A spectral method to the stochastic Stokes equations on the sphere

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(Received 23 January 2000; revised 16 May 2000)

Abstract

We construct numerical solutions to the stochastic Stokes equations on the unit sphere with additive noise. By characterising the noise as a tangential vector field, the weak formulation is derived and a spectral method is used to obtain a numerical solution. The theory is illustrated through a numerical experiment.

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DOI:10.21914/anziamj.v60i0.13987 gives this article, Published June 26, 2019, as part of the Proceedings of the 18th Biennial Computational Techniques and Applications Conference . ISSN 1445-8810.

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1 Introduction

A Stokes flow is a fluid flow characterised by strong viscous forces and a lack of advection (e.g. lava flow). Accordingly, the Stokes equations are used to model a Stokes flow and generally involve a set of parameters determined by the type of fluid, initial/boundary conditions and domain geometry. In many systems these parameters contain a random fluctuation that cannot be described exhaustively. An effective approach to circumvent this problem is the inclusion of stochastic effects in the Stokes equations to form a deterministic part and a perturbed part represented by random noise.

This work considers the stochastic Stokes problem which has been subject to active research in recent years due to the development of efficient stochastic computational methods [2, 5]. Further, the stochastic Stokes problem is directly applicable to optimal control in fluid dynamics and, more specifically, the movement of microorganisms and the flow of lava in unknown terrain [9, 3, 1]. The stochastic Stokes equations on the unit sphere $\mathbb{S}$ in $\mathbb{R}^3$ are

$$\begin{cases} -\nu\Delta\mathbf{u} + \operatorname{grad} p = \mathbf{f} + \sigma\boldsymbol{W} & \text{in } \mathbb{S}\,, \\ \operatorname{div}\mathbf{u} = 0\,, \end{cases} \tag{1}$$

where $\mathbf{u} = (u^\theta, u^\phi)$ is a tangential vector field on the surface of the unit sphere, grad is the surface gradient, Δ is the Laplace–de Rham operator, $\nu > 0$ is the viscosity, p is the pressure, $\mathbf{f} \in L^2(\mathbb{S})$ is the external force, σ is a positive constant and $\boldsymbol{W}$ is some vector valued random noise on the sphere.

For this problem, a spatial discretization is applied via a spectral method based on divergence free vector spherical harmonics [6]. This is done by first using the Hodge decomposition theorem to define the random tangential vector field on the sphere in terms of curl free and divergence free components. Following this, a bilinear form is defined and a spectral Galerkin method is used to obtain a numerical divergent free solution. In Section 5, a numerical experiment is conducted on a test case for illustration of the solution and convergence.

2 Preliminaries

To allow for a better understanding of the mathematical methods and numerical simulations presented in this work, this first section gives a preliminary overview of relevant operators on the unit sphere and the orthonormal basis used throughout.

2.1 Differential operators on $\mathbb{S}$

For the two-dimensional unit sphere $\mathbb{S} \subset \mathbb{R}^3$, the spherical co-ordinate parametrization

$$\mathbf{x} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,, \quad 0 \leqslant \theta \leqslant \pi\,, \quad 0 \leqslant \phi < 2\pi\,,$$

is employed with the associated orthonormal basis for the tangent space $T\mathbb{S}$

$$\begin{aligned} \boldsymbol{e}^\theta &= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)\,, \\ \boldsymbol{e}^\phi &= (-\sin\phi, \cos\phi, 0)\,, \end{aligned}$$

in order to express the tangential vector field on the surface of the unit sphere $\mathbb{S}$ as

$$\boldsymbol{u} = (u^\theta, u^\phi) = u^\theta \boldsymbol{e}^\theta + u^\phi \boldsymbol{e}^\phi \,.$$

With this convention, the surface divergence and gradient are defined as

$$\begin{aligned} \operatorname{div} \boldsymbol{u} &= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta u^\theta) + \frac{1}{\sin\theta}\frac{\partial u^\phi}{\partial\phi}\,, \\ \operatorname{grad} f &= \frac{\partial f}{\partial\theta}\boldsymbol{e}^\theta + \frac{1}{\sin\theta}\frac{\partial f}{\partial\phi}\boldsymbol{e}^\phi\,. \end{aligned}$$

