A three-field formulation of the Poisson problem with Nitsche approach

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July 17, 2018

Abstract

We modify a three-field formulation of the Poisson problem with Nitsche approach for approximating Dirichlet boundary conditions. Nitsche approach allows us to weakly impose Dirichlet boundary condition but still preserves the optimal convergence. We use a biorthogonal system for efficient numerical computation and introduce a stabilisation term so that the problem is coercive on the whole space. Numerical examples are presented to verify the algebraic formulation of the problem.

Subject class: 65N30, 65N50

Keywords: Three-field formulation, Poisson problem, Nitsche approach

DOI:10.21914/anziamj.v59i0.12645, © Austral. Mathematical Soc. 2018. Published July 17, 2018, as part of the Proceedings of the 13th Biennial Engineering Mathematics and Applications Conference. ISSN 1445-8810. (Print two pages per sheet of paper.) Copies of this article must not be made otherwise available on the internet; instead link directly to the DOI for this article.

Contents

1 Introduction

The finite element method is a powerful and efficient method to handle complicated geometries and impose the associated boundary conditions. However, in some cases, the treatment of the Dirichlet-type boundary conditions compromise the stability and accuracy of the standard finite element method [12].

In order to relax the Dirichlet boundary condition constraint, we need to modify the standard finite element approach. Generally, we can do this by imposing the Dirichlet boundary condition as a penalty term [1, 2]. One of such methods is Nitsche's method [15], which imposes the Dirichlet boundary condition weakly in the formulation without the need of a Lagrange multiplier. Moreover, compared to other penalty method, Nitsche's method adds the consistency, symmetry and stability terms so that this method can achieve optimal convergence. There are so many applications of Nitsche's method in many areas, such as elasticity [3], interface problems [7], potential flows [10] and plasticity [16].

Nitsche approach for a mixed finite element method for the Poisson problem has been proposed earlier [6, 11] using a two-field formulation, which is not suitable for the approach with a biorthogonal system. In this article, we modify a mixed finite element method, based on the three-field formulation [8], with Nitsche approach to solve a Poisson problem. A similar three-field formulation, known as Hu-Washizu formulation, is popular in linear elasticity field [13]. The three-field formulation allows us to apply a biorthogonal system which leads to a very efficient finite element method. In order to overcome the difficulty of coercivity condition, we introduce a stabilisation term [8] of the associated bilinear form so that it is coercive on the whole space.

2 A Three-field Formulation for Poisson Problem

Sobolev Spaces

Let $V = H^1(\Omega)$ and $L = [L^2(\Omega)]^2$. The Sobolev spaces $H^k(S)$ for $S \subset \Omega$ or $S \subset \Gamma$, and $k \geqslant 0$ are defined in the standard way [5]. We introduce the space $H^{-1/2}(\Gamma)$, the dual space of $H^{1/2}(\Omega)$, with the norm

$$\|\mu\|_{-1/2,\Gamma} = \sup_{z \in H^{1/2}(\Gamma)} \frac{\langle \mu, z\rangle}{\|z\|_{1/2,\Gamma}},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing. For functions $v \in H^1(\Omega)$ with $\Delta v \in L^2(\Omega)$, it holds [2] $\frac{\partial v}{\partial n} \in H^{-1/2}(\Gamma)$ with

$$\left\| \frac{\partial u}{\partial n} \right\|_{-1/2,\Gamma} \leqslant C \left(\|v\|_1 + \|\Delta v\|_0 \right).$$

We will also introduce the mesh-dependent norms

$$\|v\|_{1/2,h}^2 = \sum \frac{1}{h_e} \|v\|_{0,e}^2 \quad \text{for } v \in H^1(\Omega),$$
$$\|z\|_{-1/2,h}^2 = \sum h_e \|z\|_{0,e}^2 \quad \text{for } z \in L^2(\Gamma),$$

and for these norms it holds

$$\langle v, z \rangle \leqslant \|v\|_{1/2,h} \|z\|_{-1/2,h} \quad \text{for} \quad (v, z) \in H^1(\Omega) \times L^2(\Gamma). \tag{1}$$

For the rest of the article, we denote

$$\|u\|_{1,h} = \|u\|_{1,\Omega} + \|u\|_{1/2,h} \quad \text{for } u \in H^1(\Omega).$$

