A gradient recovery method based on an oblique projection and boundary modification

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Abstract

The gradient recovery method is a technique to improve the approximation of the gradient of a solution by using post-processing methods. We use an L^2-projection based on an oblique projection, where the trial and test spaces differ, for efficient numerical computation. We modify our oblique projection by applying the boundary modification method to obtain higher order approximation on the boundary patch. Numerical examples are presented to demonstrate the efficiency and optimality of the approach.

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Contents

1 Introduction

The gradient recovery method is a popular technique for approximating the gradient of a solution. The main idea is based on using various post-processing techniques on a computed solution to obtain an approximation to the gradient of the solution. There are many gradient recovery techniques, such as patch recovery [8, 10], polynomial preserving recovery [1, 9] and local and global L^2-projection [8]. While all these methods display superconvergence on the interior domain, the rate of convergence deteriorates on the boundary patch.

We propose a new gradient recovery approach based on an oblique projection using boundary modification. This approach improves the result by Lamichhane [4] to obtain higher order approximation on the boundary patch. In Subsection 2.2 we introduce our formulation of the orthogonal and oblique projection, and show that a significant reduction in computational time using an oblique projection can be obtained. In Subsection 2.3 we introduce the boundary modification method and its construction that leads to a linear extrapolation. In Section 3 the approximation property theorem of our method is presented, which is based on the classical paper by Křížek and Neittaanmäki [3] combined with the L^2-projection approach from Xu and Zhang [8] and Lamichhane [4]. The proof of the theorem requires that the triangulation has to satisfy certain conditions so that the extrapolation still preserves approximation properties on the interior domain. Two numerical examples, a transcendental solution and a less-regular solution, are given in Section 4 to verify the convergence rate of our gradient recovery method. Both examples follow the predicted theoretical rate of convergence.

2 Formulation

2.1 Finite element discretisation

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary $\partial\Omega$. Let $\mathcal{T}_h$ be a uniformly congruent triangulation of the polygonal domain Ω . We use the standard linear finite element space $V_h \subset H^1(\Omega)$ defined on the triangulation $\mathcal{T}_h$, where

$$V_h := \{v \in C^0(\Omega) : v|_T \in \mathcal{P}_1(T), T \in \mathcal{T}_h\}.$$

Let $S_h = \text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ be the standard finite element basis for V_h. A piecewise linear interpolant of a continuous function u is written as $u_h \in V_h$ with

$$u_h(x) = \sum_{i=1}^{N} u(x_i)\, \varphi_i(x) \ ,$$

where $N = \dim S_h$. We denote the set of mesh nodes by $\mathcal{N}$, the boundary nodes by $\mathcal{N}^{bd}$ and the set of interior mesh nodes by $\mathcal{N}^{in} = \mathcal{N} \backslash \mathcal{N}^{bd}$. With similar notation, we denote the set of elements, the boundary elements (the elements that have one or more nodes on the boundary) and the set of interior elements by $\mathcal{T}, \mathcal{T}^{\mathrm{bd}}$ and $\mathcal{T}^{\mathrm{in}}$, respectively. With this notation, we also define $\Omega^{\mathrm{in}} = \cup\mathcal{T}^{\mathrm{in}}$ and $\Omega^{\mathrm{bd}} = \cup\mathcal{T}^{\mathrm{bd}}$ as the interior domain and boundary patch, respectively.

2.2 Projection and biorthogonal basis

Computing the gradient recovery requires a projection of ∇u_h onto the finite element space V_h. There are two types of projection: orthogonal and oblique. The orthogonal projection operator P_h projects ∇u_h onto V_h by finding $g_h^k = P_h\left(\frac{\partial u_h}{\partial x_k}\right) \in V_h$ for $k = 1, 2$, that satisfies

$$\int_\Omega g_h^k \varphi_j \, dx = \int_\Omega \frac{\partial u_h}{\partial x_k} \varphi_j \, dx, \quad 1 \leqslant j \leqslant N\,. \tag{1}$$

Let $g_h^k = \sum_{i=1}^N g_i^k \varphi_i$ for $k = 1, 2$, then equation (1) can be written as

$$\sum_{i=1}^N g_i^k \int_\Omega \varphi_i \varphi_j \, dx = \int_\Omega \frac{\partial u_h}{\partial x_k} \varphi_j \, dx, \quad 1 \leqslant j \leqslant N\,. \tag{2}$$