Since the outward unit normal on the unit sphere at $\boldsymbol{x}$ is itself, the curl operator is decomposed into tangential and normal components. That is, the curl of a normal vector field $\boldsymbol{w} = w\boldsymbol{x}$ and a tangential vector field $\boldsymbol{v} = v_1\boldsymbol{e}^\theta + v_2\boldsymbol{e}^\phi$ are, respectively [7],

$$\begin{aligned} \operatorname{curl} \boldsymbol{w} &= -\boldsymbol{x} \times \operatorname{grad} w\,, \\ \operatorname{curl}_x \boldsymbol{v} &= -\boldsymbol{x} \operatorname{div}(\boldsymbol{x} \times \boldsymbol{v})\,. \end{aligned}$$

The surface diffusion operator Δ (also known as the Laplace–de Rham operator) acting on an arbitrary tangential vector field $\boldsymbol{v}$ is defined as

$$\Delta\boldsymbol{v} = \operatorname{grad}\operatorname{div}\boldsymbol{v} - \operatorname{curl}\operatorname{curl}_x \boldsymbol{v}\,. \tag{2}$$

If the Laplace–de Rham operator is restricted to tangential divergence-free vector fields on the sphere, then it reduces to the Stokes operator

$$A = \operatorname{curl}\operatorname{curl}_x\,. \tag{3}$$

The eigenvalues λ_ℓ and the corresponding eigenvectors of the Stokes operator A for positive integers ℓ are [10]

$$\begin{aligned} \lambda_\ell &= \ell(\ell+1)\,, \\ \boldsymbol{z}_{\ell,m} &= \lambda_\ell^{-1/2} \operatorname{curl} Y_{\ell,m}(\theta,\phi)\,, \quad m = -\ell, \dots, \ell\,, \end{aligned} \tag{4}$$

where $Y_{\ell,m}$ is the spherical harmonic function of degree ℓ and order m. For all $\ell = 1, 2, \dots$ and $m = -\ell, \dots, \ell$,

$$\operatorname{div} \boldsymbol{z}_{\ell,m} = \lambda_\ell^{-1/2} \operatorname{div}\operatorname{curl} Y_{\ell,m} = 0\,. \tag{5}$$

2.2 Orthogonality of $\boldsymbol{z}_{\ell,m}$

The space of square integrable scalar functions and vector fields on $\mathbb{S}$ are defined by $L^2(\mathbb{S})$ and $L^2(T\mathbb{S})$, respectively, with inner products

$$\begin{aligned} \langle v_1, v_2\rangle = \langle v_1, v_2\rangle_{L^2(\mathbb{S})} &= \int_{\mathbb{S}} v_1 \bar{v}_2 \, dS\,, \quad v_1, v_2 \in L^2(\mathbb{S})\,. \\ \langle \boldsymbol{v}_1, \boldsymbol{v}_2\rangle = \langle \boldsymbol{v}_1, \boldsymbol{v}_2\rangle_{L^2(T\mathbb{S})} &= \int_{\mathbb{S}} \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 \, dS\,, \quad \boldsymbol{v}_1, \boldsymbol{v}_2 \in L^2(T\mathbb{S})\,. \end{aligned}$$

The norms for the spaces $L^2(\mathbb{S})$ and $L^2(T\mathbb{S})$ are denoted by $\|\cdot\|$. In the Sobolev spaces $H^\mu(\mathbb{S})$ and $H^\mu(T\mathbb{S})$ with $\mu > 0$ the norms in the scalar and vector valued function are denoted by $\|\cdot\|_{H^\mu(\mathbb{S})}$ and $\|\cdot\|_{H^\mu(T\mathbb{S})}$, respectively. In particular,

$$\|\boldsymbol{v}\|^2_{H^\mu(T\mathbb{S})} = \|\boldsymbol{v}\|^2 + \left\|(-\Delta)^{\mu/2}\boldsymbol{v}\right\|^2\,. \tag{6}$$

Throughout this article, we identify a normal vector field $\boldsymbol{w}$ with a scalar field w via $\boldsymbol{w} = \boldsymbol{x}w$. Hence for two normal vector fields $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$,

$$\langle \boldsymbol{w}_1, \boldsymbol{w}_2\rangle := \langle w_1, w_2\rangle_{L^2(\mathbb{S})}\,, \quad \boldsymbol{w}_1 = \boldsymbol{x}w_1\,, \quad \boldsymbol{w}_2 = \boldsymbol{x}w_2\,, \quad w_1, w_2 \in L^2(\mathbb{S})\,. \tag{7}$$

