

Efficient coupling of finite elements and boundary elements—adaptive procedures and preconditioners

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Dedicated to Professor Ian H. Sloan on the occasion of his 70th birthday.

Abstract

The article is split into four parts. First, we present the symmetric finite element/boundary element-coupling method. Second, we address the choices of appropriate preconditioners for the resulting discrete system when h- and p-versions are performed. Third, we discuss contact problems which are reduced to variational inequalities. Finally, we show the practical applicability of the finite element/boundary element-coupling method by applying it to a metal turning process. Here the viscoplastic work piece is modelled with finite elements and the linear elastic work tool (milling cutter) is modelled with boundary elements. This leads to an efficient and fast numerical method to simulate the metal turning process and to predict failure of the thermal shrink fit which holds the milling cutter.

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1 Introduction

For a model non-linear transmission problem we present in Section 2 a combined approach with finite elements (FE) and boundary elements (BE). We perform the so-called symmetric coupling which renders all boundary conditions on the interface manifold Γ to be natural and allows for a non-linear elliptic differential operator in the bounded domain Ω . Our solution procedure makes use of an integral equation method for the exterior problem and of an energy (variational) method for the interior problems, and consists of coupling both methods via the transmission conditions on the interface. We solve the resulting symmetric coupling formulation with the Galerkin method using finite elements in Ω and boundary elements on Γ . We present in Theorem 2 hierarchical error estimators for the FE/BE-coupling and give corresponding numerical results in Table 1 and Figure 1. More details and mixed FE/BE-coupling methods are described by Stephan [24]. Section 3 comments on the solvers and preconditioners for the resulting discrete systems of the FE/BE-coupling. Here, additive Schwarz preconditioners play an

important role. Then Section 4 considers contact problems with Tresca friction and applies penalty or mortar methods to solve the governing variational inequalities. Then we present error estimates and corresponding adaptive numerical experiments for h- and p-versions. Finally, in Section 5 we apply the FE/BE-coupling to a viscoplastic thermo-mechanical problem describing a metal forming process [9]. Here the viscoplastic work piece is modelled with finite elements and the linear elastic work tool (milling cutter) is modelled with boundary elements.

2 Symmetric FE/BE-coupling

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$ and $\Omega^c = \mathbb{R}^d \setminus \bar{\Omega}$ with normal \mathbf{n} on Γ pointing into Ω^c . For given $f \in L^2(\Omega)$, $\mathbf{u}_0 \in H^{1/2}(\Gamma)$, $\psi_0 \in H^{-1/2}(\Gamma)$ and $\mathbf{a} \in \mathbb{R}$ find $\mathbf{u}_1 \in H^1(\Omega)$ and $\mathbf{u}_2 \in H^1_{loc}(\Omega^c)$ such that

$$\begin{aligned} -\operatorname{div} \mathbf{A}(\nabla \mathbf{u}_1) &= f \quad \text{in } \Omega, \\ -\Delta \mathbf{u}_2 &= 0 \quad \text{in } \Omega^c, \\ \mathbf{u}_1 - \mathbf{u}_2 &= \mathbf{u}_0 \quad \text{on } \Gamma, \\ \mathbf{A}(\nabla \mathbf{u}_1) \cdot \mathbf{n} - \frac{\partial \mathbf{u}_2}{\partial \mathbf{n}} &= \psi_0 \quad \text{on } \Gamma, \end{aligned} \tag{1}$$

$$\mathbf{u}_2(\mathbf{x}) = \begin{cases} \mathbf{a} \log |\mathbf{x}| + \mathbf{o}(1), & d = 2, \\ \mathbf{O}(|\mathbf{x}|^{-1}), & d = 3, \end{cases} \quad (|\mathbf{x}| \rightarrow \infty).$$

