A fast, spectrally accurate solver for the Falkner–Skan equation

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Abstract

We present a new numerical technique, the Gegenbauer homotopy analysis method, which allows for the construction of iterative solutions to nonlinear differential equations. This technique is a numerical extension of the semi-analytic homotopy analysis method that exhibits spectral convergence while performing sparse matrix operations in Gegenbauer space. This technique is used to present solutions to the Falkner–Skan equation, a well known problem in boundary layer fluid dynamics. These solutions are compared to previously published works, and the convergence properties exhibited by this new technique are considered.

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1 Introduction

When it comes to investigating the steady-state dynamics of flows in the presence of a solid boundary subject to nonslip boundary conditions, it is sometimes possible to map the governing Navier–Stokes equations to equations that are more suitable for analytic and numerical analysis. Two of the most well known equations for this flow regime are the Blasius and Falkner–Skan equations, which describe the steady flow of an ideal fluid past a flat plate. For the Blasius equation [2] the plate is aligned with the direction of a uniform freestream flow, whereas Falkner and Skan [4] developed their eponymously named equation in order to model the steady flow past a flat plate inclined at an angle of attack relative to the freestream flow direction.

One of the primary insights that can be gained by modelling these flows in the manner of Blasius, and Falkner and Skan is the calculation of the skin-friction drag coefficient, which corresponds to the derivative of the fluid velocity with respect to the dimensionless wall-normal distance. Historically, these equations have been solved using a range of analytic and numeric techniques, with Blasius initially attempting to solve the equation by computing a series solution in the near wall region, and seeking values of the skin-friction drag that allowed for the series solution to be matched to an asymptotic solution far from the wall—an approach that was ultimately unsuccessful. From a numerical perspective, a common approach has been to apply the shooting method, however this has been shown to introduce floating point overflow into the calculated solution [1]. In this work, we present a novel technique to solve the Falkner–Skan equations iteratively, based upon the homotopy analysis method (HAM), and which only requires a single matrix inversion for each position in parameter space.

2 Problem description

The Falkner–Skan equation is the result of reducing the equations of flow past a flat plate into the two-point boundary value problem

$$\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} + \beta\left[1 - \left(\frac{df}{d\eta}\right)^2\right] = 0, \tag{1}$$

$$f(0) = \frac{df}{d\eta}(0) = 0, \frac{df}{d\eta}(\eta) \to 1 \text{ as } \eta \to \infty. \tag{2}$$

Here η and f are similarity variables, and the wedge angle of attack is $\pi\beta/2$, with the Blasius equation for flow past a flat plate corresponding to the case where $\beta = 0$. The Falkner–Skan equation is well known to have a single solution for $\beta \geqslant 0$, and two solutions—known as the normal and reverse flow solutions, based upon the sign of the second derivative on the boundary—for $\beta_{\min} \leqslant \beta < 0$, where Fazio [5] showed $\beta_{min} \approx -0.1988$.

3 Numerical method

3.1 The Gegenbauer homotopy analysis method

The homotopy analysis method, as first proposed by Liao [6], is an iterative technique for solving nonlinear differential equations based upon the idea of a homotopy—a smooth deformation from one space onto another. For differential equations, this process involves using an initial guess to the solution of an equation and deforming it onto the solution to the full nonlinear problem.

To solve a nonlinear equation of the form $\mathcal{N}[u(x)] = \phi(x)$ we can construct a homotopy between this and some arbitrary linear problem $\mathcal{L}[u(x)] = 0$ by constructing a homotopy equation in the form of

$$(1-q)\mathcal{L}[U(x;q) - u_0(x)] = q\hbar(\mathcal{N}[U(x;q)] - \phi(x)). \quad (3)$$

Here $\hbar \in (-2, 0)$ is a convergence control parameter and q is the homotopy parameter, which at $q = 0$ gives us the arbitrary linear problem, and at $q = 1$ the full nonlinear problem.

