Sdrawkcab scitamehtam: The case for understanding mathematics backwards

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Abstract

Despite the widespread acknowledgement of the need for graduates with quantitative problem-solving skills, many students enter university having relied heavily on pattern recognition techniques for high school mathematics. While these can often lead students to obtaining correct solutions for problems similar to those which they have practised, they do not lead to a deeper understanding of the material and, critically, may not develop more widely-applicable skills. Even when correct solutions are obtained, students can sometimes not understand or explain why their solution is indeed correct. Here, I present an argument in favour of avoiding predictability in question structures and, in particular, asking questions "backwards" to how they might traditionally be asked. Some mathematical topics readily lend themselves to such approaches — the Fundamental Theorem of Calculus tells us that an integral problem is

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inherently linked to an antiderivative problem — whereas other require much more care and subtlety to avoid routine or predictable assessments. I will discuss some preliminary results from a first year probability subject at the University of Technology Sydney (UTS) which suggest that student understanding of material can be increased when they feel that prioritising pattern recognition over problem solving is unlikely to be rewarded with high marks.

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Contents

1 Introduction

The development of quantitative problem-solving skills has been identified by the Office of the Chief Scientist as being a national priority, recognising "the critical role they have in ensuring the continued prosperity of Australia" [3]. Despite this, many students' primary learning strategies for mathematics are founded in pattern recognition techniques [9, 13]. That is, they prioritise learning routines for questions in the belief that those on which they will be assessed may be very similar to those for which they have already seen solutions. This can lead to a situation whereby students may be able to correctly answer questions on material but have minimal understanding of why their remembered routine works or how it might need to be adapted to address less similar problems [11].

There are many reasons for this. Some of these study habits are formed in high school, when pressured to cover a large number of topics quickly with the aim of maximising performance in exams for university admission. From the perspective of university curriculum development, such issues cannot easily be addressed. There are, however, approaches which can minimise the effectiveness of such shallow learning techniques and, I would argue, encourage greater engagement with understanding of mathematics and original enquiry than with rote learning [14, 12].

One major issue which can be readily addressed with a more careful approach to undergraduate learning and assessment is avoiding unnecessary predictability in question formulation. Many leading textbooks will ask all questions on certain topics from the same approach. That is, these questions are all essentially the same routine or argument, albeit with different numbers and perhaps a different context.

For the purpose of this study, I will characterise questions on a mathematical topic into being asked either forwards or backwards. A question asked forwards is defined as one for which a well-defined problem is posed for which there is a single correct answer. Most questions which arise, either in a real-world context or from a textbook problem are forwards problems. These are important in instilling routines of mathematical solution to students. In many cases, however, the solution can be recalled as a pattern of steps, even if the underlying reason or logic behind those steps is not fully comprehended [8]. The other category of question, a backwards problem, is defined as one for which there may be multiple correct answers whereby the student has to demonstrate a much broader understanding of the topic, often having to display comprehension of multiple mathematical ideas to produce a correct solution. As such, backwards questions are, in general, both inverse problems [6] and also open questions [15] and hence blend many of the well-studied benefits [2] of both educational approaches.

There are, of course, topics in a common mathematics syllabus which lend themselves easily to both forwards and backwards approaches. For example, when learning basic differentiation, this is often accompanied by solving simple differential equations by inspection. This is effectively giving students the result of having differentiated a given function and asking them what form the original undifferentiated function must have taken. Many other topics, however, require a more creative and novel approach to problem formulation than is sometimes displayed to ensure that students are encouraged to understand problems more fully than by simply remembering one routine.

2 Discussion

As previously mentioned, one issue is that, for many topics, leading textbooks and available educational resources formulate the vast majority — if not all — problems on a topic in a similar fashion. That is to say that all questions are asked in a forwards direction. As this project draws upon student data from a first year probability subject, I will focus primarily on this discipline area.