The operator \mathbf{A} is assumed to be uniformly monotone and Lipschitz continuous; operators of this type are considered by Stephan [22] and Zeidler [30]. By using Green's formula together with the decaying condition for $|\mathbf{x}| \rightarrow \infty$ in (1) one is led to the representation formula for \mathbf{u}_2 in Ω^c

$$\mathbf{u}_2(\mathbf{x}) = \int_{\Gamma} \left(\mathbf{u}_2(\mathbf{y}) \frac{\partial}{\partial \mathbf{n}_y} G(\mathbf{x}, \mathbf{y}) - G(\mathbf{x}, \mathbf{y}) \frac{\partial \mathbf{u}_2}{\partial \mathbf{n}_y} \right) ds_y, \quad \mathbf{x} \in \Omega^c, \tag{2}$$

with the fundamental solution of the Laplacian

$$G(x, y) = \begin{cases} -\frac{1}{2\pi} \log |x - y|, & d = 2, \\ \frac{1}{4\pi} |x - y|^{-1}, & d = 3. \end{cases} \quad (3)$$

By using the boundary integral operators

$$V\psi(x) := 2 \int_{\Gamma} G(x, y) \psi(y) \, ds_y, \quad x \in \Gamma, \quad (4)$$

$$K\psi(x) := 2 \int_{\Gamma} \frac{\partial}{\partial n_y} G(x, y) \psi(y) \, ds_y, \quad x \in \Gamma, \quad (5)$$

$$K'\psi(x) := 2 \frac{\partial}{\partial n_x} \int_{\Gamma} G(x, y) \psi(y) \, ds_y, \quad x \in \Gamma, \quad (6)$$

$$W\psi(x) := -2 \frac{\partial}{\partial n_x} \int_{\Gamma} \frac{\partial}{\partial n_y} G(x, y) \psi(y) \, ds_y, \quad x \in \Gamma, \quad (7)$$

together with their well-known jump relations (as $x \rightarrow \Gamma$) we obtain the integral equations on Γ

$$2 \frac{\partial u_2}{\partial n} = -Wu_2 + (I - K') \frac{\partial u_2}{\partial n}, \quad (8)$$

$$0 = (I - K)u_2 + V \frac{\partial u_2}{\partial n}. \quad (9)$$

In the interior domain Ω we apply integration by parts and obtain

$$\int_{\Omega} A(\nabla u_1) \cdot \nabla v = \int_{\Gamma} A(\nabla u_1) \cdot n v + \int_{\Omega} f v. \quad (10)$$

Now inserting (8) into (10) and taking the weak form of (9) yield, together with the transmission conditions in (1), the weak form of the transmission problem (1)

$$\begin{aligned} 2 \int_{\Omega} A(\nabla u_1) \cdot \nabla v \, dx - \left\langle \frac{\partial u_2}{\partial n}, v \right\rangle + \left\langle K' \frac{\partial u_2}{\partial n}, v \right\rangle + \left\langle Wu_1, v \right\rangle \\ = 2(f, v) + 2\langle \psi_0, v \rangle + \langle Wu_0, v \rangle, \end{aligned} \quad (11)$$

$$-\langle \mathbf{u}_1, \psi \rangle - \langle \mathbf{V} \frac{\partial \mathbf{u}_2}{\partial \mathbf{n}}, \psi \rangle + \langle \mathbf{K} \mathbf{u}_1, \psi \rangle = -\langle \mathbf{u}_0, \psi \rangle + \langle \mathbf{K} \mathbf{u}_0, \psi \rangle. \quad (12)$$

Here we use the inner products

$$\langle \mathbf{u}_1, \psi \rangle = \int_{\Gamma} \mathbf{u}_1 \psi, \quad (f, v) = \int_{\Omega} f v.$$

In short (11) and (12) read: find $\mathbf{u} = \mathbf{u}_1 \in H^1(\Omega)$ and $\phi = \partial \mathbf{u}_2 / \partial \mathbf{n} \in H^{-1/2}(\Gamma)$

$$\mathcal{A}(\mathbf{u}, \phi; v, \psi) := 2 \int_{\Omega} \mathbf{A}(\nabla \mathbf{u}) \cdot \nabla v \, dx + \mathbf{B}(\mathbf{u}, \phi; v, \psi) = \mathcal{L}(v, \psi) \quad (13)$$

for all $v \in H^1(\Omega)$ and $\psi \in H^{-1/2}(\Gamma)$ where $\mathbf{B}(\mathbf{u}, \phi; v, \psi)$ and $\mathcal{L}(v, \psi)$ are defined by the left hand sides and the right hand sides of (11) and (12), respectively.