Nitsche Formulation for the Poisson Problem

The mixed formulation is obtained by introducing $\sigma = \nabla u$. Given $f \in L^2(\Omega)$ and $g_D = u|_\Gamma$, the (Nitsche) minimisation problem can be written as

$$\underset{\substack{(u,\sigma)\in V\times L \\ \sigma=\nabla u}}{\operatorname{argmin}} \; \frac{1}{2}\|\sigma\|_{0,\Omega}^2 + \frac{\alpha}{2}\|u - g_D\|_{1/2,h}^2 - \langle \sigma \cdot \mathbf{n}, u - g_D \rangle - \int_\Omega f u \, dx. \tag{2}$$

We write a variational equation for $\sigma = \nabla u$ using the Lagrange multiplier space $M = L$ to obtain the saddle-point problem of the minimisation problem (2). The saddle point formulation is to find $(u, \sigma, \varphi) \in V \times L \times M$ such that

$$\begin{aligned} \tilde{a}\left[(u,\sigma),(v,\tau)\right] + b\left[(v,\tau),\varphi\right] &= \ell(v,\tau), \quad (v,\tau) \in V \times L, \\ b\left[(u,\sigma),\psi\right] &= 0, \qquad\quad \psi \in M, \end{aligned} \tag{3}$$

where

$$\begin{aligned} \tilde{a}\left[(u,\sigma),(v,\tau)\right] &= \int_\Omega \sigma \cdot \tau \, dx + \alpha \langle u, v \rangle_{1/2,h} - \langle \sigma \cdot \mathbf{n}, v \rangle - \langle \tau \cdot \mathbf{n}, u \rangle, \\ b\left[(u,\sigma),\psi\right] &= \int_\Omega (\sigma - \nabla u)\, \psi \, dx, \\ \ell(v,\tau) &= \int_\Omega f v \, dx - \langle \tau \cdot \mathbf{n}, g_D \rangle + \alpha \langle g_D, v \rangle_{1/2,h}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes duality pairing between $H^{1/2}(\Omega)$ and $H^{-1/2}(\Gamma)$.

3 Finite Element Discretisation

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of the polygonal domain Ω. We use the standard linear finite element space $V_h \subset H^1(\Omega)$ defined on the triangulation $\mathcal{T}_h$, where

$$V_h := \{v \in C^0(\Omega) : v|_T \in \mathcal{P}_1(T), T \in \mathcal{T}_h\}.$$

The finite element space for the gradient of the solution is $L_h = [V_h]^2$. Let $\{\rho_1, \rho_2, \ldots, \rho_N\}$ be the finite element basis for V_h. Starting with the standard basis for V_h, we construct a space Q_h spanned by the basis $\{\mu_1, \mu_2, \ldots, \mu_N\}$ so that the basis functions of V_h and Q_h satisfy the biorthogonality condition

$$\int_\Omega \rho_i \mu_j \, dx = c_j \delta_{ij}, \quad c_j \neq 0, \quad 1 \leqslant i, j, \leqslant N,$$

where δ_{ij} is the Kronecker symbol, and c_j a scaling factor. Therefore, the sets of basis functions of V_h and Q_h form a biorthogonal system. The basis functions of Q_h are constructed locally on a reference element $\hat{T}$ so that the basis functions of V_h and Q_h have the same support, and in each element the sum of all the basis functions of Q_h is one [13]. We let $M_h = [Q_h]^2$, thus our problem is to find $(u_h, \sigma_h, \varphi_h) \in V_h \times L_h \times M_h$ such that

$$\begin{array}{lcll} \tilde{a}\left[(u_h, \sigma_h), (v_h, \tau_h)\right] + b\left[(v_h, \tau_h), \varphi_h\right] & = & \ell(v_h, \tau_h), & (v_h, \tau_h) \in V_h \times L_h, \\ b\left[(u_h, \sigma_h), \psi_h\right] & = & 0, & \psi_h \in M_h. \end{array} \tag{4}$$

To show that the saddle-point problem has a unique solution, we need to show that the following well-posedness conditions are satisfied.

1. The linear form $\ell(\cdot)$, the bilinear forms $\tilde{a}[\cdot,\cdot]$ and $b[\cdot,\cdot]$ are continuous on the spaces in which they are defined.