Consider the example of assessing a student's understanding of the expectation of a discrete random variable. The vast majority of questions set on this topic will give a valid probability mass function and assess whether or not the student knows that the expectation is calculated via a weighted sum of each possible value of the variable with weights equal to the probabilities of each outcome. Indeed, using the freely-available resources on this topic on the Khan Academy website [1] (as of November 2017), there are ten such questions on this topic and all ten are identical in structure and would be classified here as forwards questions. While such resources are unquestionably useful for developing the most commonly-required routines for solutions, they nonetheless risk students developing shallow understanding which may be reinforced by frequently obtaining correct solutions without understanding the topic completely.

Keeping with the example of assessing a student's understanding of the expectation of a discrete random variable, such forwards questions only assesses whether a student can remember and implement the fact that, for a random variable X with probability mass function f_X, the expectation of this variable $E(X) = \sum_k k f_X(k)$.

Consider the case of two students, one with only superficial understanding of random variables and the other with a more complete understanding. Given a discrete random variable X with well-defined probability mass function

$$f_X(k) = P(X = k) = \begin{cases} 0.8 & k = 1, \\ 0.2 & k = 3, \\ 0 & \text{otherwise,} \end{cases}$$

both students may be able to answer that $E(X) = (1 \times 0.8) + (3 \times 0.2) = 1.4$. If, however, the problem is not well-defined and has a meaningless invalid probability mass function, say,

$$f_X(k) = P(X = k) = \begin{cases} 1.1 & k = 1, \\ -0.1 & k = 3, \\ 0 & \text{otherwise,} \end{cases}$$

the student who fully understands the topic may note that this is meaningless, whereas the student with superficial understanding may well implement the remembered formula and state that $E(X) = (1 \times 1.1) + (3 \times -0.1) = 0.8$.

By reversing the question, and asking the student to give an example of a probability mass function which has a given expectation, an assessor is able to gain considerably deeper insight into a student's understanding of this topic. The student needs not only to display an understanding of how to calculate an expectation, but also what constitutes a valid probability mass function and how to write one correctly. Table 1 compares the understanding required to answer both a forwards and backwards question.

Table 1: Comparison of required topic understanding for answering a question on discrete probability mass functions and the associated expectation of their random variables. The first column is a more traditional question approach and the second column is for the backwards question.

	Question	
	"Given probability mass function f_X, calculate $E(X)$"	"Write down a possible probability mass function f_X such that $E(X) = \ldots$"
Recollection and implementation of $E(X) = \sum_k k f_X(k)$	REQUIRED	REQUIRED
Understanding that random variable is discrete, hence probability mass function, not a continuous density function	IMPLICITLY GIVEN BY QUESTION	REQUIRED
Understanding that probability mass function is always non-negative i.e. $f_X(k) \geqslant 0$ for all $k \in \mathbb{R}$.	IMPLICITLY GIVEN BY QUESTION	REQUIRED
Understanding that probability mass function must sum to one i.e. $\sum_k f_X(k) = 1$	IMPLICITLY GIVEN BY QUESTION	REQUIRED

3 Results

To assess whether backwards questions can more accurately capture students' deeper understanding of mathematical material, a small pilot study was undertaken in a first year undergraduate subject delivered at UTS. For each of five topic areas, 83 students were asked to self-identify their own level of understanding of that topic. Each was asked to self-assess his/her understanding of each topic into one of the four categories, presented as below:

4 = "I believe I understand this topic fully and can correctly answer all or almost all questions on it."

3 = "I believe can correctly answer all or almost all questions on this topic, but don't fully understand the theory."

2 = "I believe I understand some of the theory and can answer some questions correctly."

1 = "I struggle to answer questions on this topic correctly."

Responses for this survey were carried out after students had already attempted two forwards questions (and seen whether or not their work had been marked as correct) but before attempting one additional backwards question.

The most major limitation with this is the small sample size. With 83 student responses for each of 5 topics, with 3 categories for forward questions, 2 for backwards questions and 4 for the self-stated understanding, there are far too few observations to fit all possible models. This would require a 5 way (83 by 5 by 3 by 2 by 4) contingency table, but would have many extremely small expected counts, with some almost certainly zero. For example, unsurprisingly, there were no students who got both forwards questions and the backwards question correct, yet self-identified as having the lowest possible level of understanding of a given topic. Even merging some cells would not be sufficient to properly assess all marginal and conditional dependence models.