Remark 1 The above derivation shows that if \mathbf{u}_1 and \mathbf{u}_2 solve (1) then \mathbf{u}_1 and $\partial \mathbf{u}_2 / \partial \mathbf{n}$ satisfy (11) and (12). Conversely, provided \mathbf{u}_1 and $\partial \mathbf{u}_2 / \partial \mathbf{n}$ solve (11) and (12) then \mathbf{u}_1 and \mathbf{u}_2 , defined by (2), are solutions of the transmission problem (1). Furthermore, under the assumptions in (1), the variational formulation is uniquely solvable (as shown by Costabel and Stephan [8, 24]).

Let X_M and Y_N be finite dimensional approximating subspaces of $H^1(\Omega)$ and $H^{-1/2}(\Gamma)$, respectively, then the finite element/boundary element Galerkin coupling reads: find $\mathbf{u}_M \in X_M$ and $\phi_N \in Y_N$ such that

$$\mathcal{A}(\mathbf{u}_M, \phi_N; v, \psi) := 2 \int_{\Omega} \mathbf{A}(\nabla \mathbf{u}_M) \cdot \nabla v \, dx + \mathbf{B}(\mathbf{u}_M, \phi_N; v, \psi) = \mathcal{L}(v, \psi) \quad (14)$$

for all $v \in X_M$ and $\psi \in Y_N$.

As shown by Costabel and Stephan [8, 24] every Galerkin scheme (14) converges with optimal order; that is, with the exact solution of (13) and the Galerkin solution \mathbf{u}_M and ϕ_N of (14) there holds

$$e_{\mathbf{u}, \phi} := \|\mathbf{u} - \mathbf{u}_M\|_{1, \Omega} + \|\phi - \phi_N\|_{-1/2, \Gamma} \lesssim \inf_{\hat{\mathbf{u}} \in X_M} \|\mathbf{u} - \hat{\mathbf{u}}\|_{1, \Omega} + \inf_{\hat{\phi} \in Y_N} \|\phi - \hat{\phi}\|_{-1/2, \Gamma}.$$

This gives for the *hp-version* of the Galerkin scheme (14) on quasiuniform meshes $\mathbf{e}_{\mathbf{u},\phi} \lesssim h^\alpha p^{-2\alpha}$ with some $\alpha \in \mathbb{R}$ and on geometrically refined meshes

$$\mathbf{e}_{\mathbf{u},\phi} \lesssim \begin{cases} \exp(-\mathbf{b}_1 M^{1/3}) + \exp(-\mathbf{b}_2 N^{1/2}), & d = 2, \\ \exp(-\mathbf{b}_1 M^{1/5}) + \exp(-\mathbf{b}_2 N^{1/4}), & d = 3, \end{cases}$$

with some $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}$ where $M = \text{dof of } X_M, N = \text{dof of } Y_N$. These results were derived by Guo and Stephan [10], Babuška et al. [2], Maischak and Stephan [16, 23].

Next we restrict ourselves to the *h-version* of (14), and we introduce a hierarchical error estimator for the FE/BE-coupling. For this purpose we consider regular triangulations ω_H of Ω and partitions γ_H of Γ . Our test and trial spaces are

$$\mathbb{T}_H := \{v_H : \Omega \rightarrow \mathbb{R} ; v_H \text{ p.w. linear on } \omega_H, v_H \in \mathcal{C}^0(\Omega)\}, \quad (15)$$

$$\tau_H := \{\psi_H : \Gamma \rightarrow \mathbb{R} ; \psi_H \text{ p.w. constant on } \gamma_H\}. \quad (16)$$

On locally refined meshes we distinguish between old hat functions \mathbf{b}_{old} (being one at the vertices of the triangles) and new hat functions \mathbf{b}_{new} (being one at the midpoints of the edges of the triangles). On the boundary mesh γ_H we consider piecewise constant functions β with $\beta_{\text{old}}^i = 1$ on interval Γ_i and

$$\beta_{\text{new}}^i = \begin{cases} 1, & \text{on left half of } \Gamma_i, \\ -1, & \text{on right half of } \Gamma_i. \end{cases}$$