Equation (2) is equivalent to the system of linear equations $M\vec{g}^k = \vec{f}^k$ for $k = 1, 2$, where M is the mass matrix, $\vec{g}^k = \left(g_1^k, \ldots, g_N^k\right)^T$ and $\vec{f}^k = \left(f_1^k, \ldots, f_N^k\right)^T$, with $f_j^k = \int_\Omega \frac{\partial u_h}{\partial x_k}\varphi_j \, dx$. Solving this equation requires the computationally expensive inversion of the matrix M. The computation time can be significantly reduced by diagonalizing the matrix using a suitable oblique projection instead of an orthogonal projection.

The oblique projection operator Q_h projects ∇u_h onto V_h by finding $g_h^k = Q_h\left(\frac{\partial u_h}{\partial x_k}\right) \in V_h$ for $k = 1, 2$, that satisfies

$$\int_\Omega g_h^k \mu_j \, dx = \int_\Omega \frac{\partial u_h}{\partial x_k} \mu_j \, dx, \quad 1 \leqslant j \leqslant N\,, \tag{3}$$

where $\mu_j \in M_h$. The piecewise polynomial space M_h has the following construction method. Starting with the standard bases for V_h, we construct a space M_h spanned by the basis $\{\mu_1, \mu_2, \ldots, \mu_N\}$ so that the basis functions of V_h and M_h satisfy a biorthogonality relation

$$\int_\Omega \varphi_i \mu_j \, dx = c_j \delta_{ij}, \quad 1 \leqslant i, j, \leqslant N,$$

where δ_{ij} is the Kronecker symbol, and c_j a nonzero scaling factor. The basis functions of M_h are constructed locally on a reference element $\hat{T}$ so that for every i, the basis functions φ_i and μ_i have the same support [5]. The projection operator Q_h is well defined due to the stability condition [4]: there is a constant $\beta > 0$ such that

$$\beta = \inf_{\varphi_h \in V_h} \left[\sup_{\mu_h \in M_h} \frac{\int_\Omega \varphi_h \mu_h}{\|\varphi_h\|_{L^2(\Omega)} \|\mu_h\|_{L^2(\Omega)}} \right].$$

Let $g_h^k = \sum_{i=1}^N g_i^k \varphi_i$ for $k = 1, 2$, then equation (3) can be written as

$$\sum_{i=1}^N g_i^k \int_\Omega \mu_i \varphi_j \, dx = \int_\Omega \frac{\partial u_h}{\partial x_k} \mu_j \, dx, \quad 1 \leqslant j \leqslant N. \tag{4}$$

Equation (4) is equivalent to the system of linear equations $D\vec{g}^k = \vec{f}^k$ for $k = 1, 2$, where D is a diagonal matrix, $\vec{g}^k = \left(g_1^k, ..., g_N^k\right)^T$ and $\vec{f}^k = \left(f_1^k, ..., f_N^k\right)^T$, with $f_j^k = \int_\Omega \frac{\partial u_h}{\partial x_k} \mu_j \, dx$.

2.3 Boundary modification

The boundary modification method comes from the context of mortar finite element [7]. The modification allows us to remove the basis functions associated to the nodes belonging to $\mathcal{N}^{bd}$ and add a multiple of them onto the basis functions associated to the nodes belonging to $\mathcal{N}^{in}$.

We construct our boundary modified space as follows. For each boundary node $n_i \in \mathcal{N}^{bd}$ with global coordinate x_i, select an interior element $T \in \mathcal{T}^{in}$ with interior nodes $n_{i_m} \in \mathcal{N}^{in}$ with global coordinates x_{i_m}, $1 \leqslant m \leqslant 3$, respectively. We remove the basis functions $\varphi_i \in V_h$ and modify $\varphi_{i_m}, 1 \leqslant m \leqslant 3$, as

$$\tilde{\varphi}_{i_m} = \varphi_{i_m} + \alpha_m \varphi_i, \quad 1 \leqslant m \leqslant 3,$$

where $\alpha_m, 1 \leqslant m \leqslant 3$, are scalar numbers that satisfy

$$\sum_{i=1}^{3} \alpha_m p\left(x_{i_m}\right) = p\left(x_i\right), \quad p \in \mathcal{P}_1\left(\Omega\right),$$

that is, $(\alpha_m)_{m=1,2,3}$ are the barycentric coordinates of x_i with respect to the chosen interior element.