Using these identities [7]

$$\begin{aligned} \langle \operatorname{grad} v\,, \boldsymbol{v}\rangle_{L^2(T\mathbb{S})} &= -\langle v, \operatorname{div} \boldsymbol{v}\rangle_{L^2(\mathbb{S})}\,, \\ \langle \operatorname{curl} \operatorname{curl}_x \boldsymbol{w}\,, \boldsymbol{z}\rangle_{L^2(T\mathbb{S})} &= \langle \operatorname{curl}_x \boldsymbol{w}\,, \operatorname{curl}_x \boldsymbol{z}\rangle_{L^2(\mathbb{S})}\,. \end{aligned} \tag{8}$$

The vector fields $\{\boldsymbol{z}_{\ell,m} : \ell = 1, 2, \ldots; m = -\ell, \ldots, \ell\}$ form an orthonormal set in $L^2(T\mathbb{S})$ since

$$\begin{aligned} \langle \boldsymbol{z}_{\ell,m}, \boldsymbol{z}_{L,M}\rangle &= \left\langle \lambda_\ell^{-1/2} \operatorname{curl} Y_{\ell,m}\,, \lambda_L^{-1/2} \operatorname{curl} Y_{L,M}\right\rangle \\ &= \left\langle \lambda_\ell^{-1/2} \operatorname{curl}_x \operatorname{curl} Y_{\ell,m}\,, \lambda_L^{-1/2} Y_{L,M}\right\rangle \\ &= \left\langle \lambda_\ell^{-1/2} (-\Delta Y_{\ell,m})\,, \lambda_L^{-1/2} Y_{L,M}\right\rangle \\ &= \left\langle \lambda_\ell^{-1/2} \lambda_\ell Y_{\ell,m}\,, \lambda_L^{-1/2} Y_{L,M}\right\rangle = \delta_{\ell,L}\delta_{m,M}\,, \end{aligned}$$

where the orthogonality of the spherical harmonic function was utilized to obtain the Kronecker deltas.

3 Random tangential vector fields on $\mathbb{S}$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $T\mathbb{S}$ be the space of tangential vector fields on the sphere surface $\mathbb{S}$. A measurable function $\mathbb{S} \times \Omega \to T\mathbb{S}$ is called a real random vector field on the sphere.

The Hodge decomposition theorem states that any smooth tangential field on $\mathbb{S}$ can be decomposed into the sum of curl free and divergence free components [4]. That is,

$$C^\infty(T\mathbb{S}) = C^\infty(T\mathbb{S}; \mathrm{grad}) \oplus C^\infty(T\mathbb{S}; \mathrm{curl}) , \tag{9}$$

where

$$\begin{aligned} C^\infty(T\mathbb{S}; \mathrm{grad}) &= \{\mathrm{grad}\, \psi : \psi \in C^\infty(\mathbb{S})\} , \\ C^\infty(T\mathbb{S}; \mathrm{curl}) &= \{\mathrm{curl}\, \psi : \psi \in C^\infty(\mathbb{S})\} . \end{aligned} \tag{10}$$

A random tangential vector field on the sphere is thus written as

$$\boldsymbol{W}(\boldsymbol{x}, \omega) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell,m}(\omega) \boldsymbol{y}_{\ell,m}(\boldsymbol{x}) + b_{\ell,m}(\omega) \boldsymbol{z}_{\ell,m}(\boldsymbol{x}) , \tag{11}$$

where $a_{\ell,m}$ and $b_{\ell,m}$ are random coefficients, $\boldsymbol{z}_{\ell,m}$ is the divergence free vector field defined in (4) and the curl free vector field is

$$\boldsymbol{y}_{\ell,m} = \lambda_\ell^{-1/2}\, \mathrm{grad}\, Y_{\ell,m} .$$

The Fourier truncation of $\boldsymbol{W}$ to degree N is defined by

$$\boldsymbol{W}_N(\boldsymbol{x}, \omega) = \sum_{\ell=1}^{N} \sum_{m=-\ell}^{\ell} [a_{\ell,m}(\omega) \boldsymbol{y}_{\ell,m}(\boldsymbol{x}) + b_{\ell,m}(\omega) \boldsymbol{z}_{\ell,m}(\boldsymbol{x})] , \tag{12}$$

and this is used when implementing the numerical experiments.

