A numerical solution for moving boundary shallow water flow above parabolic bottom topography

Joe Sampson[1]

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Abstract

A numerical method has been applied to the nonlinear shallow water wave equations for unforced linear frictional flow above parabolic bottom topography and is found to be accurate. This solution involves moving shorelines. The motion decays over time. The numerical scheme used is adapted from the Selective Lumped Mass numerical scheme. The wetting and drying algorithm used in the numerical scheme is different to that in the Selective Lumped Mass scheme. The numerical scheme is finite element in space, using fixed triangular elements, finite difference in time and is explicit. The numerical solution compares well with an analytical solution.

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1 Introduction

We consider the case where the motion of shallow water in a basin is governed by the equations [12]

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \tau U + g\frac{\partial \zeta}{\partial x} = 0\,, \tag{1}$$

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + \tau V + g\frac{\partial \zeta}{\partial y} = 0\,, \tag{2}$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (h+\zeta)U}{\partial x} + \frac{\partial (h+\zeta)V}{\partial y} = 0\,, \tag{3}$$

where $\zeta(x,y,t)$ is the height of the water surface above mean water level, $z=-h(x,y)$ is the bottom surface, $U(x,y,t)$ is the depth averaged velocity component of the water current to the East, $V(x,y,t)$ is the depth averaged velocity component of the water current to the North, g is the acceleration due to gravity, τ is the bottom friction parameter and t is the time. The bottom friction parameter, τ, is considered to be constant, which implies that the bottom friction force varies linearly with velocity. In tidal modelling the bottom friction parameter is usually taken to be proportional to the magnitude of the velocity but occasionally it is accurate to consider it a constant [6]. Equations (1) and (2) are equations of the form, force equals mass times acceleration, where the force terms are friction and hydrostatic pressure, while (3) is a statement of mass conservation.

Assume that the flow is unforced and takes place above parabolic topography, defined by depth

$$h = h_0 \left(1 - \frac{x^2}{a^2}\right) , \tag{4}$$

with h_0 and a constant.

A numerical solution is found. Initial values of the velocity components, U and V, and initial height of the water surface, ζ, are assumed. These initial values are the same as in the analytical solution by Sampson, Easton and Singh [8], briefly discussed in Section 3.

2 Numerical solution

The numerical method used was introduced by Sampson et al. [9] and its results compared against an analytical solution; the numerical method was tested against another analytical solution by Sampson et al. [10]. The article leads on from the previous work to compare the results of the method against a new analytical solution involving linear friction and a dry/wet interface. This article differs from previous work in that the numerical method is verified for a problem involving moving boundary unforced flow, whereas the other articles involved moving boundary forced flow; in addition, this article, unlike previous articles, provides a table of errors for different discretisation sizes, compares the numerical and analytical values of the U-velocity at a given time and the water level, ζ, over time for a given node and compares solutions at a number of y values.

The only other discussions of the numerical solutions of the nonlinear shallow water equations involving moving boundary frictional flow above parabolic bottom topography are by Balzano [1], Holdahl, Holden and Lie [3], Lewis and Adams [5], Peterson et al. [7] and Yoon and Cho [13], who compared numerical solutions of the nonlinear shallow water wave equations with some of the analytical solutions by Thacker [11] for moving boundary frictionless flow above parabolic bottom topography.

The numerical scheme used is adapted from the SLM (Selective Lumped Mass) numerical scheme of Kawahara, Hirano and Tsubota [4]. The SLM method applies the Galerkin weighted residual finite element method to the shallow water equations, with fixed triangular elements, resulting in a nonlinear system of first order differential equations. A modified Euler's method, which is second order accurate, is further modified using selectively lumped mass matrices and applied to these equations, resulting in finite difference equations that are explicit. The finite element method has the advantage over the finite difference method that it can represent non-rectangular domains of flow more accurately. As the shoreline moves over time, in the numerical solution there are some nodes that are wet part of the time and dry part of the time. The wetting and drying algorithm used in the numerical scheme, discussed in detail by Sampson et al. [9], is different to that in the SLM model. In the numerical scheme, a decision on whether a node is dry or wet is made at the end of each time step Δt. Some nodes are initially made wet while others are initially made dry. Some nodes change from dry to wet or wet to dry at the end of a time step, while some nodes remain wet or remain dry. There are wet and dry elements, with an element being wet if all the nodes are wet and dry if at least one node is dry. At any time step or half time step the SLM calculations are made only for elements that were wet at the end of the last time step. At the end of each time step each element that contains only one dry node is tested to determine whether conditions are favourable for wetting that node.

