Numerical solutions to a boundary integral equation with spherical radial basis functions

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(Received 15 August 2008; revised 23 October 2008)

Dedicated to Professor Ian H. Sloan on the occasion of his 70th birthday.

Abstract

The Laplace equation in the exterior of the unit sphere with a Dirichlet boundary condition arises from geodesy, oceanography and meteorology. This problem is reformulated into a weakly singular integral equation on the sphere. We study the use of spherical radial basis functions to find approximate solutions to this integral equation using collocation methods. Experiments with data collected by a NASA satellite are performed to clarify the method. Our results illustrate how scattered data can be handled when solving boundary value problems in the exterior of the sphere.

http://anziamj.austms.org.au/ojs/index.php/ANZIAMJ/article/view/1464 gives this article, Published November 10, 2008. ISSN 1446-8735. (Print two pages per sheet of paper.)

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1 Introduction

The Dirichlet problem in the exterior of the unit sphere (which has applications in geodesy, oceanography, and meteorology [1, 8]) is known to be equivalent to a weakly singular integral equation on the sphere; see Section 2. Demands for efficient solutions to this equation are increasing when the Dirichlet boundary condition is given as scattered data collected by satellites. We introduce the use of *spherical radial basis functions* to find an approximate solution to this integral equation using collocation methods.

The use of spherical radial basis functions results in meshless methods which, over the past few years, became more popular [9, 10]. These methods are alternative to finite element methods. The advantage of spherical radial basis functions is in the construction of the finite dimensional subspace; it is independent of the dimension of the geometry. In particular, when scattered data are given, the data can be used as centres to define the spherical radial basis functions without resorting to interpolation.

Morton and Neamtu [2] use spherical radial basis functions to solve more general pseudo-differential equations on the sphere, with more emphasis on operators of positive orders. Their method is practically a Galerkin method, namely, it may require a double integral over the sphere if the equation is written in form of a boundary integral equation.

We propose to use a set of radial basis functions (see Section 3) which is different from that used by Morton and Neamtu [2]. Our method turns out to be a genuine collocation method, the advantage of which is a simpler way to compute the approximate solution (24) in Section 4. In the implementation, we are particularly interested in data collected by NASA satellite MAGSAT. These data points are used as centres to define radial basis functions. This is an advantage of our method compared to the finite element method, which requires a triangulation of the sphere using these points as nonstructured grid points. Error analysis for the approximation (of more general pseudo-differential equations of negative orders) will be reported elsewhere [6].

2 The problem

2.1 The Dirichlet problem

Throughout this article we denote $\mathbb{S} := \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\}$, $\mathbb{B} := \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| < 1\}$, and $\mathbb{B}_e := \mathbb{R}^3 \setminus \bar{\mathbb{B}}$. Here, $\|\mathbf{x}\|$ denotes the Euclidean norm in $\mathbb{R}^3$. We consider the Laplace equation with the Dirichlet boundary condition and the vanishing condition at infinity:

$$\Delta U = 0 \quad \text{in } \mathbb{B}_e \,, \tag{1}$$

$$u = u_D \quad \text{on } \mathbb{S}\,, \tag{2}$$
$$u(x) = O(1/\|x\|) \quad \text{as } \|x\| \to \infty\,, \tag{3}$$

where Δ denotes the Laplacian operator and u_D is a given function on $\mathbb{S}$.

In Subsection 2.2, we suggest a solution technique in which the boundary value problem (1)–(3) is reformulated into a boundary integral equation and the solution of which is approximated by spherical radial basis functions.

2.2 Boundary integral equation

For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{S})$ is defined as usual [5, e.g.]. Let $\mathcal{S}$ and $\mathcal{D}$ denote, respectively, the single layer and double layer potentials, namely,

$$\mathcal{S}v(x) = \frac{1}{4\pi}\int_{\mathbb{S}} v(y)\frac{1}{\|x-y\|}\,d\sigma_y\,,$$
$$\mathcal{D}v(x) = \frac{1}{4\pi}\int_{\mathbb{S}} v(y)\frac{\partial}{\partial \nu_y}\frac{1}{\|x-y\|}\,d\sigma_y\,,$$

for $x \notin \mathbb{S}$, where $\partial/\partial\nu$ denotes the normal derivative with respect to the outward normal vector ν to the sphere. Associated with these potentials, we define the following boundary integral operators

