unit ball B_1 if whenever

$$A(x), q(x) = \begin{cases} I, 1 & \text{if } x \in \Omega \setminus B_2, \\ A_c, q_c & \text{if } x \in B_2 \setminus B_1, \\ \text{arbitrary} & \text{if } x \in B_1, \end{cases} \tag{5}$$

then

$$\Lambda_{A,q} = \Lambda_{I,1}.$$

Here I is the identity matrix and the equality $A, q = I, 1$ means $A = I$ and $q = 1$. Thus, the boundary measurements (Dirichlet and Neumann data) on $\partial\Omega$ with respect to $A(x), q(x)$ are identical to those obtained when $A = I$ and $q = 1$. Physically the field U appears uniformly on Ω regardless of the content on B_1. Or, light waves at the boundary $\partial\Omega$ behave identically regardless of the content on B_1, giving the impression that B_1 is cloaked.

A change of variable scheme was proposed by Schurig et al. [9] to construct a cloak A_c, q_c. The scheme relies on the following [4]:

Let $F : \Omega \to \Omega$ be a differentiable, orientation-preserving, surjective and invertible map such that $F(x) = x$ on $\partial\Omega$. Let DF be the Jacobian matrix and let

$$F_*A(y) = \frac{DF(x)A(x)DF^T(x)}{\det(DF(x))}, \quad F_*q(y) = \frac{q(x)}{\det(DF(x))}, \quad x = F^{-1}(y).$$

Then

$$u(x) \text{ solves } \nabla_x \cdot (A(x)\nabla_x u) + k^2 q(x) u = 0,$$

if and only if

$$w(y) = u(F^{-1}(y)) \text{ solves } \nabla_y \cdot (F_*A(y)\nabla_y w) + k^2 F_*q(y) w = 0.$$

Moreover, A, q and F_*A, F_*q give the same boundary measurements, i.e.,

$$\Lambda_{A,q} = \Lambda_{F_*A, F_*q}.$$

An example of the map F is given by $F = F_\varepsilon$ [9], where

$$F_\varepsilon(x) = \begin{cases} \frac{x}{\varepsilon} & \text{if } |x| \leqslant \varepsilon, \\ \left(\frac{2-2\varepsilon}{2-\varepsilon} + \frac{|x|}{2-\varepsilon}\right)\frac{x}{|x|} & \text{if } \varepsilon \leqslant |x| \leqslant 2, \\ x & \text{if } |x| > 2. \end{cases} \tag{6}$$

Therefore, F_ε maps B_ε to the unit ball B_1, the annulus $B_2 \setminus B_\varepsilon$ to the annulus $B_2 \setminus B_1$ and outside B_2 the map F_ε is simply the identity map.

The inverse map F_ε^{-1} is

$$F_\varepsilon^{-1}(y) = \begin{cases} \varepsilon y & \text{if } |y| \leqslant 1. \\ y\left(2 - \varepsilon - \frac{2(1-\varepsilon)}{|y|}\right) & \text{if } 1 \leqslant |y| \leqslant 2. \end{cases} \tag{7}$$

It has been suggested [3, 9] that if we take $F_0 = \lim_{\varepsilon\to 0} F_\varepsilon$, i.e., F_0 is the singular map that blows the origin up to the ball B_1, and define

$$A_c = (F_0)_* I, \quad q_c = (F_0)_* 1,$$

then the ball B_1 would be cloaked. Hence for small ε, then $(F_\varepsilon)_* I,\ (F_\varepsilon)_* 1$ should nearly cloak B_1, which means that if

$$A(y), q(y) = \begin{cases} I,\ 1 & \text{if } y \in \Omega \setminus B_2, \\ (F_\varepsilon)_* I, (F_\varepsilon)_* 1 & \text{if } y \in B_2 \setminus B_1, \\ \text{arbitrary} & \text{if } y \in B_1, \end{cases} \tag{8}$$

then $\Lambda_{A,q} \approx \Lambda_{I,1}$. However, due to resonance the statement is not true for $k \neq 0$ [4, Section 2.5].