The numerical solution is compared with an analytical solution, developed by Sampson et al. [8], with errors small and reducing with reduced element size.

3 The analytical solution used for comparison

Sampson et al. [8] obtained analytical solutions of the shallow water equations (1), (2) and (3) above the parabolic topography (4), by assuming that

$$U = u_0(t) \quad \text{and} \quad V = 0\,. \tag{5}$$

A solution obtained is

$$u_0(t) = Be^{-\tau t/2}\sin st\,, \tag{6}$$

where B is a constant and where

$$s = \frac{\sqrt{8gh_0 - \tau^2 a^2}}{2a} \tag{7}$$

and

$$\begin{aligned} \zeta(x,t) &= \frac{a^2B^2e^{-\tau t}}{8g^2h_0}\left(-s\tau\sin 2st + \left[\frac{\tau^2}{4} - s^2\right]\cos 2st\right) - \frac{B^2e^{-\tau t}}{4g} \\ &\quad - \frac{e^{-\tau t/2}}{g}\left(Bs\cos st + \frac{\tau B}{2}\sin st\right)x. \end{aligned} \tag{8}$$

Equation (8) implies that at any time t the water surface is a plane. The projection of the moving shorelines on the xy-plane is two parallel straight lines

$$x = \frac{a^2e^{-\tau t/2}}{2h_0 g}\left(-Bs\cos st - \frac{\tau B}{2}\sin st\right) \pm a. \tag{9}$$

4 The analytical solution versus the numerical solution

The numerical solution has been tested against the analytical solution with good agreement between the numerical and analytical solutions.

For the numerical model the values chosen were $h_0 = 10\,\mathrm{m}$, $a = 3000\,\mathrm{m}$, $\tau = 0.001\,\mathrm{s}^{-1}$ and $B = 2\,\mathrm{m\,s}^{-1}$ with the initial values of ζ, U and V set to those of the analytical model. The period of the trigonometric terms in the motion, T, is $1353.49\,\mathrm{s}$. The initial velocity is $0\,\mathrm{m\,s}^{-1}$. The calculation, using a program written in Visual C++, was done over eight periods ($10827.90\,\mathrm{s}$).

Three different triangular meshes were used, each one covering a rectangular region of width $8640\,\mathrm{m}$ in the x direction, but with different breadths in the y direction. The breadth of the region used in the first and coarsest mesh, Mesh 1, is 2160 metres. The breadth of the second mesh, Mesh 2, which is finer than Mesh 1, is 720 metres. The breadth of the finest mesh, Mesh 3, is 240 metres. Each triangle in each mesh is an isosceles right angled triangle. Mesh 1 has 1105 nodes and 2048 elements, Mesh 2 has 3281 nodes and 6144 elements and Mesh 3 contains 9809 nodes and 18432 elements. There is no condition on the solution on any of the boundaries. The SLM scheme involves a selective lumping parameter, s_r. The scheme is restricted by the Courant–Friedrichs–Lewy stability requirement