$$Sv(x) = \frac{1}{4\pi}\int_{\mathbb{S}} v(y)\frac{1}{\|x-y\|}\,d\sigma_y\,,$$
$$Dv(x) = \frac{1}{4\pi}\int_{\mathbb{S}} v(y)\frac{\partial}{\partial \nu_y}\frac{1}{\|x-y\|}\,d\sigma_y\,,$$

for $x \in \mathbb{S}$. The following formulae are known [5, Theorem 3.1.2]

$$(\mathcal{S}v)|_{\mathbb{S}} = Sv \quad \text{and} \quad (\mathcal{D}v)|_{\mathbb{S}}^{\pm} = \pm\frac{1}{2}v + Dv\,, \tag{4}$$

where $+$ denotes the limit taken from the exterior domain, and $-$ denotes the limit taken from the interior domain.

If $U \in H^1_{loc}(\mathbb{R}^3 \setminus \mathbb{S})$ satisfies $\Delta U = 0$ in $\mathbb{R}^3 \setminus \mathbb{S}$ and $U(\mathbf{x}) = O(1/\|\mathbf{x}\|)$ as $\|\mathbf{x}\| \to \infty$, then for $\mathbf{x} \notin \mathbb{S}$ the following representation formula holds

$$U(\mathbf{x}) = \mathcal{S}([\partial_\nu U])(\mathbf{x}) - \mathcal{D}([U])(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathbb{R}^3 \setminus \mathbb{S},$$

where $[v] := v^- - v^+$. In particular, a solution to (1) satisfies

$$U(\mathbf{x}) = \mathcal{D}(U)(\mathbf{x}) - \mathcal{S}(\partial_\nu^+ U)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{B}_e. \tag{5}$$

By taking the limit on both sides of (5) as $\mathbf{x}$ approaches a point on $\mathbb{S}$, and using (2) and (4) we obtain

$$U_D(\mathbf{x}) = \frac{1}{2}U_D(\mathbf{x}) + DU_D(\mathbf{x}) - S(\partial_\nu^+ U)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S},$$

implying

$$S(\partial_\nu^+ U)(\mathbf{x}) = -\frac{1}{2}U_D(\mathbf{x}) + DU_D(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}. \tag{6}$$

Therefore, solving the Dirichlet problem (1)–(3) is equivalent to solving the weakly singular integral equation

$$Su = f \quad \text{on } \mathbb{S}, \quad \text{where } f = -\frac{1}{2}U_D + DU_D. \tag{7}$$

Due to (5) and (6), the solution U of (1)–(3) is computed from the solution u of (7) by $U = \mathcal{D}U_D - \mathcal{S}u$.

In Subsection 2.3, we write the operator S in terms of spherical harmonics.

2.3 Representation in terms of spherical harmonics

A spherical harmonic of order ℓ on $\mathbb{S}$ is the restriction to $\mathbb{S}$ of a homogeneous harmonic polynomial of degree ℓ in $\mathbb{R}^3$. The space of all spherical harmonics of order ℓ is the eigenspace of the Laplace–Beltrami operator $\Delta_{\mathbb{S}}$ corresponding to the eigenvalue $\lambda_\ell = -\ell(\ell+1)$. The dimension of this space being $2\ell+1$ (see for example the book by Müller [3]), one may choose for it an orthonormal basis $\{Y_{\ell,m}\}_{m=-\ell}^{\ell}$. The collection of all the spherical harmonics $Y_{\ell,m}$, $m = -\ell, \dots, \ell$ and $\ell = 0, 1, \dots$, forms an orthonormal basis for $L^2(\mathbb{S})$. Let (r, θ, φ) be the spherical coordinates of a point $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, where $r = \|\mathbf{x}\|$, and θ and φ are the polar and azimuthal angles. For any function $v \in L^2(\mathbb{S})$, its associated Fourier series,

$$v = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \hat{v}_{\ell,m} Y_{\ell,m}(\theta, \varphi) \quad \text{where} \quad \hat{v}_{\ell,m} = \int_{\mathbb{S}} v(\theta, \varphi) \overline{Y_{\ell,m}}(\theta, \varphi)\, d\sigma\,,$$

converges in $L^2(\mathbb{S})$. Here $d\sigma$ is the element of surface area. Müller [3] gives more details on spherical harmonics.