To explain this point further, let $\Omega = B_2$ and consider

$$A_\varepsilon, q_\varepsilon = \begin{cases} I, 1 & \text{if } x \in B_2 \setminus B_\varepsilon, \\ \tilde{A}_\varepsilon, \tilde{q}_\varepsilon & \text{if } x \in B_\varepsilon, \end{cases}$$

where $\tilde{A}_\varepsilon$ and $\tilde{q}_\varepsilon$ are real-valued constants. The general solution of the associated two-dimensional Helmholtz equation can be expressed in polar coordinates as

$$u = \begin{cases} \sum_{\ell=-\infty}^{\infty} \alpha_\ell J_\ell \left(kr\sqrt{\tilde{q}_\varepsilon/\tilde{A}_\varepsilon} \right) e^{i\ell\theta} & \text{if } r \leqslant \varepsilon, \\ \sum_{\ell=-\infty}^{\infty} [\beta_\ell J_\ell(kr) + \gamma_\ell H_\ell^{(1)}(kr)] e^{i\ell\theta} & \text{if } \varepsilon < r \leqslant 1, \end{cases}$$

for appropriate choices of α_ℓ, β_ℓ, and γ_ℓ. Here J_ℓ and $H_\ell^{(1)}$ are the classical Bessel and Hankel functions of the first kind, respectively. When we solve a Neumann problem, the three unknowns for mode ℓ $(\alpha_\ell, \beta_\ell, \gamma_\ell)$ are determined by three linear equations: agreement with the Neumann data at $r = 2$ and satisfaction of the two transmission conditions at $r = \varepsilon$. However, for any $k \neq 0$ and any ℓ, this linear system has zero determinant at selected values of $\tilde{A}_\varepsilon, \tilde{q}_\varepsilon$. When the linear system is degenerate (for some ℓ) the homogeneous Neumann problem has a nonzero solution, and the boundary map $\Lambda_{A_\varepsilon, q_\varepsilon}$ is not even well-defined. In other words, no matter how small the value of ε, for any $k \neq 0$ there are cloak-busting choices of $\tilde{A}_\varepsilon, \tilde{q}_\varepsilon$ for which the ball with such an inclusion is resonant at frequency k.

To deal with the resonance problem, a near-cloak mechanism was introduced by Kohn et al. [4], which has a new damping parameter $\beta > 0$. The near-cloak is defined as

$$A(y), q(y) = \begin{cases} I, 1 & \text{if } y \in \Omega \setminus B_2, \\ (F_{2\varepsilon})_* I, (F_{2\varepsilon})_* 1 & \text{if } y \in B_2 \setminus B_1, \\ (F_{2\varepsilon})_* I, (F_{2\varepsilon})_* (1 + i\beta) & \text{if } y \in B_1 \setminus B_{1/2}, \\ \text{arbitrary real, elliptic} & \text{if } y \in B_{1/2}. \end{cases} \tag{9}$$

With $\beta > 0$, the following problem is well-posed [4, Proposition 3.5]

$$\begin{cases} \nabla(A_\varepsilon \nabla u) + k^2 q_\varepsilon u = 0 & \text{if } x \in \Omega \\ \partial u / \partial \nu = \psi & \text{if } x \in \partial\Omega, \end{cases} \tag{10}$$

where

$$\begin{cases} A_\varepsilon = I, \; q_\varepsilon = 1 & \text{if } x \in \Omega \setminus B_{2\varepsilon}, \\ A_\varepsilon = 1, \; q_\varepsilon = 1 + i\beta & \text{if } x \in B_{2\varepsilon} \setminus B_\varepsilon, \\ A_\varepsilon, q_\varepsilon \text{ arbitrary real, elliptic} & \text{if } x \in B_\varepsilon. \end{cases}$$

Furthermore, when $\beta \sim \varepsilon^{-2}$, then their construction approximately cloaks $B_{1/2}$ in the sense that

$$\|\Lambda_{A,q} - \Lambda_{I,1}\| \leqslant C \begin{cases} 1/|\log \varepsilon| & \text{in space dimension 2,} \\ \varepsilon & \text{in space dimension 3.} \end{cases} \tag{11}$$

The theoretical estimate (11) is pessimistic in two dimensions since the proof relies on the fundamental solution of the two-dimensional Laplace equation. However, numerical experiments show that when $\varepsilon \to 0$, the approximate cloaking scheme performs reasonably well.