$$\Delta t \leqslant d_m \Delta x/\sqrt{gh}, \tag{10}$$

where Δx is the smallest space step, Δt is the maximum time step and d_m is a function of s_r, obtained from a table by Goraya [2]. For Mesh 1 $\Delta t = 8.46\,\mathrm{s}$, for Mesh 2 $\Delta t = 1.57$ s and for Mesh 3 $\Delta t = 0.392\,\mathrm{s}$; these values are close to the maximum values according to the stability requirement. For each mesh the values of ζ and U at seven different x values at four different times were calculated for nodes on the base of each mesh, top of each mesh and on a line parallel to the base and midway between the base and the top of each region; for any given x value for a given mesh the value of ζ was the same to $0.01\,\mathrm{m}$ for each of the y values chosen, similarly for U to $0.01\,\mathrm{m\,s}^{-1}$. Table 1 shows the analytical value for water height, ζ, and the difference between the analytical value and numerical value at seven x values along the base of each rectangle at time $T/2 = 6746.747\,\mathrm{s}$ for the three meshes: the finer the mesh the smaller the error; the errors are quite small by Mesh 3.

The values of water elevation, ζ, velocity U and the x-coordinate of the left

Table 1: Errors, $\Delta\zeta$, at time $t = T/2 = 676.747$ s.

x (m)	Analytical Values (m)	Errors (m) Mesh 1	Mesh 2	Mesh 3
-2565	-1.833	-0.051	-0.013	-0.002
-2025	-1.469	0.022	0.009	0.003
-1080	-0.831	-0.035	-0.009	-0.008
0	-0.102	-0.062	-0.013	-0.007
1080	0.626	0.030	-0.005	0.000
2025	1.264	-0.055	-0.001	0.003
2565	1.628	-0.009	0.007	0.004

hand shoreline discussed below are for nodes sitting on a line parallel to the base of the Mesh 3 rectangular region and half way between the base and top of the region.

Figure 1 plots the numerical and analytical x-coordinates of the left hand shoreline as a function of time over two periods. The analytical solution is shown in the diagram as a continuous curve while the numerical solution is a number of points; these points are so close together that they appear to be a number of straight lines parallel to the time axis. As the distance between successive nodes is 15 m, the distance between successive apparent straight lines is 15 m, which means that numerically when the shoreline moves it moves 15 m in one time step. There is good agreement between the analytical and numerical values.

Figure 2 compares the numerical and analytical values for the water level, ζ, against x at time $t = T$. The values are close.

Figure 3 compares the numerical and analytical values for U-velocity, against x at time $t = T/4$. The values are moderately close for most points except for close to the shoreline.

Figure 4 compares the numerical and analytical values of the water surface, ζ, over 3.5 periods at $x = -2685$ m. The values are close.

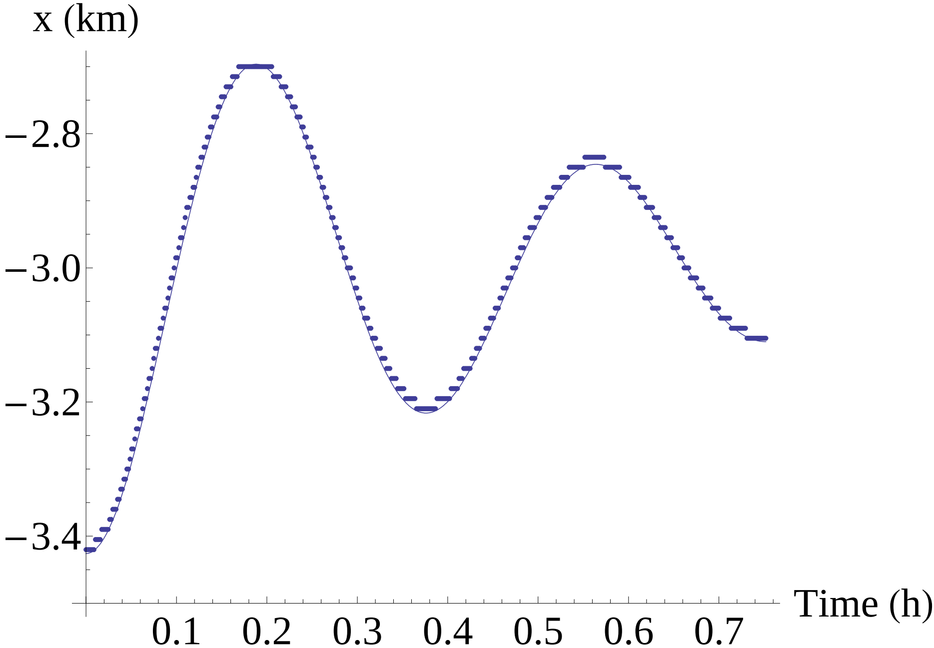

Figure 1: A plot of the numerical and analytical values of the x-coordinate of the left hand shoreline as a function of time over two periods. The analytical solution is a continuous curve while the numerical solution is a number of points.