In order to derive the representation of S in terms of spherical harmonics, we introduce the *interior* Dirichlet problem, namely, equation

$$\Delta U = 0 \quad \text{in } \mathbb{B}\,, \tag{8}$$

together with condition (2). For this section only, we denote by U_{int} the solution to (8) and (2), and by U_{ext} the solution to (1)–(3).

Similarly to (6), we now have

$$S(\partial_\nu^- U_{\text{int}})(\mathbf{x}) = \frac{1}{2} U_D(\mathbf{x}) + D U_D(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}\,. \tag{9}$$

If the Dirichlet data U_D has an expansion as a sum of spherical harmonics

$$U_D(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \widehat{(U_D)}_{\ell,m} Y_{\ell,m}(\theta, \varphi),$$

then [5]

$$U_{\text{int}}(r, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} r^\ell \widehat{(U_D)}_{\ell,m} Y_{\ell,m}(\theta, \varphi), \tag{10}$$

and

$$u_{\text{ext}}(r,\theta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{r^{\ell+1}}\widehat{(u_D)}_{\ell,m}Y_{\ell,m}(\theta,\varphi). \tag{11}$$

For any given u_D defined on $\mathbb{S}$, by using (10) and (11) we define the interior and exterior Dirichlet-to-Neumann operators by

$$T_{\text{int}} : u_D \mapsto u_{\text{int}} \mapsto \partial_\nu^- u_{\text{int}} \quad \text{and} \quad T_{\text{ext}} : u_D \mapsto u_{\text{ext}} \mapsto \partial_\nu^+ u_{\text{ext}}\,.$$

By differentiating (10) and (11) with respect to r (that is with respect to the normal vector), and setting $r = 1$ we obtain

$$\begin{aligned} T_{\text{int}}u_D &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\ell\widehat{(u_D)}_{\ell,m}Y_{\ell,m}\,,\\ T_{\text{ext}}u_D &= -\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(\ell+1)\widehat{(u_D)}_{\ell,m}Y_{\ell,m}\,. \end{aligned} \tag{12}$$

From (9) and (6) we deduce

$$T_{\text{int}}u_D = \partial_\nu^- u_{\text{int}} = \frac{1}{2}S^{-1}(u_D) + S^{-1}D(u_D)$$

and

$$T_{\text{ext}}u_D = \partial_\nu^+ u_{\text{ext}} = -\frac{1}{2}S^{-1}(u_D) + S^{-1}D(u_D),$$

so that

$$S^{-1}(u_D) = T_{\text{int}}u_D - T_{\text{ext}}u_D\,.$$

By using (12) we infer

$$S^{-1}u_D = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(2\ell+1)\widehat{(u_D)}_{\ell,m}Y_{\ell,m}\,,$$

which in turn yields

$$SU_D = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \widehat{(U_D)}_{\ell,m} Y_{\ell,m} \,.$$

Therefore, the weakly singular integral operator S has the following representation

$$Sv = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \hat{v}_{\ell,m} Y_{\ell,m} \quad \text{for all } v \in H^s(\mathbb{S}) \text{ and } s \in \mathbb{R} \,. \qquad (13)$$

In Section 3, we approximate the solution of (7) by using spherical radial basis functions. These functions are defined via positive definite kernels.

3 Spherical radial basis functions

The finite dimensional subspaces that we use in our approximation are defined by positive definite kernels on $\mathbb{S}$ and spherical radial basis functions.

3.1 Positive definite kernels

A continuous function $\Phi : \mathbb{S} \times \mathbb{S} \to \mathbb{C}$ is called a *positive definite kernel* on $\mathbb{S}$ if $\Phi(\mathbf{x}, \mathbf{y}) = \overline{\Phi(\mathbf{y}, \mathbf{x})}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{S}$, and if for every set of distinct points $\{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ on $\mathbb{S}$, the $N \times N$ matrix A with entries $A_{i,j} = \Phi(\mathbf{x}_i, \mathbf{x}_j)$ is positive semidefinite. If the matrix A is positive definite, then Φ is called a *strictly positive definite* kernel [7, 11]. We define the kernel Φ in terms of a univariate function $\phi : [-1, 1] \to \mathbb{R}$,

$$\Phi(\mathbf{x}, \mathbf{y}) = \phi(\mathbf{x} \cdot \mathbf{y}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{S} \,.$$