2. The bilinear form $\tilde{a}[\cdot,\cdot]$ is coercive on the kernel space K_h defined as

$$K_h = \{(u_h, \sigma_h) \in V_h \times L_h : b\left[(u_h, \sigma_h), \psi_h\right] = 0, \text{ for all } \psi_h \in M_h\}.$$

3. The bilinear form $b[\cdot,\cdot]$ satisfies the *inf-sup* condition

$$\inf_{\psi_h \in M_h} \sup_{(v_h,\tau_h) \in V_h \times L_h} \frac{b\left[(v_h, \tau_h), \psi_h\right]}{\|v_h, \tau_h\|_{V_h \times L_h} \|\psi_h\|_{0,\Omega}} \geqslant \gamma, \quad \gamma > 0.$$

The mesh-dependent norm for the product space $V_h \times L_h$ is defined by

$$\|u_h, \sigma_h\|^2_{V_h \times L_h} = \|u_h\|^2_{1,h} + \|\sigma_h\|^2_{0,\Omega}, \quad (u_h, \sigma_h) \in V_h \times L_h.$$

With the introduction of M_h, the bilinear form $\tilde{a}[\cdot,\cdot]$ is not coercive on the kernel subspace $K_h \subset V_h \times L_h$. Thus, we need to modify the bilinear form $\tilde{a}[\cdot,\cdot]$ so that it is coercive on the kernel space K_h or even the whole space $V_h \times L_h$. In this article, we modify the bilinear form $\tilde{a}[\cdot,\cdot]$ by adding a stabilisation term so that it is coercive on the whole space $V_h \times L_h$ [8].

$$\begin{aligned} a\left[(u_h, \sigma_h), (v_h, \tau_h)\right] = {} & r \int_\Omega \sigma_h \cdot \tau_h \, dx + (1-r) \int_\Omega \nabla u_h \cdot \nabla v_h \, dx \\ & + \alpha \langle u_h, v_h \rangle_{1/2,h} - \langle \sigma_h \cdot \mathbf{n}, v_h \rangle - \langle \tau_h \cdot \mathbf{n}, u_h \rangle, \end{aligned}$$

for $0 < r < 1$.

We use the following inverse estimate result [12] to show the continuity condition of $\ell(\cdot)$ and also continuity and coercivity condition of the bilinear form $a[\cdot,\cdot]$,

$$C_I \left\| \frac{\partial v_h}{\partial n} \right\|_{-1/2,h} \leqslant \|\nabla v_h\|_{0,\Omega} \text{ for } v_h \in V_h. \tag{5}$$

The continuity of the linear form $\ell(\cdot)$, and the bilinear forms $a[\cdot,\cdot]$ and $b[\cdot,\cdot]$ then follows from the Cauchy-Schwarz inequality, the duality pairing (1) and the inverse estimate (5).

For the coercivity condition, using the inverse estimate (5) and the following Poincare-Friedrichs inequality,

$$\|u_h\|_{1,\Omega}^2 = \|u_h\|_{0,\Omega}^2 + \|\nabla u_h\|_{0,\Omega}^2 \leqslant \left(c^2+1\right)\|\nabla u_h\|_{0,\Omega}^2 \,,$$