Let \mathbf{n} denote the number of new nodes, \mathbf{m} the number of intervals on the fine boundary mesh and let $\mathbb{T}_j = \text{span}\{\mathbf{b}_{\text{new}}^j\}, 1 \leq j \leq \mathbf{n}, \tau_k = \text{span}\{\beta_{\text{new}}^k\}, 1 \leq k \leq \mathbf{m}$. Then one has stable two level subspace decompositions $\mathbb{T}_{H/2} = \mathbb{T}_H \oplus \mathbb{T}_1 \oplus \dots \oplus \mathbb{T}_{\mathbf{n}}, \tau_{H/2} = \tau_H \oplus \tau_1 \oplus \dots \oplus \tau_{\mathbf{m}}$ of the finite element and boundary element spaces. Now one considers sequences of nested spaces $\tilde{\mathbb{T}}_k \times \tilde{\tau}_k \subset \tilde{\mathbb{T}}_{k+1} \times \tilde{\tau}_{k+1}$ starting from $\tilde{\mathbb{T}}_0 = \mathbb{T}_H, \tilde{\mathbb{T}}_1 = \mathbb{T}_{H/2}, \tilde{\tau}_0 = \tau_H, \tilde{\tau}_1 = \tau_{H/2}$. Let (\mathbf{u}, ϕ) be the exact solution and $(\mathbf{u}_k, \phi_k) \in \tilde{\mathbb{T}}_k \times \tilde{\tau}_k$ be the Galerkin

solution at level $k \in \mathbb{N}_0$. Under the saturation assumption, that is, there exists $\kappa < 1$ such that

$$\|\mathbf{u} - \mathbf{u}_{k+1}\|_{1,\Omega} + \|\Phi - \Phi_{k+1}\|_{-1/2,\Gamma} \leq \kappa \left(\|\mathbf{u} - \mathbf{u}_k\|_{1,\Omega} + \|\Phi - \Phi_k\|_{-1/2,\Gamma} \right), \quad (17)$$

there holds the following theorem. Here and in Table 1 we use the notation

$$\|\mathbf{u} - \mathbf{u}_L, \Phi - \Phi_L\|_{\mathcal{H}}^2 := \|\mathbf{u} - \mathbf{u}_k\|_{1,\Omega}^2 + \|\Phi - \Phi_k\|_{-1/2,\Gamma}^2.$$

Theorem 2 (Mund, Stephan [20]) *Assuming (17) there holds*

$$\|\mathbf{u} - \mathbf{u}_L, \Phi - \Phi_L\|_{\mathcal{H}}^2 \approx \eta^2 := \sum_{i=1}^n \Theta_{k,i}^2 + \sum_{j=1}^m \vartheta_{k,j}^2,$$

where

$$\Theta_{k,i} := \frac{|\mathcal{L}(\mathbf{b}_{k+1,i}, \mathbf{0}) - \mathcal{A}(\mathbf{u}_k, \Phi_k; \mathbf{b}_{k+1,i}, \mathbf{0})|}{\|\mathbf{b}_{k+1,i}\|_{H^1(\Omega)}},$$

$$\vartheta_{k,j} := \frac{|\mathcal{L}(\mathbf{0}, \beta_{k+1,j}) - \mathcal{A}(\mathbf{u}_k, \Phi_k; \mathbf{0}, \beta_{k+1,j})|}{\|\beta_{k+1,j}\|_{H^{-1/2}(\Gamma)}},$$

with basis functions $\mathbf{b}_{i,k+1} \in \tilde{\mathbb{T}}_{k+1} \setminus \tilde{\mathbb{T}}_k$ and $\beta_{k+1,j} \in \tilde{\tau}_{k+1} \setminus \tilde{\tau}_k$.

Remark 3 A simple adaptive algorithm uses $\Theta_{k,i}$ and $\vartheta_{k,j}$ for local FE/BE-refinements and it refines if $\eta_k^r \geq \theta \max_{1 \leq l \leq N_k} \eta_k^l$ as derived by Mund and Stephan [20], where $\eta_k^r := \eta_k$ on the r th triangle Δ_k^r at level k , $1 \leq r \leq N_k$, and θ is a preset value.