If ϕ has a series expansion in terms of Legendre polynomials P_ℓ,

$$\phi(t) = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell+1)\hat{\phi}(\ell) P_\ell(t), \tag{14}$$

where $\hat{\phi}(\ell) = 2\pi \int_{-1}^{1} \phi(t) P_\ell(t)\, dt$, then by using the well known addition formula for spherical harmonics [3],

$$\sum_{m=-\ell}^{\ell} Y_{\ell,m}(\mathbf{x}) \overline{Y_{\ell,m}(\mathbf{y})} = \frac{2\ell+1}{4\pi} P_\ell(\mathbf{x}\cdot\mathbf{y}) \quad \text{for all } \mathbf{x},\mathbf{y} \in \mathbb{S}\,, \tag{15}$$

we write

$$\Phi(\mathbf{x},\mathbf{y}) = \sum_{\ell=0}^{\infty} \hat{\phi}(\ell) \sum_{m=-\ell}^{\ell} Y_{\ell,m}(\mathbf{x}) \overline{Y_{\ell,m}(\mathbf{y})}. \tag{16}$$

In Subsection 3.2, we assume that

$$c_1(\ell+1)^{-2\tau} \le \hat{\phi}(\ell) \le c_2(\ell+1)^{-2\tau}, \quad \ell = 0,1,2,\ldots, \tag{17}$$

for some positive constants c_1 and c_2, and some $\tau > 1$. In particular, this assumption implies $\hat{\phi}(\ell) > 0$, which in turn yields the strict positive definiteness of the kernel Φ [7].

3.2 Finite dimensional subspace

Let $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ be a set of data points on the sphere. The *spherical radial basis functions* Φ_j, $j = 1,\ldots,N$, associated with X and the kernel Φ are defined by

$$\Phi_j(\mathbf{x}) := \Phi(\mathbf{x},\mathbf{x}_j) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \hat{\phi}(\ell) \overline{Y_{\ell,m}(\mathbf{x}_j)} Y_{\ell,m}(\mathbf{x}), \tag{18}$$

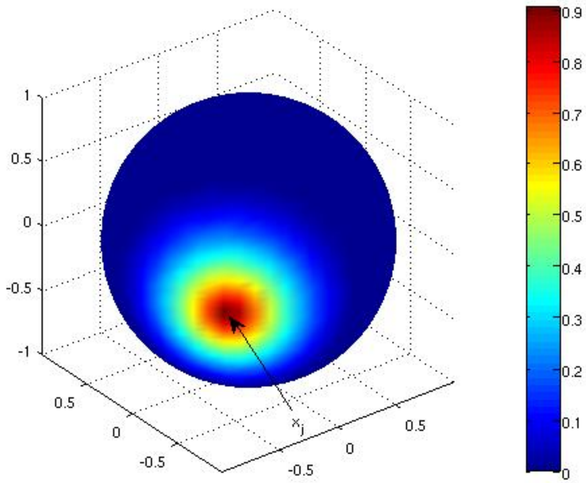

Figure 1: Spherical basis function Φ_j centred at $\mathbf{x}_j = \left(-1/\sqrt{2}, -1/\sqrt{2}, 0\right)$.

and the finite dimensional space is

$$V_X^\phi := \operatorname{span}\{\Phi_1, \dots, \Phi_N\}.$$

The functions Φ_j are radial functions because their values at $\mathbf{x}$ depend only on the geodesic distance between $\mathbf{x}$ and $\mathbf{x}_j$, namely, the angle between the two vectors defining these two points.

The function ϕ will be defined from Wendland's function [10]; see Section 4 for more detail. Figure 1 depicts a basis function Φ_j centred at $\mathbf{x}_j = (-1/\sqrt{2}, -1/\sqrt{2}, 0)$ and defined from $\phi(t) = \left(1 - \sqrt{2-2t}\right)_+^2$, where

$$(1 - \sqrt{2-2t})_+^2 := \left(\max\left\{1 - \sqrt{2-2t}, 0\right\}\right)^2.$$

We note that if (17) holds then

$$\Phi_j \in H^s(\mathbb{S}) \quad \text{for all } s < 2\tau - 1\,. \tag{19}$$

Since $\tau > 1$, the Sobolev embedding theorem gives $V_X^\phi \subset C(\mathbb{S})$.