4 Weak formulation

For $\mu \geqslant 0$, let $C^{\infty,\mu}(T\mathbb{S};\mathrm{curl})$ denote the closure of $C^{\infty}(T\mathbb{S};\mathrm{curl})$ in the $H^{\mu}(T\mathbb{S})$ norm. The space [7]

$$V = \text{closure of } C^{\infty}(T\mathbb{S};\mathrm{curl}) \text{ in } H^{1}(T\mathbb{S}) \tag{13}$$

is crucial for uncoupling the unknown velocity and pressure terms in (1).

Using the identities in (8) with the divergent free definition of V, we obtain the bounded, symmetric, and coercive bilinear weak formulation of the spherical surface Stokes equations

$$\nu \langle \mathrm{curl}_x \boldsymbol{u} , \mathrm{curl}_x \boldsymbol{v} \rangle = \langle \mathbf{f} + \sigma \boldsymbol{W} , \boldsymbol{v} \rangle , \quad \boldsymbol{v} \in V . \tag{14}$$

The existence and uniqueness of the solution $\boldsymbol{u} \in V$ of (14) follows from the Lax–Milgram theorem, for all $\mathbf{f} \in V^*$ the adjoint of V.

For a spectral Galerkin method we define

$$V_N := \mathrm{span}\{\boldsymbol{z}_{\ell,m} : \ell = 1, 2, \dots, N\}.$$

The Ritz–Galerkin approximation problem for (14) is to find $\boldsymbol{u}_N \in V_N$ so that

$$\nu \langle \mathrm{curl}_x \boldsymbol{u}_N, \mathrm{curl}_x \boldsymbol{v} \rangle = \langle \mathbf{f} + \sigma \boldsymbol{W} , \boldsymbol{v} \rangle , \quad \boldsymbol{v} \in V_N . \tag{15}$$

Since $\boldsymbol{u}_N \in V_N$, let

$$\boldsymbol{u}_N = \sum_{\ell=1}^{N} \sum_{m=-\ell}^{\ell} c_{\ell,m} \boldsymbol{z}_{\ell,m} ,$$

and choosing $\boldsymbol{v} = \boldsymbol{z}_{LM}$ as the basis functions in (15) gives

$$\sum_{\ell=1}^{N} \sum_{|m|\leqslant\ell} \nu c_{\ell,m} \langle \mathrm{curl}_x \boldsymbol{z}_{\ell,m} , \mathrm{curl}_x \boldsymbol{z}_{L,M} \rangle = \langle \mathbf{f} + \sigma \boldsymbol{W} , \boldsymbol{z}_{LM} \rangle . \tag{16}$$

From the properties of the Laplace–de Rham operator in (2) and the identities in (8),

$$\begin{aligned} \langle \operatorname{curl}_x \boldsymbol{z}_{\ell,m} , \operatorname{curl}_x \boldsymbol{z}_{L,M} \rangle &= \langle \operatorname{curl} \operatorname{curl}_x \boldsymbol{z}_{\ell,m} , \boldsymbol{z}_{L.M} \rangle \\ &= \lambda_\ell \langle \boldsymbol{z}_{\ell,m} , \boldsymbol{z}_{L,M} \rangle \,, \end{aligned}$$

and since $\{\boldsymbol{z}_{\ell,m}\}$ is an orthonormal basis, from (16) we obtain

$$c_{\ell,m} = \frac{1}{\nu\lambda_\ell} \langle \mathbf{f} + \sigma \boldsymbol{W} , \boldsymbol{z}_{\ell,m} \rangle \,. \tag{17}$$

5 Numerical experiments

The spectral Galerkin method on the Stokes equations with additive noise is now demonstrated through a test case vector field. This is done by first utilizing the derivatives of the spherical harmonic functions $Y_{\ell,m}$ to obtain the orthonormal basis $\{\boldsymbol{z}_{\ell,m}\}$ in $L^2(T\mathbb{S})$. The solution of the test case $\boldsymbol{u}_N$ is then obtained and illustrated in the parameter space. Lastly, the convergence of the truncated solution is shown graphically.

5.1 Computing the orthonormal Basis

The orthonormal set $\{\boldsymbol{z}_{\ell,m}\}$ in $L^2(T\mathbb{S})$ is defined as

$$\boldsymbol{z}_{\ell,m} = \lambda_\ell^{-1/2} \left(\frac{1}{\sin\theta} \frac{\partial Y_{\ell,m}}{\partial \phi} \boldsymbol{e}^\theta - \frac{\partial Y_{\ell,m}}{\partial \theta} \boldsymbol{e}^\phi \right), \quad m = -\ell, \dots, \ell \,, \tag{18}$$

where $Y_{\ell,m}$ is the spherical harmonic function of degree ℓ and order m. Varshalovich [10] defines the derivatives of the spherical harmonic function in the definition of $\boldsymbol{z}_{\ell,m}$.