Remark 4 Residual-type error estimators were derived by Carstensen and Stephan [3] for the h -version of 2D and 3D FE/BE-coupling, namely

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|\Phi - \Phi_h\|_{-1/2,\Gamma} \leq C(R_1 + R_2 + R_3 + R_4),$$

where

$$R_1^2 = \sum_{\Delta \in \omega_h} h^2 \int_{\Delta} |f + \operatorname{div}(\sigma)|^2 dx,$$

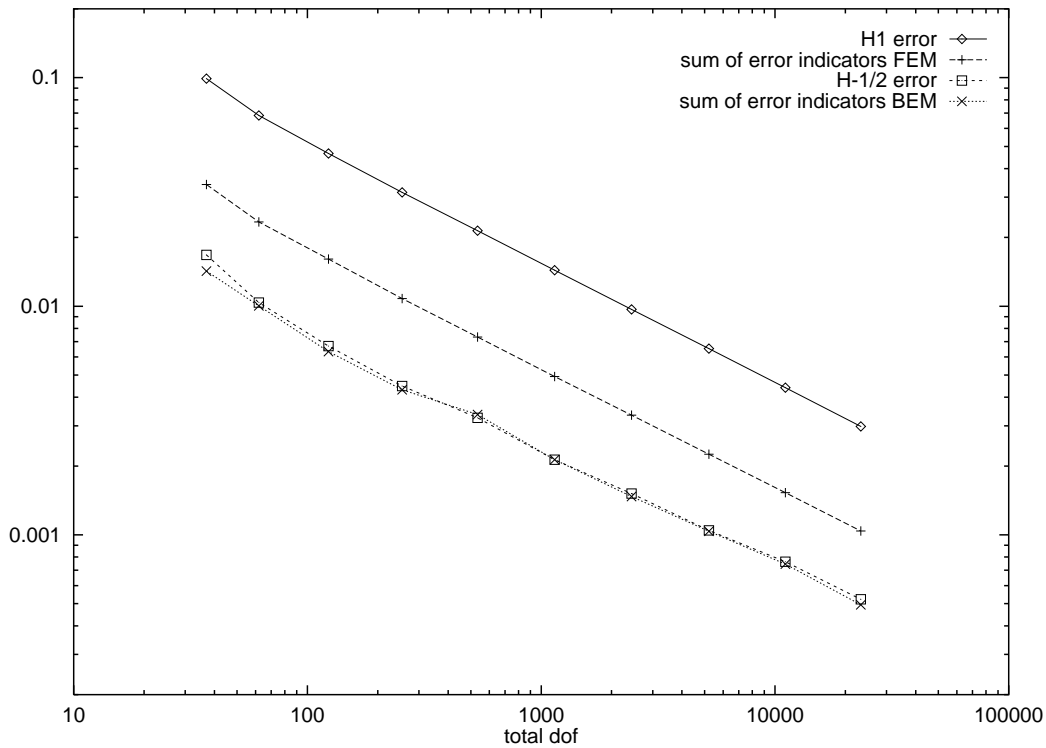


FIGURE 1: Nonlinear case: error in energy norm/local error indicators [20].

TABLE 1: Errors in energy norm E_L and error indicator η_L versus number of unknowns. $L =$ level, $\mathcal{N}_L = \dim T_L + \dim \tau_L$, $E_L := \|(\mathbf{u} - \mathbf{u}_L, \phi - \phi_L)\|_{\mathcal{H}}$.

L	\mathcal{N}_L	$\dim T_L$	$\dim \tau_L$	E_L	η_L	η_L/E_L	κ_L	α_L
0	37	21	16	0.10040	0.03693	0.368	—	—
1	62	44	18	0.06912	0.02546	0.368	0.688	0.505
2	123	103	20	0.04708	0.01729	0.367	0.681	0.451
3	254	232	22	0.03181	0.01163	0.366	0.676	0.483
4	534	510	24	0.02164	0.00807	0.373	0.680	0.489
5	1142	1112	30	0.01453	0.00537	0.369	0.671	0.511
6	2436	2400	36	0.00981	0.00365	0.372	0.675	0.511
7	5217	5175	42	0.00660	0.00248	0.375	0.673	0.515
8	11088	11036	52	0.00447	0.00170	0.381	0.676	0.516
9	23283	23217	66	0.00302	0.00115	0.381	0.677	0.525