By varying q from 0 to 1 the solution of the linear problem can be deformed onto the full nonlinear equation. To do this, we differentiate equation (3) m times and then rescale, so that the homotopy equation can now be recast as a sequence of linear differential equations of the form

$$\mathcal{L}[U_m(x) - \chi_{m-1}U_{m-1}(x)] = \frac{\hbar}{(m-1)!}\left.\frac{\partial^{m-1}(\mathcal{N}[U(x;q)] - \phi(x))}{\partial q^{m-1}}\right|_{q=0}, \quad (4)$$

with

$$\chi_m = \begin{cases} 0 & \text{if } m \leqslant 1, \\ 1 & \text{if } m > 1. \end{cases}$$

The solution to the original equation is $F(x) = U(x;1) = \sum_{m=0}^{\infty} U_m(x)$. Unlike many other perturbative techniques, HAM does not rely upon any small perturbation parameter to exist in the problem, nor one to be introduced artificially. Furthermore, as the linear operator does not depend upon the solution at the previous step, the scheme only requires a single matrix inversion after the discretization of the equations, which has significant implications for the computational cost of the scheme.

3.2 Linear solver

In previous papers addressing the solution of various nonlinear problems with numerical analogues of the homotopy analysis method, the linear equations have been cast in terms of Chebyshev differentiation matrices, which results in a technique known as the spectral homotopy analysis method (SHAM) [7, 9, 10]. While this approach is well understood, it produces dense matrices that often are ill-conditioned for variable coefficient boundary value problem. Recently, Olver and Townsend [8] developed a spectral method for linear ordinary differential equations which results in almost banded, sparse matrices that can solve linear differential equations in $\mathcal{O}(m^2 n)$ operations, where m and n are respectively the numbers of Chebyshev points to resolve the differential operators and the solution to the differential equation.

A general variable coefficient boundary value problem on $[-1, 1]$

$$\mathcal{L}[u(x)] = a^N(x)\frac{d^N u}{dx^N} + \cdots + a^1(x)\frac{du}{dx} + a^0 u = f(x), \tag{5}$$

where $\{a^0, \ldots, a^N, f\}$ are suitably smooth functions on $[-1, 1]$, can be discretized in terms of the set of Gegenbauer polynomials, of which the Chebyshev polynomials are a subset. The linear operator in (5) can be represented as a matrix operator through

$$\mathcal{L} = \mathcal{M}_N[a^N]D_N + \sum_{\lambda=1}^{N-1}\left(\prod_{i=\lambda}^{N-1} S_i\right)M_\lambda[a^\lambda]D_\lambda + \left(\prod_{i=0}^{N-1} S_i\right)M_0[a^0]. \tag{6}$$

Here $M_\lambda[a^\lambda]$ are multiplication operators that take the functions $a^\lambda \lambda^{(P)}(x)$ in Chebyshev space, and separates out the contribution of a^λ, creating a matrix operator. The differential operators D_λ act on Chebyshev polynomials and differentiate in Gegenbauer space, and S_λ are conversion operators from Chebyshev to Gegenbauer space. Full details on the calculation of these matrices can be found in the paper by Olver and Townsend [8]. Numerically implementing the homotopy analysis method using the aforementioned linear discretization gives the Gegenbauer homotopy analysis method (GHAM)

3.3 Solution technique

To solve the Falkner–Skan equation on a Chebyshev grid, the computational domain $\eta \in [0, \infty)$ is truncated to $[0, L]$, and in turn is mapped to the Chebyshev domain $x \in [-1, 1]$ by the linear mapping $x = 2\eta/L - 1$. In order to have homogeneous boundary conditions the substitution $f(\eta) = F(x) + \eta + e^{-\eta} - 1$ is made. The boundary conditions for equation (1) become $F(-1) = 0$, $F'(1) = F'(0) = 0$. This transform results in a variable coefficient boundary value problem with the modified linear operator

$$\mathcal{L}[F(x)] = \left(\frac{2}{L}\right)^3 \frac{d^3F(x)}{dx^3} + \left(\frac{2}{L}\right)^2 (\eta + e^{-\eta} - 1)\frac{d^2F(x)}{dx^2} - 2\left(\frac{2}{L}\right)\beta(1 - e^{-\eta})\frac{dF(x)}{dx} + (e^{-\eta})F. \quad (7)$$