5.2 Test Case

In solving equation (1) we use a test case adapted from Narcowich et al. [8] to examine the accuracy of the spectral method and the convergence of the solution. Here, $\mathbf{u} \in V \cap H^\alpha(T\mathbb{S})$ with $\alpha > 2$ is an infinitely smooth vector field motivated by the atmospheric low pressure flow field at a point $\mathbf{x}_c$ with $\theta_c = \pi/4$, $\phi_c = 0$. A stream function $\psi \in H^{\alpha+1}(\mathbb{S})$ is used to define the divergence free vector field

$$\mathbf{u}(\mathbf{x}) = \frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}\mathbf{e}^\theta - \frac{\partial\psi}{\partial\theta}\mathbf{e}^\phi$$

which satisfies $\mathbf{u} \in V \cap H^\alpha(T\mathbb{S})$.

The stream function is chosen to be

$$\psi(\mathbf{x}) = \tfrac{2}{3}\sin^{15}\theta - \psi_{\mathbf{x}_c}(\mathbf{x}), \quad \psi_{\mathbf{x}_c}(\mathbf{x}) = \exp\left(-[8\rho_{\mathbf{x}_c}(\mathbf{x})]^2\right),$$

where $\rho_{\mathbf{x}_c}(\mathbf{x}) = \arccos(\mathbf{x}\cdot\mathbf{x}_c)$ is the geodesic distance from $\mathbf{x}_c$ to $\mathbf{x}$. The forcing field $\mathbf{f}$ is chosen so that the divergent free vector field $\mathbf{u}$ is the unique solution of the Stokes equations (1) with $\nu = 1$. Substituting this into equation (17) with the truncated noise $\mathbf{W}_N$ yields the simplified expression

$$c_{\ell,m} = \frac{1}{\lambda_\ell}\langle \mathbf{f}, \mathbf{z}_{\ell,m}\rangle + \frac{\sigma}{\lambda_\ell} b_{\ell,m}, \tag{19}$$

with $\ell = 1, 2, \ldots$, $m = -\ell, \ldots, \ell$ and $b_{\ell,m}$ chosen to be random variables with a standard normal distribution of mean zero and variance $\ell^{-\beta}$ for $\beta > 2$, to ensure the noise has finite variance. To compute the inner product in (19), a Gauss–Legendre quadrature is used, that is

$$\langle \mathbf{f}, \mathbf{z}_{\ell,m}\rangle = \int_S \mathbf{f}\cdot\mathbf{z}_{\ell,m}\, dS \approx \sum_{j=1}^{M} w_j \frac{\pi}{M}\sum_{k=1}^{2M} \mathbf{f}(\theta_j, \phi_k)\cdot\mathbf{z}_{\ell,m}(\theta_j, \phi_k),$$

where M is the number of quadrature points and w_j are the Gauss–Legendre weights. For each value N, the average mean square error is computed over a

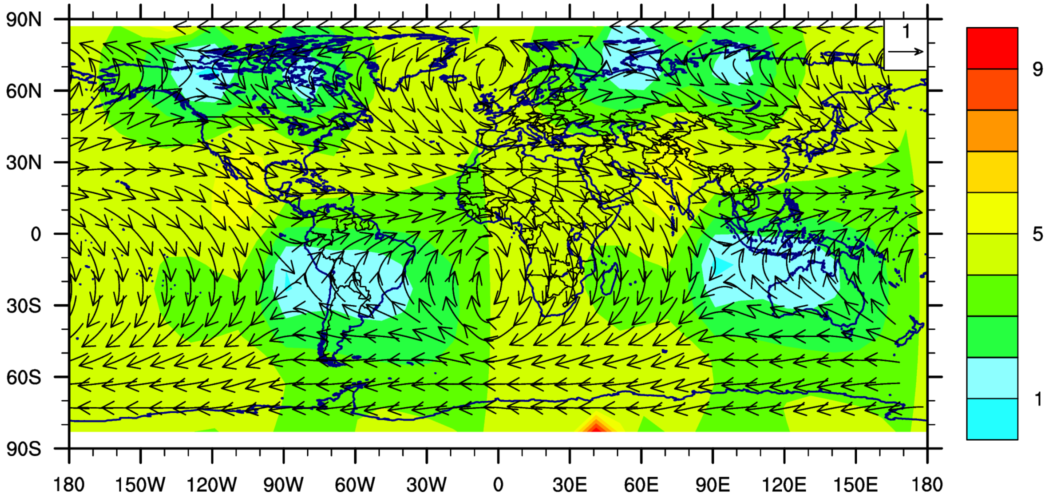

Figure 1: A divergence free random vector field on the unit sphere.