$$R_2^2 = \sum_{\text{interface } E} h_E \int_E |[\sigma \cdot \mathbf{n}_E]|^2 ds,$$

$$R_3^2 = \sum_{E \text{ on } \Gamma} h_E \| \{\sigma \cdot \mathbf{n} - \psi_0 + W(\mathbf{u}_h - \mathbf{v}_0) + (\mathbf{K}' - \mathbf{I})\phi_h\} \|_{L^2(E)}^2,$$

$$R_4^2 = \sum_{E \text{ on } \Gamma} h_E \left\| \frac{\partial}{\partial s} ((\mathbf{K} - \mathbf{I})(\mathbf{u}_h - \mathbf{u}_0) - \mathbf{V}\phi_h) \right\|_{L^2(E)}^2,$$

with $\sigma := \rho(|\nabla \mathbf{u}_h|) \nabla \mathbf{u}_h$.

In numerical experiments reported by Carstensen, Mund and Stephan [20, 3], hierarchical and residual estimators behave similarly, but the computation of R_3 and R_4 is expensive. Table 1 lists the error $E_L^2 := \|\mathbf{u} - \mathbf{u}_L\|_{1,\Omega}^2 + \|\phi - \phi_L\|_{-1/2,\Gamma}^2$, the error indicator η_L , the experimental saturation constant κ_L , and the experimental convergence rate α_L for problem (1) on an L-shaped domain Ω , when $\mathbf{A}(\nabla \mathbf{u}_1) = \rho(|\nabla \mathbf{u}_1|) \nabla \mathbf{u}_1$ with $\rho = \frac{1}{6} (1 + \frac{5}{1+5t})$. Figure 1 shows the respective errors in the energy norm and the sums of the local error indicators plotted versus the number of unknowns \mathcal{N} . Here $\sum_{i=1}^n \Theta_{k,i}^2$ is the sum of the error indicators FEM; $\sum_{j=1}^m \vartheta_{k,j}^2$ is the sum of error indicators

BEM. Further details are investigated by Mund and Stephan [20].

3 Preconditioners for FE/BE-coupling

Here we present preconditioners for the *hp-version* of the symmetric FE/BE-coupling. As iterative solver Heuer, Maischak, Stephan [13] apply the minimum residual method (MINRES); its stable formulation, the hybrid modified conjugate residual method (HMCR), is considered by Mund and Stephan [19]. The linearized Galerkin FE/BE-coupling system (14) is in matrix form

$$\underbrace{\begin{pmatrix} \mathbf{A} & \mathbf{B}^\top & \mathbf{0} \\ \mathbf{B} & \mathbf{C} + \mathbf{W} & \mathbf{K}^\top - \mathbf{I} \\ \mathbf{0} & \mathbf{K} - \mathbf{I} & -\mathbf{V} \end{pmatrix}}_{= \mathcal{A}} \begin{pmatrix} \mathbf{u}_{\Omega_1} \\ \mathbf{u}_\Gamma \\ \phi_\Gamma \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix} \quad (18)$$

where the FEM block $\mathbf{A}_N = \begin{pmatrix} \mathbf{A} & \mathbf{B}^\top \\ \mathbf{B} & \mathbf{C} \end{pmatrix}$ corresponds to a Neumann problem for the Laplacian. In (18) \mathbf{W} denotes the matrix block belonging to the hypersingular operator and so on; thus we use same letters for matrix and operator. The components \mathbf{u}_{Ω_1} , \mathbf{u}_Γ and ϕ_Γ denote the coefficient vectors of the Galerkin approximations of \mathbf{u} in Ω_1 , \mathbf{u} on Γ and $\partial \mathbf{u}_2 / \partial \mathbf{n} = \phi$ on Γ , whereas \mathbf{b}_i denotes the i th component of the right side in (14).