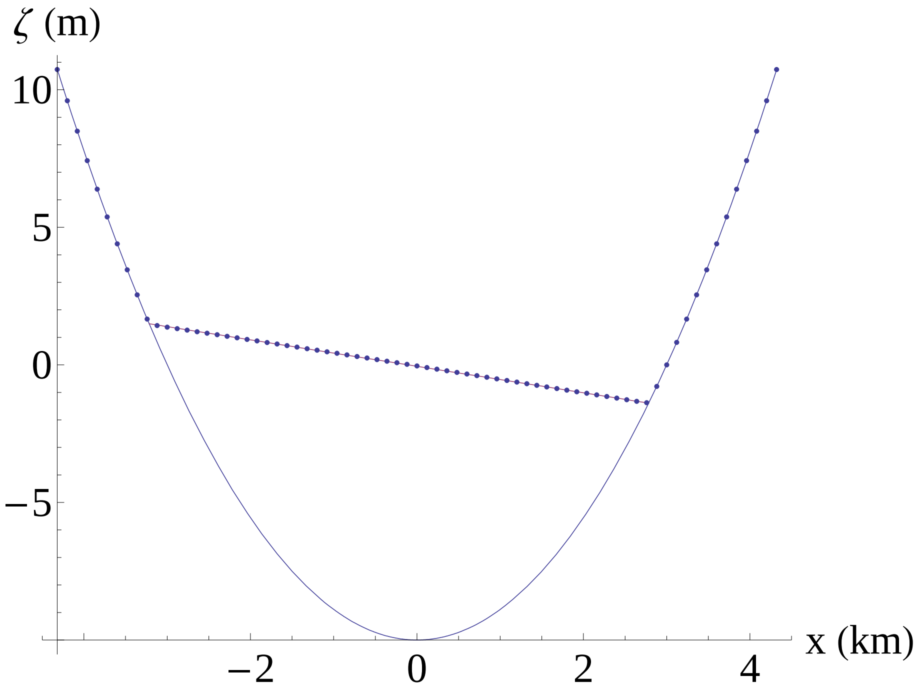

Figure 2: A comparison of the numerical and analytical values of the water surface at time $t = T$. The analytical solution is a continuous line whereas the numerical solution is a series of dots; the results for every eighth node are shown.