sample of 100 random solutions $\mathbf{u}_N$, that is

$$\mathbf{E}\|\mathbf{u}_N - \mathbf{u}\| \approx \frac{1}{100}\sum_{j=1}^{100} \|\mathbf{u}_N^{(j)} - \mathbf{u}^{(j)}\|\,.$$

Figure 1 illustrates a tangential vector field on the unit sphere. The quiver arrows are normalised to be of equal length to highlight the divergence free nature of the vector field. Consequently the low pressure cells are less prominent in the upper and lower middle of the image due to other weaker cells in the light aqua regions. The color bar gives the normalized value of the tangential vector field at different positions.

Figure 2 plots the average mean square error between the exact solution and its approximation at each truncation level N. As N increases, the error converges to machine error.

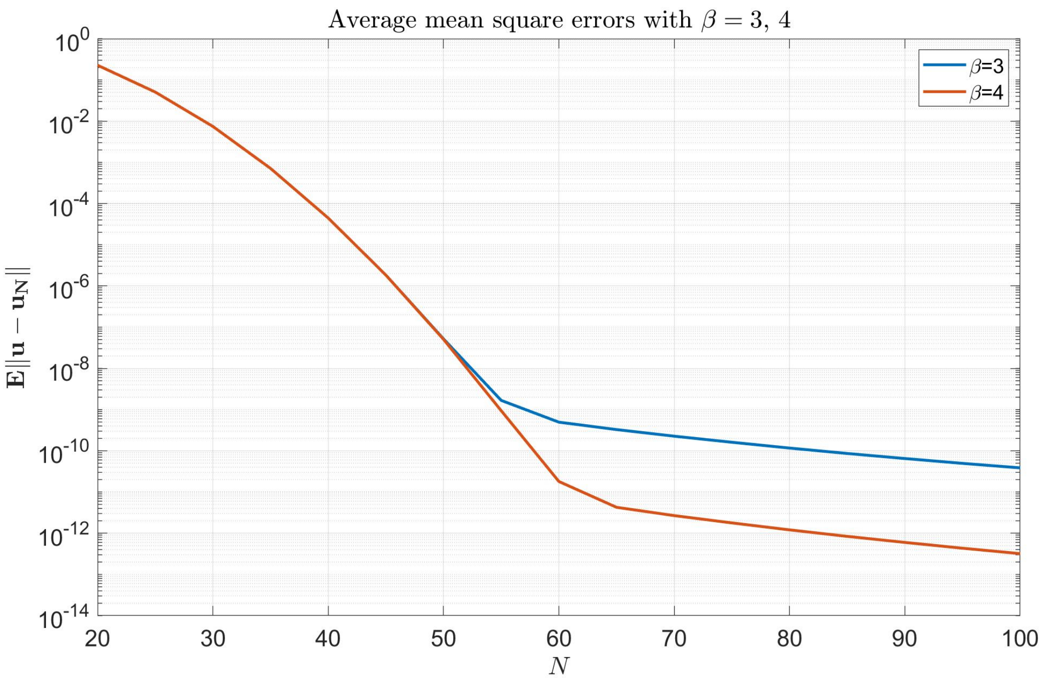

Figure 2: The average mean square errors in L^2 are plotted in increments of five from $N = 20$ to $N = 100$.

6 Conclusion

A spectral method based on divergence free vector spherical harmonics was applied to obtain numerical solutions to the stochastic Stokes equation. This approach is distinguished from existing methods by treating the stochastic Stokes equations on the sphere, defining the random noise as a random tangential vector field on the sphere and solving the resultant bilinear weak form via the spectral Galerkin method. A numerical experiment was conducted to illustrate the solution as a divergent free vector field on the sphere surface $\mathbb{S}$. The average error between the exact solution and numerical solutions was shown to converge to machine error with increasing N.

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