3 Using finite element methods

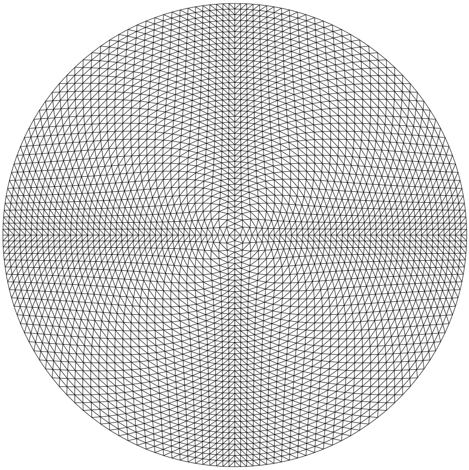
In this section, we will describe how to solve the near-cloaking problem using finite element methods. The weak formulation of (10) is: find $u \in H^1(\Omega)$ so that

$$\int_\Omega [A_\varepsilon(x)\nabla_x u(x) \cdot \nabla_x v(x) - k^2 q_\varepsilon u(x)v(x)]\,dx = \int_{\partial\Omega} A_\varepsilon \psi v\, dy, \quad \text{for all } v \in H^1(\Omega). \tag{12}$$

The weak formulation of the push-forward problem is: find $w \in H^1(\Omega)$ so that

$$\int_\Omega [F_*(A_\varepsilon)\nabla_y w(y) \cdot \nabla_y \phi(y) - k^2 q_\varepsilon w(y)\phi(y)]\,dy = \int_{\partial\Omega} F_* A_\varepsilon \psi\phi\, dy, \quad \text{for all } \phi \in H^1(\Omega). \tag{13}$$

Figure 1: A uniform mesh for computing u_ε on B_2.



Introducing the bilinear form

$$a(w,\phi) = \int_\Omega [F_*(A_\varepsilon)\nabla_y w(y) \cdot \nabla_y \phi(y) - k^2 q_\varepsilon w(y)\phi(y)]\, dy,$$

and defining the finite dimensional space

$$V_h = \operatorname{span}\{\phi_1, \phi_2, \ldots, \phi_N\} \subset H^1(\Omega),$$

the Ritz–Galerkin approximation problem to the push-forward problem (13) is written as: find $w \in V_h$ so that

$$a(w,\chi) = \int_{\partial\Omega} A_\varepsilon \psi \chi \, dy, \qquad \text{for all } \chi \in V_h(\Omega).$$

A uniform mesh that is used to construct the piecewise linear finite elements when $\Omega = B_2$ is shown in Figure 1.

4 Numerical experiments

In this section, we describe some initial numerical experiments on the interior two-dimensional Dirichlet example introduced by Kohn et al. [4]. Consider then the problem

$$\begin{cases} \nabla \cdot (F_*(A_\varepsilon)\nabla U_\varepsilon(y)) + k^2 F_*(q_\varepsilon)U_\varepsilon(y) = 0 & y \in B_2, \\ \partial_\nu U_\varepsilon(y) = \partial_\nu u_0(2,\theta) \quad y \text{ on } \Gamma = \partial B_2, \end{cases} \tag{14}$$

where

$$u_0(r,\theta) = \sum_{\ell=-30}^{30} J_\ell(kr)e^{i\ell\theta},$$

and J_ℓ is the classical Bessel function of order ℓ.

In two dimensions,

$$\begin{cases} F_*(A_\varepsilon)(y) = \left.\frac{DF(x)DF^T(x)}{\det(DF(x))}\right|_{x=F^{-1}(y)} & \text{if } 1 < |y| \leqslant 2, \\ F_*(q_\varepsilon)(y) = \left.\frac{1}{\det(DF(x))}\right|_{x=F^{-1}(y)} & \text{if } 1 < |y| \leqslant 2, \\ F_*(A_\varepsilon)(y) = 1, \qquad F_*(q_\varepsilon)(y) = 4\varepsilon^2(1+i\beta) & \text{if } \frac{1}{2} < |y| \leqslant 1, \\ F_*(A_\varepsilon)(y) = A_\varepsilon, \qquad F_*(q_\varepsilon)(y) = 4\varepsilon^2 q_\varepsilon & \text{if } |y| \leqslant \frac{1}{2}. \end{cases}$$

We now compute the Jacobian $F' = DF(x) = (\partial F_i/\partial x_j)$. The computation for the special case $\varepsilon = 0$ is considered by Kohn et al. [5]. For the general case

$$DF(x) = \left[\left(\frac{1-2\varepsilon}{1-\varepsilon}\right)\frac{1}{|x|} + \frac{1}{2(1-\varepsilon)}\right] I - \left(\frac{1-2\varepsilon}{1-\varepsilon}\right)\frac{\hat{x}\hat{x}^T}{|x|}, \tag{15}$$

where $\hat{x} = x/|x|$ and I is the identity matrix.