we can write

$$\begin{aligned}
&|a\left[(u_h,\sigma_h),(u_h,\sigma_h)\right]| \\
&= r\|\sigma_h\|_{0,\Omega}^2 + (1-r)\|\nabla u_h\|_{0,\Omega}^2 + \alpha\|u_h\|_{1/2,h}^2 - 2\langle\sigma_h\cdot\mathbf{n},u_h\rangle\,, \\
&\geqslant r\|\sigma_h\|_{0,\Omega}^2 + (1-r)\|\nabla u_h\|_{0,\Omega}^2 - 2\|\sigma_h\cdot\mathbf{n}\|_{-1/2,h}\|u_h\|_{1/2,h} + \alpha\|u_h\|_{1/2,h}^2\,, \\
&\geqslant r\|\sigma_h\|_{0,\Omega}^2 + (1-r)\|\nabla u_h\|_{0,\Omega}^2 - \left(\frac{1}{\varepsilon}\|\sigma_h\cdot\mathbf{n}\|_{-1/2,h}^2 + \varepsilon\|u_h\|_{1/2,h}^2\right) \\
&\qquad + \alpha\|u_h\|_{1/2,h}^2\,, \\
&\geqslant \left(r-\frac{1}{\varepsilon C_I}\right)\|\sigma_h\|_{0,\Omega}^2 + (1-r)\|\nabla u_h\|_{0,\Omega}^2 + (\alpha-\varepsilon)\|u_h\|_{1/2,h}^2\,, \\
&\geqslant \left(r-\frac{1}{\varepsilon C_I}\right)\|\sigma_h\|_{0,\Omega}^2 + \frac{1-r}{c^2+1}\|u_h\|_{1,\Omega}^2 + (\alpha-\varepsilon)\|u_h\|_{1/2,h}^2\,, \\
&\geqslant C\|(u_h,\sigma_h)\|_{V_h\times L_h}^2\,,
\end{aligned} \tag{6}$$

where C is the minimum of $\left(r-\frac{1}{\varepsilon C_I}\right)$, $\frac{1-r}{c^2+1}$ and $(\alpha-\varepsilon)$. We also require $\frac{1}{C_I} < r\varepsilon < \varepsilon < \alpha$ and $0 < r < 1$. The choice of the penalty parameter α and stabilisation parameter r is important for the application of this method. However, the choice of the optimal parameter is beyond the scope of this article. From this point forward, we use constant C as a mesh-independent generic constant.

Now the *inf-sup* condition for the bilinear form $b[\cdot,\cdot]$ can be shown as in [8]. Thus we have proved the following theorem.

Theorem 1. *The saddle point problem* (4) *with stabilised* $a[\cdot,\cdot]$ *has a unique solution* $(u_h,\sigma_h,\varphi_h)\in V_h\times L_h\times M_h$. *The solution also satisfies*

$$\|(u_h,\sigma_h)\|_{V_h\times L_h} + \|\varphi_h\|_{0,\Omega} \leqslant C\|f\|_{0,\Omega}\,.$$

4 Algebraic Formulation

In order to present an algebraic formulation of the problem, we use $(x_u, x_\sigma, x_\varphi)$ for the vector representation of the solution $(u_h, \sigma_h, \varphi_h)$ as elements in $V_h \times L_h \times M_h$. Let $\mathbf{S}$, $\mathbf{D}$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{M}$ be the matrices associated with bilinear forms $\int_\Omega \nabla u_h \cdot \nabla v_h \, dx$, $\int_\Omega \tau_h \cdot \varphi_h \, dx$, $\int_\Gamma (\sigma_h \cdot \mathbf{n}) \, u_h \, ds$, $\int_\Omega \nabla v_h \cdot \varphi_h \, dx$, $\sum \frac{1}{h_e} \int_e u_h v_h \, ds$ and $\int_\Omega \sigma_h \cdot \tau_h \, dx$, respectively. For the right hand side, we write $\mathbf{f}_1$ and $\mathbf{f}_2$ to represent the discrete forms of

$$\int_\Omega f \, v_h \, dx \, v_h \, ds + \alpha \, \langle g_D, v_h \rangle_{1/2,h} \, ,$$

and $\langle \tau_h \cdot \mathbf{n}, g_D \rangle$, respectively. Then the algebraic formulation of the problem is

$$\begin{bmatrix} (1-r)\,\mathbf{S} + \alpha \mathbf{C} & -\mathbf{A} & -\mathbf{B} \\ -\mathbf{A}^T & r\mathbf{M} & \mathbf{D} \\ -\mathbf{B}^T & \mathbf{D} & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_u \\ x_\sigma \\ x_\varphi \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ -\mathbf{f}_2 \\ \mathbf{0} \end{bmatrix}, \tag{7}$$

where the first two equations of (7) correspond to first equation of (4) with stabilised $a\,[\cdot,\cdot]$, by setting $\tau_h = 0$ and $v_h = 0$, respectively. After statically condensing out degrees of freedom associated with σ_h and ϕ_h in (7), we arrive at the following system

$$\mathbf{K} x_u = \mathbf{F}$$

where

$$\begin{aligned} \mathbf{K} &= (1-r)\,S + \alpha C - AD^{-1}B^T - BD^{-1}A^T + rBD^{-1}MD^{-1}B^T, \\ \mathbf{F} &= \mathbf{f}_1 - BD\mathbf{f}_2. \end{aligned}$$

Due to the choice of a biorthogonal system, matrix $\mathbf{D}$ is diagonal. As a result, the statically condensed system matrix is sparse.