The infinite series of linear differential equations that results from solving the Falkner–Skan equation are

$$\begin{cases} \mathcal{L}[U_m(x) - (\hbar + \chi_{m-1})U_{m-1}(x)] = \hbar R_{m-1}, \\ R_{m-1} = \displaystyle\sum_{i=0}^{m-1} U_i(x)\frac{d^2U_{m-1-i}}{dx^2} - \beta \sum_{i=0}^{m-1} \frac{dU_i}{dx}\frac{dU_{m-1-i}}{dx} \\ \qquad - (1 - \chi_{m-1})\left[\beta(1 - (1 - e^{-\eta})^2\right] - e^{-\eta}(\eta + e^{-\eta} - 2). \end{cases} \quad (8)$$

This system of linear boundary value problems can be solved using a sparse, Chebyshev based matrix solver following Olver and Townsend [8]. The auxiliary parameter $\hbar$ is calculated by applying bounded minimization on the solution residuals. Solutions to the Blasius equation are calculated in the same numerical scheme as above, with β simply being set to 0 .

4 Discussion

The numerical scheme described above was solved in Matlab on a 512 point grid with Chebyshev spacing with solutions up to the 20th order being calculated using GHAM. The domain of $x \in [0, \infty)$ was truncated to $[0, 20]$, then mapped to the Chebyshev grid by setting $L = 20$. This domain truncation was justified by the rate in which $f'(\eta)$ approaches 1 as η increases, which suggests that the domain truncation is appropriate in the context of the original boundary condition that $f'(\eta) \to 1$ as $\eta \to \infty$.

Solutions to the Falkner–Skan equation using GHAM were found over $-0.198 < \beta < 5$, generating a range of solutions, with the solutions for the two indicative cases of $\beta = -0.1$ and 1 presented in Figure 1. A representative sample of these results, and comparison results using SHAM and Matlab's BVP4C routine are presented in Table 1, where it can be seen that with the exception of $\beta = -0.195$ there is strong agreement between the solutions generated by all three techniques. To calculate these solutions for a 512 point Chebyshev grid at $\hbar = -1$ for GHAM and SHAM, and a 50 point grid in BVP4C, it took an average of 0.039 , 0.22 and 2.16 seconds per calculation respectively. This disparity between the number of grid points for the homotopy based codes and BVP4C is a product of the latter's significantly slower computational performance.

A more comprehensive comparison of the relationship between β and the work of Cebeci and Keller [3] is shown in Figure 2. With the exception of $\beta = 0.6$, the solutions align to at least 3 significant figures. The disparity between the

Figure 1: Normal flow solutions to the Falkner–Skan equation, and their first and second derivatives, for a) $\beta = 1$ and b) $\beta = -0.1$