Considering separately the finite element functions on the interface boundary and in the interior domain, and taking the boundary element functions which discretize the weakly singular operator, we have a splitting of the ansatz space into subspaces. They induce a three-block decomposition of the Galerkin matrix which will be the *three-block* preconditioner. By this decomposition the strong coupling of edge and interior functions is neglected. Therefore, this three-block splitting allows only for sub-optimal preconditioners. Already in case of exactly inverting the blocks one gets $\mathcal{O}(\mathfrak{h}^{-3/4} \mathfrak{p}^{3/2})$ iteration numbers.

In detail we employ a preconditioner of the form

$$\mathcal{M}_3 = \begin{pmatrix} \tilde{\mathbf{A}} & 0 & 0 \\ 0 & \tilde{\mathbf{C}} & 0 \\ 0 & 0 & \tilde{\mathbf{V}} \end{pmatrix} \quad (19)$$

where $\tilde{\mathbf{A}}$, $\tilde{\mathbf{C}}$ and $\tilde{\mathbf{V}}$ are spectrally equivalent matrices to \mathbf{A} , \mathbf{C} and \mathbf{V} , respectively.

Considering the Neumann block as a whole, that is, by taking together finite element functions on the interface and in the interior, we obtain a two-block Jacobi method which has bounded iteration numbers for exact inversion of the two blocks and therefore, allows for almost optimal *two-block* preconditioners.

Our preconditioning matrix is

$$\mathcal{M}_2 = \begin{pmatrix} \tilde{\mathbf{A}}_{\mathbf{M}} & 0 \\ 0 & \tilde{\mathbf{V}} \end{pmatrix} \quad (20)$$

where $\tilde{\mathbf{A}}_{\mathbf{M}}$ is spectrally equivalent to $\mathbf{A}_{\mathbf{N}} + \mathbf{W} + \mathbf{M}$ and $\tilde{\mathbf{V}}$ is spectrally equivalent to \mathbf{V} . Here \mathbf{M} is an additional mass matrix which is added to make $\mathbf{A}_{\mathbf{N}} + \mathbf{W}$ positive definite. As shown by Heuer et al. [13] the iteration numbers of the two-block HMCR are bounded.

The *additive Schwarz* preconditioner \mathcal{M}_{ASM} extends the two-block methods by replacing the main blocks by block-diagonal matrices. Here we proceed as follows. First we construct discrete harmonic functions by applying the Schur complement method for the finite element block of the Galerkin matrix. Then, for the finite element part, we decompose the test and trial functions in nodal, edge and interior functions. This amounts to a block Jacobi (Additive Schwarz) preconditioner for the finite element block. We split the boundary element block, belonging to the weakly singular integral operator, into unknowns from a coarse grid space (consisting of piecewise constant functions), and into individual subspaces for each element (consisting of all

polynomials up to degree p without the constants). Our preconditioner \mathbf{M}_{diag} is obtained by further splitting the subspaces of edge functions (for both finite elements and boundary elements) into one dimensional subspaces according to the edge basis functions. For the two-block method we obtain in this way two different preconditioned linear systems which need respectively $\mathcal{O}(\log^2 p)$ and $\mathcal{O}(p \log^2 p)$ minimum residual iterations to be solved up to a given accuracy.

In the first case we apply the Schwarz preconditioner based on decomposing the FE-subspaces \mathbf{X}_M (as described by Babuška et al. [1]) and the BE-subspace \mathbf{Y}_N (as described by Tran and Stephan [27]). Here we decompose the FE ansatz space into piecewise constant functions on the boundary mesh and local subspaces on each element spanned by Legendre polynomials. Respective details are given by Heuer et al. [13]. The resulting additive Schwarz preconditioner is called \mathcal{M}_{ASM} . In the second case we take partially diagonal scaling resulting from further refining the subspace decompositions. This block diagonal preconditioner consists of blocks belonging separately to the piecewise linear functions, the interior functions for individual elements and the piecewise constant functions. For the remaining functions we simply take the diagonal of the stiffness matrix \mathcal{A} . This method combines the decomposition of \mathbf{X}_M , proposed by Babuška et al. [1], and of \mathbf{Y}_N , proposed by Heuer et al. [15], and gives the preconditioner $\mathcal{M}_{\text{diag}}$.