This boundary modification technique is a linear extrapolation and the boundary modified space preserves the approximation property from the original space V_h [7]. We denote the oblique projection operator on the boundary modified space as Q_h^*.

Remark 1. This study is restricted to the two-dimensional case, but this technique can be extended to the three-dimensional case by selecting corresponding oblique projections and boundary modifications.

3 Approximation property

Theorem 2. *Let Ω be a bounded polygonal domain with Lipschitz boundary $\partial\Omega$ in $\mathbb{R}^2$ and $\mathcal{T}_h$ be a uniformly congruent triangulation of Ω. Given $u \in H^3\left(\Omega\right)$ with its Lagrange interpolant $u_h \in V_h$, there exists a constant $C > 0$ independent of mesh-size h such that*

$$\|\nabla u - Q_h^*\left(\nabla u_h\right)\|_{0,\Omega} \leqslant Ch^2 \left|u\right|_{3,\Omega}.$$

Proof: Define the average gradient operator G_h and the linear interpolant $u_h \in V_h$. For uniformly congruent triangulation and any $x \in \mathcal{N}$,

$$G_h\left(u_h\right)(x) = \frac{1}{6} \sum_{\substack{K \in \mathcal{T}^{\text{in}} \\ \overline{K} \cap \{x\} \neq \emptyset}} \nabla u_h|_K \,,$$

where $\nabla u_h|_K$ is the value of the gradient ∇u_h on the triangle K and $\overline{K}$ denotes the closure of triangle K. Křížek and Neittaanmäki [3] showed that given $\Omega^{\text{in}} \subset \Omega$ such that $\max \operatorname{dist}_{x \in \partial\Omega^{\text{in}}}(x, \partial\Omega) = O\left(h^2\right)$ and $u|_{\Omega^{\text{in}}} \in H^3\left(\Omega_h\right)$,

$$\left\|\nabla u - G_h\left(u_h\right)\right\|_{0,\Omega^{\text{in}}} \leqslant Ch^2 \left|u\right|_{3,\Omega^{\text{in}}} \,.$$

Since [4, 8]

$$G_h\left(u_h\right) = Q_h\left(\nabla u_h\right) \,,$$

we obtain a similar approximation property for the oblique projection operator

$$\left\|\nabla u - Q_h\left(\nabla u_h\right)\right\|_{0,\Omega^{\text{in}}} \leqslant Ch^2 \left|u\right|_{3,\Omega^{\text{in}}} \,.$$

Using that the boundary modification technique is a linear extrapolation that preserves the linear approximation property [7], we conclude

$$\left\|\nabla u - Q_h^*\left(\nabla u_h\right)\right\|_{0,\Omega} \leqslant Ch^2 \left|u\right|_{3,\Omega} \,.$$

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4 Numerical results

We provide two numerical examples to verify the convergence rate and robustness of our gradient recovery method. We survey the L^2-error and rate of convergence for the standard oblique projection and the modified boundary oblique projection. We denote the error from the boundary modified oblique projection as $E^{*}u = \|\nabla u - Q_h^{*}(\nabla u_h)\|_{L^2(\Omega)}$, where u_h is the Galerkin solution of the standard Poisson problem

$$\Delta u = -f\,,$$

and the right-hand side function f is derived from the exact solution. We also denote the error from the standard oblique projection on the whole and interior domain, respectively as $Eu = \|\nabla u - Q_h(\nabla u_h)\|_{L^2(\Omega)}$ and $E^{in}u = \|\nabla u - Q_h(\nabla u_h)\|_{L^2(\Omega^{in})}$. For comparison we present the error from the gradient of the solution, that is σ_h, using the stabilised mixed finite element method [2]. This error is denoted by $E^{mx}u = \|\nabla u - \sigma_h\|_{L^2(\Omega)}$.