To find the determinant of $DF(x)$, we note that $\hat{x}$ is an eigenvector of $DF(x)$ with eigenvalue $0.5/(1-\varepsilon)$ and $\hat{x}^\perp$ is an $n-1$ dimensional eigenspace with eigenvalue

$$\frac{(1-2\varepsilon)}{(1-\varepsilon)}\frac{1}{|x|} + \frac{1}{2(1-\varepsilon)}.$$

So, the determinant of $\mathsf{DF}(x)$ is

$$\det(\mathsf{DF}(x)) = \frac{1}{2(1-\varepsilon)}\left[\frac{(1-2\varepsilon)}{(1-\varepsilon)}\frac{1}{|x|} + \frac{1}{2(1-\varepsilon)}\right]^{n-1}.$$

For $x = F^{-1}(y)$, we have

$$\frac{1}{\det(\mathsf{DF}(F^{-1}(y)))} = \frac{2}{|y|^2}\big(2|y|(1-\varepsilon) - 2(1-2\varepsilon)\big)^2.$$

Consequently, we can calculate the product $\mathsf{DF}(x)(\mathsf{DF}(x))^T$

$$\begin{aligned}\mathsf{DF}(x)(\mathsf{DF}(x))^T = {} & \left(\frac{(1-2\varepsilon)^2}{(1-\varepsilon)^2}\frac{1}{|x|^2} + \frac{(1-2\varepsilon)}{(1-\varepsilon)^2}\frac{1}{|x|} + \frac{1}{4(1-\varepsilon)^2}\right) I \\ & - \left(\frac{(1-2\varepsilon)^2}{(1-\varepsilon)^2}\frac{1}{|x|^2} + \frac{(1-2\varepsilon)}{(1-\varepsilon)^2}\frac{1}{|x|}\right)\hat{x}\hat{x}^T.\end{aligned}$$

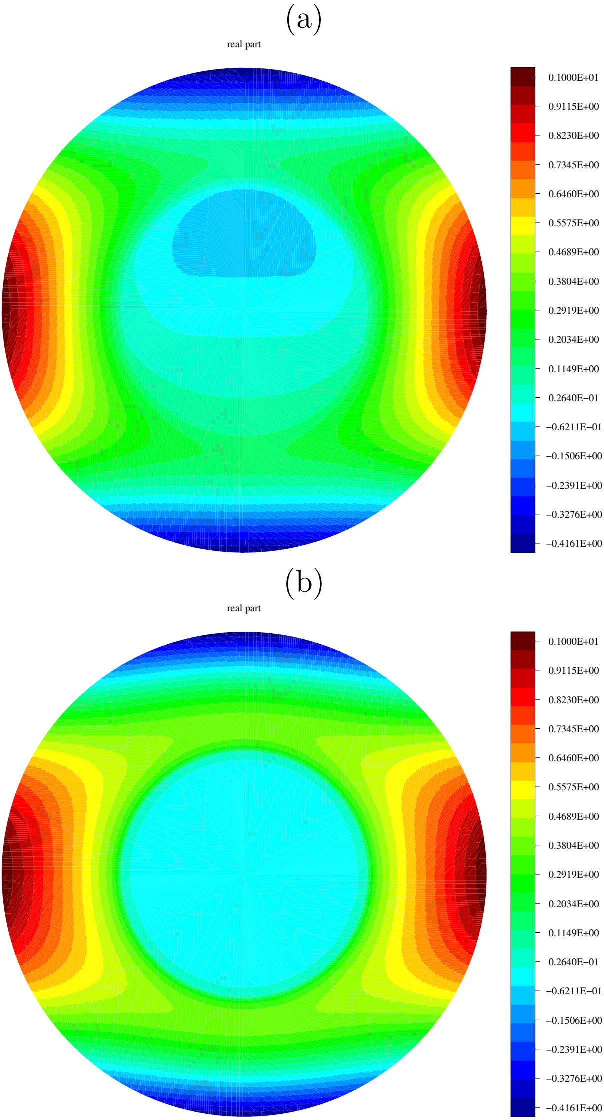
For the Dirichlet problem we have performed numerical simulations with the program package MAIPROGS [7] using FEM-2D with piecewise linear elements. We set the wavenumber $k = 1$ and $A_\varepsilon = q_\varepsilon = 1$.