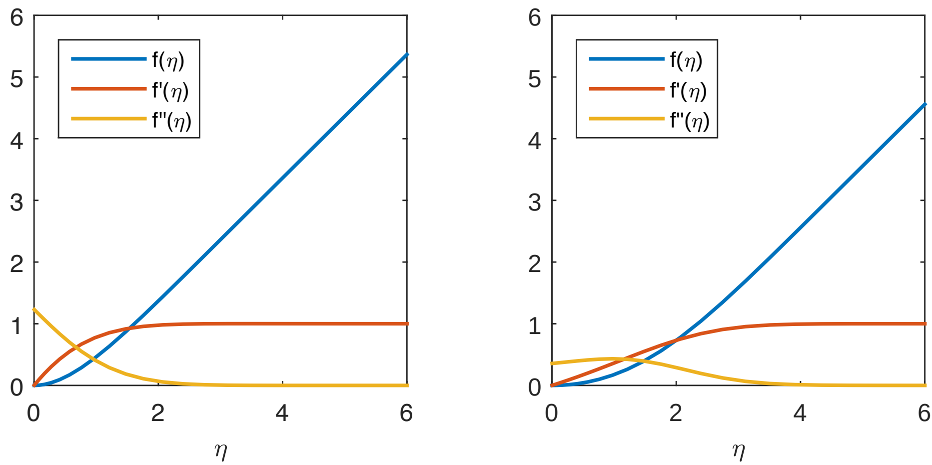
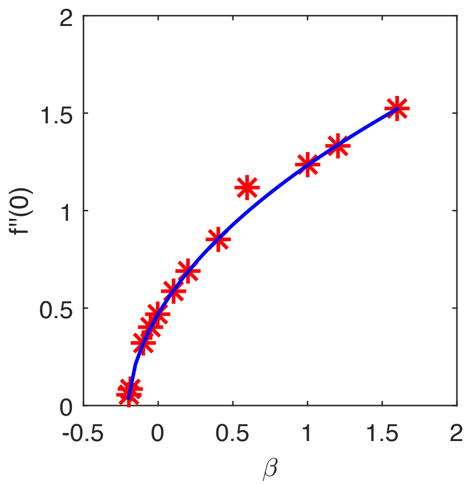


Table 1: Absolute error between calculated values of the skin friction coefficient $f''(0)$ and the reference case of SHAM for both GHAM and Matlab's BVP4C routine, presented for a range of β values.

	β						
	-0.195	0	0.2	0.6	1.0	1.2	2.0
GHAM	5.8×10^{-7}	4.8×10^{-7}	3.8×10^{-7}	3.6×10^{-7}	2.2×10^{-7}	3.1×10^{-7}	3.1×10^{-7}
BVP4C	2.3×10^{-2}	3.5×10^{-6}	9.4×10^{-6}	4.5×10^{-6}	2.8×10^{-5}	1.8×10^{-5}	2.2×10^{-5}

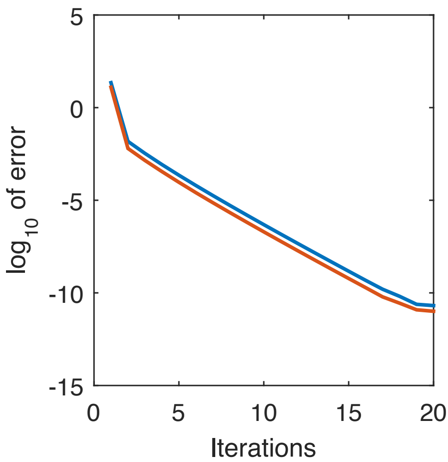
Figure 2: The relationship between β and $f''(0)$. In blue are the solutions calculated using the Homotopy based method, and in red are previously calculated solutions [3].



two appears to be the result of a tabulation mistake in the original paper, as the presented result for $\beta = 0.6$ aligns much more closely with $\beta = 0.8$ from the results calculated using GHAM.

To assess the validity of this new technique, we examined the rate of convergence of the L_2 and L_∞ norms of the residual of the GHAM solutions to the Falkner–Skan equation, as shown in Figure 3. For $\beta = 1$ and at the optimal value of $\hbar$ it took 18 iterations for the residuals to converge to near machine precision, in a manner that suggests spectral convergence.

Figure 3: Convergence of the residual at $\beta = 1$ and at the optimal $\hbar$. The L_2 and L_∞ norm of the residual in blue and red respectively.



5 Conclusion

A new numerical technique based upon the homotopy analysis method has been applied to solving the nonlinear Blasius and Falkner–Skan equations from boundary layer theory. The numerical calculations accurately reproduce the results from previous studies. This was performed using a numerical technique that only required a single matrix inversion of a sparse, mostly-banded matrix—which has a significant impact on the overall computational cost—whilst still exhibiting spectral convergence. These computational advantages translated to a significant improvement in computational speed as compared to both SHAM using Chebyshev collocation matrices and Matlab's BVP4C routine. However, a branch-following algorithm will need to be implemented in order to solve for the dual solutions that exist for $\beta_{min} \leqslant \beta \leqslant 0$.

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