3.3 Collocation method

We find an approximate solution $u_X \in V_X^\phi$ to the solution u of (7) by solving the equations

$$S u_X(x_j) = f(x_j), \quad j = 1, \ldots, N. \tag{20}$$

By writing $u_X = \sum_{i=1}^N c_i \Phi_i$, we derive the matrix equation

$$A\mathbf{c} = \mathbf{f} \tag{21}$$

from (20), where $\mathbf{c} = (c_i)_{i=1,\ldots,N}$, $\mathbf{f} = (f(\mathbf{x}_i))_{i=1,\ldots,N}$, and

$$A_{i,j} = S\Phi_i(\mathbf{x}_j) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\hat{\phi}(\ell)}{2\ell+1} \overline{Y_{\ell,m}(\mathbf{x}_i)}\, Y_{\ell,m}(\mathbf{x}_j).$$

By using the addition formula (15), we obtain

$$A_{i,j} = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} \hat{\phi}(\ell)\, P_\ell(\mathbf{x}_i \cdot \mathbf{x}_j). \tag{22}$$

Since $\hat{\phi}(\ell)/(2\ell+1) > 0$ for all $\ell \geq 0$, the matrix A is positive definite [7]. Therefore, the system of equations (20) has a unique solution. Error analysis for this approximation (for more general pseudo-differential equations of negative orders) is carried out by Pham and Tran [6]. In the next section, we mention in detail how to define the spherical radial basis functions, and present our numerical results.

Table 1: Wendland's RBFs.

m	$\rho_m(r)$	τ
0	$(1-r)_+^2$	1.5
1	$(1-r)_+^4(4r+1)$	2.5
2	$(1-r)_+^6(35r^2+18r+3)$	3.5

4 Numerical experiments

In this section, we discuss the implementation of our method with the sets X of collocation points being scattered position data points extracted from a very large data set collected by NASA satellite MAGSAT. The code was written in FORTRAN 90 and run on computers equipped with dual Opteron 2.0GHz CPU and 4GB RAM.

The univariate function ϕ defining the kernel Φ, see (16), is

$$\phi(t) = \rho_m\big(\sqrt{2-2t}\big),$$

where ρ_m are Wendland's functions [10]. Narcowich and Ward [4, Proposition 4.6] proved that condition (17) holds with $\tau = m + 3/2$. Table 1 details the functions ρ_m used in our experiments and the corresponding values for τ. The spherical radial basis functions Φ_i, $i = 1, \ldots, N$, are computed by

$$\Phi_i(x) = \rho_m\big(\sqrt{2-2x\cdot x_i}\big), \quad x \in \mathbb{S}. \tag{23}$$

Each entry $A_{i,j}$ in (22) is approximated by a partial sum using the first 500 terms in the series. The coefficients $\widehat{\phi}(\ell)$ are computed by the MATLAB built-in function `quadl` (which uses an adaptive Lobatto quadrature) with tolerance 10^{-15}. The system (21) is solved by the conjugate gradient method with relative tolerance 10^{-9}.

In order to illustrate our method, we chose in our experiment

$$U_D(\mathbf{x}) = (1.25 - x_3)^{-1/2}$$

where $\mathbf{x} = (x_1, x_2, x_3)$, so that the exact solution is $U(\mathbf{x}) = \|\mathbf{x} - \mathbf{p}\|^{-1}$ with $\mathbf{p} = (0, 0, 0.5)$. Noting (6), the exact solution to the weakly singular integral equation (7) is

$$u(\mathbf{x}) = \partial_\nu^+ U(\mathbf{x}) = \frac{-1 + \mathbf{x} \cdot \mathbf{p}}{\|\mathbf{x} - \mathbf{p}\|^3} = \frac{(0.5x_3 - 1)}{(1.25 - x_3)^{3/2}}, \quad \mathbf{x} \in \mathbb{S}.$$