We introduce two projections $P_h : L^2(\Omega) \to Q_h$ and $P_h^* : L^2(\Omega) \to V_h$ as follows for $v \in L^2(\Omega)$.

$$\int_\Omega (P_h v - v) \cdot \mu_h \, dx = 0, \quad \mu_h \in Q_h, \qquad \int_\Omega (P_h^* v - v) \cdot \varphi_h \, dx = 0, \quad \varphi_h \in V_h.$$

They satisfy the following estimates for $u \in H^1(\Omega)$:

$$\|P_h u - u\|_{0,\Omega} \leqslant Ch\|u\|_{1,\Omega}, \quad \|P_h^* u - u\|_{0,\Omega} \leqslant Ch\|u\|_{1,\Omega}. \tag{8}$$

Furthermore, P_h and P_h^* are stable in L^2-norm [14]. Using this projection, our problem is to find $u_h \in V_h$ such that,

$$A(u_h, v_h) = L(v_h), \quad v_h \in V_h, \tag{9}$$

where

$$\begin{aligned} A(u_h, v_h) &= \int_\Omega P_h(\nabla u_h) \cdot P_h(\nabla v_h)\, dx + \alpha \langle u_h, v_h \rangle_{1/2,h} \\ &\quad - \int_\Gamma (P_h(\nabla u_h) \cdot \mathbf{n})\, v_h\, ds - \int_\Gamma (P_h(\nabla v_h) \cdot \mathbf{n})\, u_h\, ds, \\ L(v_h) &= \int_\Omega f v_h\, dx - \int_\Gamma (P_h(\nabla v_h) \cdot n)\, g_D\, ds + \alpha \langle g_D, v_h \rangle_{1/2,h}. \end{aligned}$$

We also introduce two mesh-dependent norms

$$\begin{aligned} \|u_h\|_h^2 &= \|u_h\|_{1,h}^2 + \|P_h(\nabla u_h)\|_{0,\Omega}^2, & u_h \in V_h, \\ |\!|\!|u|\!|\!|_h^2 &= \|u\|_{1,h}^2 + \|P_h(\nabla u)\|_{0,\Omega}^2 + \|\nabla u \cdot n\|_{-1/2,h}^2, & u \in H^2(\Omega), \end{aligned}$$

so that

$$|A(u, v_h)| \leqslant |\!|\!|u|\!|\!|_h \|v_h\|_h, \quad u \in V \text{ and } v_h \in V_h. \tag{10}$$

We get the following estimate combining the interpolation estimate of Lemma 3.4 of [12] with that of the approximation of P_h:

$$\inf_{v_h \in V_h} |\!|\!|u - v_h|\!|\!|_h \leqslant Ch\|u\|_{2,\Omega}. \tag{11}$$

We then have the following theorem.

Theorem 2. *Let $u_h \in V_h$ be the solution to the problem (9). Suppose that $u \in H^2(\Omega)$ is the solution to the problem (3) then*

$$\|u - u_h\|_h \leqslant Ch\|u\|_{2,\Omega}.$$

Proof: From the coercivity (6) and continuity condition (10),

$$\begin{aligned}\alpha \left\|u_h - v_h\right\|_h^2 &\leqslant A\left(u_h - v_h, u_h - v_h\right),\\ &= A\left(u - v_h, u_h - v_h\right) + A\left(u_h, u_h - v_h\right) - A\left(u, u_h - v_h\right),\\ &= A\left(u - v_h, u_h - v_h\right) + L\left(u_h - v_h\right) - A\left(u, u_h - v_h\right),\\ &\leqslant \left\|\left|u - v_h\right|\right\|_h \left\|u_h - v_h\right\|_h + L\left(u_h - v_h\right) - A\left(u, u_h - v_h\right).\end{aligned}$$