Tests for independence between variables were conducted using the G-test [10] with appropriate degrees of freedom. Initially, I collapsed the data on the student and understanding variables and tested for evidence of associations between performance on the forward questions, backwards question and the topic. The saturated model was rejected ($\mathsf{G}^2 \approx 59.98$, $\chi^2_{22}(59.98) < 0.0001$) as were those with independence of forward and backwards results, both controlling ($\mathsf{G}^2 \approx 47.22$, $\chi^2_{10}(47.22) < 0.0001$) and not controlling for topic ($\mathsf{G}^2 \approx 40.06$, $\chi^2_{2}(40.06) < 0.0001$.) Unsurprisingly, this suggested that there was an association between the number of forwards questions a student had answered correctly and whether or not he/she correctly answered the backwards question. Furthermore, the five topics were seemingly of similar difficulty.

When looking for associations between answering backwards questions and understanding, many categories had to be combined. The data were collapsed on the student and topic variables. Analysing the resulting 3 (forwards) by 2 (backwards) by 4 (understanding) contingency table, all models of independence were rejected — the saturated model, all three partial independence models and all three marginal models all gave G^2 statistics corresponding to p-values < 0.005.) This gives that, even when controlling for level of performance on the forwards questions, a student's performance on the backwards question is not independent of his/her topic understanding. Critically, this suggests that a student's ability to answer these conceptually-harder questions correctly is significantly correlated to his/her topic understanding.

It should be noted that the contingency table for this has 4 of its 24 cells with expected counts below 5. It is often suggested that, when using the Pearson statistic for such situations, no more than 20% of cells should have such low counts [7], and ideally many fewer than that. Use of the G-statistic is preferred in this situation [4]. In any case, combining the lowest two categories of understanding, 1 and 2, got around this issue and did not change the conclusion. The original uncombined data have been presented here.

Table 2: Data for number of students organised by their self-stated level of topic understanding (1, 2, 3 or 4), plus by the number of forwards (zero, one or two) questions and the number of backwards (zero or one) questions they answered correctly. In brackets are the expected cell counts under the null model of complete independence (to 2 decimal places.)

Understanding		1		2		3		4	
		Backwards		Backwards		Backwards		Backwards	
		Zero	One	Zero	One	Zero	One	Zero	One
Forwards	Zero	24 (6.60)	0 (2.15)	26 (13.59)	3 (4.43)	23 (37.58)	6 (12.25)	21 (25.19)	3 (8.21)
	One	8 (10.68)	0 (3.48)	18 (22.00)	5 (7.17)	71 (60.81)	13 (19.82)	41 (40.76)	15 (13.28)
	Two	1 (7.62)	0 (2.48)	2 (15.69)	2 (5.11)	50 (43.39)	25 (14.14)	15 (29.08)	31 (9.48)

4 Conclusions

Although the quantitative analyses presented here are relatively small in scope, some striking findings stand out. It has to be acknowledged, of course, that reliance upon students' self-identified understanding is less than ideal, as it is well-documented that there are often inherent biases in such evaluations [5]. Despite these limitations, it is interesting to note that many students are able to accurately tell the difference between having fully mastered material and merely being able to answer questions on it, despite incomplete understanding.

Consider the comparison of the students who believe they can answer most questions correctly, but who may or may not state that they fully understand the underlying mathematical theory. Controlling for the level of performance on the two backwards questions, a student was more than twice as likely to answer the backwards question correctly if he/she had stated level 4 of understanding, compared to only level 3. The implication of this is that, if questions are only ever asked forwards, an assessor may fail to differentiate between a student with complete understanding of the concepts and one who has mastered or memorised a routine, but with incomplete understanding of why he/she is performing the calculations. This is not simply an issue of assessment or fair allocation of marks. If students know that all questions will be asked in a similar fashion, many may not feel incentivised to understand the material fully when partially-understood rote learnt routines may well suffice for maximum marks. Few would believe that this is a desirable outcome. I would argue that, if the primary purpose of assessment is to encourage and reward deeper comprehension of mathematical concepts, then less traditionally structured backwards questions may well achieve this outcome for more students than for those tested purely on forwards questions. The challenge, therefore, is on educators and assessors to be more creative and less predictable in problem formulation and examination.

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