The sets X are chosen such that the one with smaller cardinality is a subset of those of larger cardinalities; see Table 2. Having solved (21), we found u_X by (noting (23))

$$u_X(\mathbf{x}) = \sum_{i=1}^{N} c_i \rho_m\left(\sqrt{2 - 2\mathbf{x} \cdot \mathbf{x}_i}\right). \tag{24}$$

Denoting $E = u - u_X$, we computed the discrete ℓ_∞- and ℓ_2-norms of E by

$$\|E\|_{\ell_\infty} = \max_{\mathbf{x}_g \in \mathcal{G}} |E(\mathbf{x}_g)| \quad \text{and} \quad \|E\|_{\ell_2} = \left(\frac{1}{|\mathcal{G}|} \sum_{\mathbf{x}_g \in \mathcal{G}} |E(\mathbf{x}_g)|^2 \right)^{1/2},$$

where $\mathcal{G} = \{\mathbf{x}_g\}$ is a set of 142883 points on $\mathbb{S}$, and $|\mathcal{G}| = 142883$ is the cardinality of $\mathcal{G}$. Convergence is observed from the computed errors in Table 2. The approximate solution is smoother for larger values of m, resulting in a faster convergence.

Acknowledgements The first author is supported by the University International Postgraduate Award offered by the University of New South Wales. The third author is supported by the Australian Research Council under its Centres of Excellence program.

Table 2: Errors in discrete ℓ_∞- and ℓ_2-norms.

	$m=0$		$m=1$		$m=2$	
N	$\|E\|_{\ell_\infty}$	$\|E\|_{\ell_2}$	$\|E\|_{\ell_\infty}$	$\|E\|_{\ell_2}$	$\|E\|_{\ell_\infty}$	$\|E\|_{\ell_2}$
1079	0.27E-1	0.17E-2	0.37E-2	0.20E-3	0.13E-2	0.63E-4
2158	0.83E-2	0.42E-3	0.41E-3	0.19E-4	0.41E-4	0.21E-5
4316	0.19E-2	0.94E-4	0.24E-4	0.17E-5	0.17E-5	0.92E-7
8631	0.57E-3	0.26E-4	0.33E-5	0.21E-6	0.89E-7	0.92E-8
17262	0.17E-3	0.87E-5	0.83E-6	0.60E-7	0.32E-7	0.64E-8

References

[1] W. Freeden, T. Gervens, and M. Schreiner. *Constructive approximation on the sphere with applications to geomathematics.* Oxford: Clarendon Press, New York, 1998. C267

[2] T. M. Morton and M. Neamtu. Error bounds for solving pseudodifferential equations on spheres. *J. Approx. Theory*, **114** (2002), 242–268. doi:10.1006/jath.2001.3642 C268

[3] C. Müller. *Spherical Harmonics*, volume 17 of *Lecture Notes in Mathematics.* Springer–Verlag, Berlin, 1966. C271, C274

[4] F. J. Narcowich and J. D. Ward. Scattered data interpolation on spheres: error estimates and locally supported basis functions. *SIAM J. Math. Anal.*, **33** (2002), 1393–1410. doi:10.1137/S0036141001395054 C277

[5] J.-C. Nédélec. *Acoustic and Electromagnetic Equations.* Springer–Verlag, New York, 2000. C269, C271

[6] T. D. Pham and T. Tran. *Collocation solutions to pseudodifferential equations of negative orders on the sphere using spherical radial basis functions*. In preparation. C268, C276

[7] I. J. Schoenberg. Positive definite function on spheres. *Duke Math. J.*, **9** (1942), 96–108. doi:10.1215/S0012-7094-42-00908-6 C273, C274, C276

[8] S. Svensson. Pseudodifferential operators – a new approach to the boundary problems of physical geodesy. *Manuscr. Geod.*, **8** (1983), 1–40. C267

[9] H. Wendland. Meshless Galerkin methods using radial basis functions. *Math. Comp.*, **68** (1999), 1521–1531. doi:10.1090/S0025-5718-99-01102-3 C267

[10] H. Wendland. *Scattered Data Approximation*. Cambridge University Press, Cambridge, 2005. C267, C275, C277

[11] Y. Xu and E. W. Cheney. Strictly positive definite functions on spheres. *Proc. Amer. Math. Soc.*, **116** (1992), 977–981. doi:10.2307/2159477 C273

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