In Figure 2 the finite element approximations u_h to U_ε for $\varepsilon = 10^{-d}$, $d = 1, 6$ and $\beta = \varepsilon^{-2}$ are shown. As ε gets smaller, the numerical solution becomes more uniform on $B_{1/2}$, so the content of $B_{1/2}$ is cloaked. These are consistent with numerical results of Kohn et al. [4], which were obtained using a different numerical method.

5 Conclusions

In this work, we have summarised the approximate cloaking framework proposed in [4] and proposed a finite element method to construct numerical solutions using an open source package. This will lay the foundation for future work in error analysis and coupled finite element–boundary element methods for the approximate cloaking problem.

Figure 2: The two-dimensional push-forward FEM solutions u_ε on B_2. (a): $\varepsilon = 10^{-1}$, and (b): $\varepsilon = 10^{-6}$.



Acknowledgement This work was initiated while the last author was visiting the School of Mathematics and Statistics, UNSW, Sydney. His visit was supported by a UNSW Science Faculty Research Grant.

References

[1] A. Alu and N. Engheta. Achieving transparency with plasmonic and metamaterial coatings. *Phys. Rev. E* **72**:016623, 2005. doi:10.1103/PhysRevE.72.016623 C163

[2] W. Cai, U. K. Chettiar, A. V. Kildishev and V. M. Shalaev. Optical cloaking with metamaterials. *Nature Photonics* **1**:224–227, 2007. doi:10.1038/nphoton.2007.28 C163

[3] A. Greenleaf, Y. Kurylev, M. Lassas and G. Uhlmann. Cloaking devices, electromagnetic wormholes and transformation optics. *SIAM Rev.* **51**:3–33, 2009. doi:10.1137/080716827 C163, C166

[4] R. V. Kohn, D. Onofrei, M. S. Vogelius and M. I. Weinstein. Cloaking via change of variables for the Helmholtz equation. *Comm. Pure Appl. Math.* **63**:973–1016, 2010. doi:10.1002/cpa.20341 C163, C165, C166, C167, C170, C171

[5] R. V. Kohn, H. Shen, M. S. Vogelius and M. I. Weinstein. Cloaking via change of variables in electric impedance tomography. *Inverse Problems* **24**:015016, 2008. doi:10.1088/0266-5611/24/1/015016 C170

[6] U. Leonhardt. Optical conformal mapping. *Science* **312**:1777–1780, 2006. doi:10.1126/science.1126493 C163

[7] M. Maischak. *Book of Numerical Experiments (B.O.N.E)*. `http://www.ifam.uni-hannover.de/~maiprogs/` C171

[8] J. B. Pendry, D. Schurig and D. R. Smith. Controlling electromagnetic fields. *Science* **312**:1780–1782, 2006. doi:10.1126/science.1125907 C163

[9] D. Schurig, J. Mock, B. Justice, S. Cummer, J. Pendry, A. Starr and D. Smith. Metamaterial electromagnetic cloak at microwave frequencies. *Science* **314**:977–980, 2006. doi:10.1126/science.1133628 C163, C165, C166

Author addresses

1. **Q. T. Le Gia**, School of Mathematics and Statistics, UNSW, Sydney, NSW 2052, Australia.
   mailto:qlegia@unsw.edu.au
2. **H. Gimperlein**, Maxwell Institute for Mathematical Sciences and Department of Mathematics, Heriot-Watt University, Edinburgh, UK.
   mailto:h.gimperlein@hw.ac.uk
3. **M. Maischak**, Department of Mathematics, Brunel University London, UK.
   mailto:matthias.maischak@brunel.ac.uk
4. **E. P. Stephan**, Institut für Angewandte Mathematik, Leibniz University Hannover, Germany.
   mailto:stephan@ifam.uni-hannover.de