Using $w_h = u_h - v_h$ and divide both sides by $\|w_h\|_h$, we get

$$\alpha \left\|u_h - v_h\right\|_h \leqslant \left\|\left|u - v_h\right|\right\|_h + \frac{L\left(w_h\right) - A\left(u, w_h\right)}{\left\|w_h\right\|_h}.$$

Following exactly as in the proof of Strang's second lemma [4] we get

$$\left\|u - u_h\right\|_h \leqslant C\left(\inf_{v_h \in V_h} \left\|\left|u - v_h\right|\right\|_h + \sup_{w_h \in V_h} \frac{L\left(w_h\right) - A\left(u, w_h\right)}{\left\|w_h\right\|_h}\right). \tag{12}$$

The first term of (12) is estimated using (11). We estimate the second term of (12) using the approximation property of P_h, P_h^* and the fact that $f = -\Delta u$. For the complete proof, reader can refer to [9].

♠

5 Numerical Examples

In this section, we show two numerical examples to verify the convergence rate of our approach. We compute the error in L^2-norm and the rate of convergence for u and σ. We also compute the error in H^1-norm and the rate of convergence for u. We will use Dirichlet boundary conditions for all our examples. Both examples have Dirichlet boundary condition on Γ and are defined on $\Omega = [0,1] \times [0,1]$. For both examples, we set the parameter $\alpha = 10$ and $r = 1/2$.

Example 1

We consider the exact solution

$$u = xy\,(1-x)\,(1-y)\,,$$

for the first example. The errors for this example with Dirichlet boundary conditions are shown in Table 1.

Table 1: Discretisation errors with Dirichlet boundary conditions for example 1

elem	$\|u-u_h\|_{0,\Omega}$		$\|u-u_h\|_{1,h}$		$\|\sigma-\sigma_h\|_{0,\Omega}$	
	error	rate	error	rate	error	rate
8	3.74e-02		1.98e-01		1.73e-01	
32	8.89e-03	2.0742	1.09e-01	0.8654	5.94e-02	1.5444
128	1.92e-03	2.2083	5.53e-02	0.9754	1.81e-02	1.7175
512	4.37e-04	2.1364	2.76e-02	1.0035	5.68e-03	1.6702
2048	1.04e-04	2.0752	1.37e-02	1.0062	1.87e-03	1.6056
8192	2.52e-05	2.0392	6.85e-03	1.0042	6.33e-04	1.5590

Example 2

We consider the exact solution

$$u = e^{x^2+y^2} + y^2\cos(xy) + x^2\sin(xy)\,,$$

for our second example. The errors for this example with Dirichlet boundary conditions are shown in Table 2.

From Tables 1 and 2, we can see that the rate of convergence of errors for u in L^2-norm and $(1,h)$-norm is 2 and 1, respectively, while the rate of convergence of errors for σ in L^2-norm is 1.5. These results are very similar to the result from the three-field formulation for Poisson problem with same examples [8].

Table 2: Discretisation errors with Dirichlet boundary conditions for example 2

elem	$\|u - u_h\|_{0,\Omega}$		$\|u - u_h\|_{1,h}$		$\|\sigma - \sigma_h\|_{0,\Omega}$	
	error	rate	error	rate	error	rate
8	7.36e-01		4.23e+00		2.32e+00	
32	1.50e-01	2.2902	2.10e+00	1.0110	8.56e-01	1.4381
128	3.12e-02	2.2694	1.03e+00	1.0258	2.93e-01	1.5457
512	6.83e-03	2.1915	5.07e-01	1.0231	1.00e-01	1.5494
2048	1.57e-03	2.1179	2.51e-01	1.0149	3.45e-02	1.5384
8192	3.76e-04	2.0661	1.25e-01	1.0085	1.20e-02	1.5263

6 Conclusion

In this article, we describe a mixed finite element method to solve Poisson equation based on Nitsche's method. We add a stabilisation term so that our bilinear form is coercive on the whole space. From numerical examples, we can observe that the error and rate of convergence is very similar to our previous three-field formulation for Poisson problem. Thus we can conclude that this approach works well as an alternative to the standard formulation.

Acknowledgment

The first author is supported by UNIPRS and UNRSC50:50 scholarship for this research. He also thanks University of Newcastle HDR Funding 2017 and ANZIAM Student Support Scheme for the conference support.

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