Since the triangulation is uniformly congruent, the uniform mesh-size and the number of elements in the error approximation tables and are related by $h \propto N^{-1/2}$. Therefore each subsequent row in these tables after the first row corresponds to halving the mesh-size. The exponential rate at which the errors decrease between subsequent rows as a power of h is also shown.

4.1 Transcendental solution

Our first example is a well-behaved problem with a uniformly smooth solution. We consider as an exact solution the transcendental function

$$u = e^{x}\left(x^2 + y^2\right) + y^2\cos(xy) + x^2\sin(xy), \tag{5}$$

Dirichlet boundary conditions for this exact solution are constructed on $\partial\Omega$, where $\Omega = [0,1]^2$. The errors for this problem are shown in Table 1 as the number of elements varies.

The convergence rate of error from the nonmodified approach on the interior domain reaches 2, but this convergence rate deteriorates on the boundary and only reaches 1.5. Our boundary modified approach successfully preserves the convergence rate on the boundary. Also, even though the convergence rate of Eu and $E^{mx}u$ are similar, the mixed finite element method gives a better approximation, as indicated by the lower value of the error.

Table 1: Gradient error approximation for the exact solution (5) (see text for details of the various errors).

N	E^*u	rate	Eu	rate	$E^{in}u$	rate	$E^{mx}u$	rate
32	4.4e-1	-	8.3e-1	-	1.9e-1	-	6.3e-1	-
128	1.1e-1	1.99	3.2e-1	1.36	7.4e-2	1.37	2.2e-1	1.49
512	2.7e-2	2.04	1.2e-1	1.43	2.2e-2	1.75	7.8e-2	1.52
2048	6.7e-3	2.02	4.3e-2	1.47	6.0e-3	1.89	2.7e-2	1.53
8192	1.6e-3	2.02	1.5e-2	1.48	1.6e-3	1.95	9.4e-3	1.52
32768	4.1e-4	2.01	5.5e-3	1.49	4.0e-4	1.97	3.3e-3	1.52

Table 2: Gradient error approximation for the exact solution (6), with $\alpha = 1.6$ (see text for details of the various errors).

N	E^*u	rate	Eu	rate	$E^{in}u$	rate	$E^{mx}u$	rate
32	1.2e-0	-	3.2e-1	-	2.1e-2	-	3.1e-1	-
128	6.3e-2	4.27	1.2e-1	1.45	2.1e-2	-0.01	1.1e-1	1.55
512	2.0e-2	1.63	4.2e-2	1.47	8.0e-3	1.36	3.6e-2	1.55
2048	6.4e-3	1.66	1.5e-2	1.48	2.8e-3	1.50	1.3e-2	1.53
8192	2.1e-3	1.62	5.4e-3	1.48	9.6e-4	1.55	4.4e-3	1.51

4.2 Less-regular solution

In the second example we test the behaviour of our method for a solution with less regularity. We use a modified version of the standard re-entrant corner problem to obtain a source of singularities in the solution. This problem is defined on the slit domain $\Omega = (-1,1)^2 \setminus \{0 \leqslant x \leqslant 1, y = 0\}$ and Dirichlet boundary conditions are constructed on $\partial\Omega$. The exact solution is

$$u = r^{\alpha} \sin(\alpha\theta) \tag{6}$$

where $r = \sqrt{x^2 + y^2}$ is the distance from the corner and $\theta = \tan^{-1}(y/x)$. This problem satisfies $u \in H^{1+\alpha}(\Omega)$ and in this example $\alpha = 1.6$ to obtain $H^{2.6}$ regularity.

In Table 2 we can observe the superiority of our approach compared to the standard oblique projection method and mixed finite element method, even though it is not as great as our previous example. While the convergence rate seems similar, our approach gives the best approximation followed by the mixed finite element method and then the standard oblique projection method.

5 Conclusion

We have proposed a gradient recovery method based on an oblique projection using a boundary modification. The boundary modification technique improves the approximation for the gradient on the boundary nodes and preserves the superconvergence of the interior domain. Numerical examples show that our approach has better accuracy than the standard oblique projection approach even with a less-regular solution. Furthermore, this boundary modified gradient recovery operator can be applied to approximate the solution of a biharmonic problem [6].

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