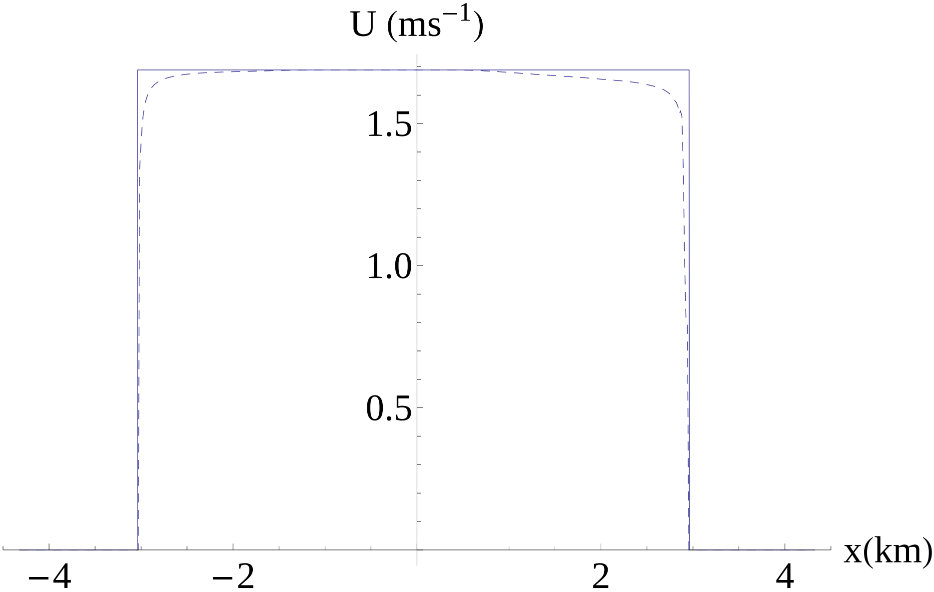

Figure 3: A comparison of the numerical and analytical values of the u-velocity at time $t = T/4$. The analytical solution is a continuous line whereas the numerical solution is a dashed line.

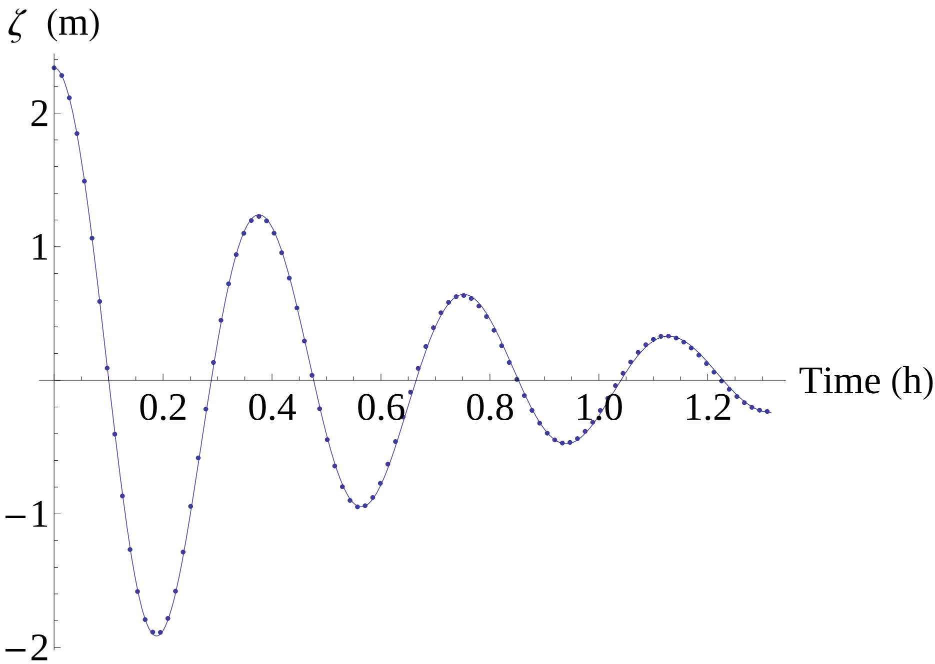

FIGURE 4: A comparison of the numerical and analytical values of the water surface over 3.5 periods at $x = -2685$ m. The analytical solution is a continuous line whereas the numerical solution is a series of dots; the results for every 128 data points are shown.

5 Conclusions

A numerical solution of the nonlinear shallow water wave equations for unforced linear flow above parabolic bottom topography has been found. This solution involves moving shorelines. The motion decays over time. The numerical scheme used is adapted from the SLM numerical scheme. The wetting and drying algorithm used in the numerical scheme is different to that in the SLM scheme. The numerical scheme is finite element in space, using fixed triangular elements, finite difference in time and is explicit. The numerical solution has been compared with an analytical solution with good agreement between the two solutions. The numerical scheme has been found to be accurate and hence would be useful for modelling actual shallow water flow involving moving shorelines, including flow in lakes, storm surges and flow in tidal flats.

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Author address

1. **Joe Sampson**, Mathematics Discipline, Faculty of Engineering and Industrial Sciences, Swinburne University of Technology, Melbourne, Australia.
mailto:jsampson@swin.edu.au