Of course we can also apply multilevel preconditioners to the coupled FE/BE-system of the *h-version*. Here we use multigrid or BPX for the FE matrix \mathbf{A} and for the BE matrices \mathbf{W} and \mathbf{V} . Whereas the FE preconditioners are standard, the BE preconditioners for the boundary integral operators were first analyzed in the concept of multilevel additive Schwarz methods by Tran and Stephan [26]. For a multigrid preconditioner we have bounded iteration numbers of HMCR solvers where for BPX the iteration numbers grow like $\mathcal{O}(\log^2 \frac{1}{h})$.

Finally we remark that overlapping Schwarz methods for the boundary integral operators (with single layer potential and hypersingular operator) have

TABLE 2: Numbers of iterations required to reduce residual by a factor of 10^{-3} (h-, p-versions) [13].

$1/h$	p	M + N	\mathcal{A}	$\mathcal{M}_3^{-1}\mathcal{A}$	$\mathcal{M}_2^{-1}\mathcal{A}$	$\mathcal{M}_{ASM}^{-1}\mathcal{A}$	$\mathcal{M}_{diag}^{-1}\mathcal{A}$
2	2	37	36	21	11	30	30
2	4	97	138	34	12	47	52
2	6	181	285	40	13	54	67
2	8	289	536	45	13	62	77
2	10	421	892	50	13	66	94
2	12	577	1374	55	13	74	111
2	14	757	2328	60	13	82	135
2	16	961	> 9999	79	17	102	197
4	1	37	17	15	10		
8	1	97	38	23	13		
16	1	289	67	32	13		
32	1	961	130	45	14		
64	1	3457	243	60	15		
128	1	13057	476	75	15		

been studied by Tran and Stephan [28] and non-overlapping Schwarz methods by Heuer, Leydecker and Stephan [12, 14].

4 Contact problems with Tresca friction

We consider two non-overlapping polygonal domains Ω^i , $i = 1, 2$, with Lipschitz boundaries Γ^i . Each Γ^i consists of three mutually disjoint measurable parts Γ_D^i , Γ_N^i and Γ_C^i , where Dirichlet, Neumann and frictional contact conditions are prescribed. Let $g(x)$, $x \in \Gamma_C := \Gamma_C^1 \cup \Gamma_C^2$, be the distance between the bodies. We define a jump $h \in \mathcal{C}(\Gamma_C)$ across Γ_C pointwise as $[h](x) := h(x) - h(\mathbf{b}(x))$, $x \in \Gamma_C$, with a bijective mapping $\mathbf{b} : \Gamma_C^1 \rightarrow \Gamma_C^2$, where the normal or the tangential displacement ($\mathbf{u}_n := \mathbf{u} \cdot \mathbf{n}$ or $\mathbf{u}_t := \mathbf{u} \cdot \mathbf{t}$) is at $h(x)$. The stress tensor σ is determined by \mathbf{u} with Hooke's law and under small strain assumptions there holds with $\lambda > 0$ and $\mu > 0$

$$\sigma(\mathbf{u}) = \lambda \operatorname{tr} \varepsilon(\mathbf{u}) + 2\mu \varepsilon(\mathbf{u}), \quad \varepsilon(\mathbf{u}) = (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2.$$

The two body contact problem reads: for given traction $\hat{\mathbf{t}}$ on Γ_N , friction $\mathcal{F} \geq 0$, and gap g , find displacement $\mathbf{u} : \Omega = \Omega^1 \cup \Omega^2 \rightarrow \mathbb{R}^3$ such that

$$\begin{aligned} \operatorname{div} \sigma(\mathbf{u}) &= 0 && \text{in } \Omega, \\ \mathbf{u} &= 0 && \text{on } \Gamma_D := \Gamma_D^1 \cup \Gamma_D^2, \\ \sigma(\mathbf{u}) \cdot \mathbf{n} &= \hat{\mathbf{t}} && \text{on } \Gamma_N := \Gamma_N^1 \cup \Gamma_N^2, \end{aligned} \quad (21)$$

$$\left. \begin{aligned} \sigma_n \leq 0, \quad [\mathbf{u}_n] \leq g, \quad \sigma_n([\mathbf{u}_n] - g) = 0 \\ |\sigma_t| \leq \mathcal{F}, \quad \sigma_t[\mathbf{u}_t] + \mathcal{F}[\mathbf{u}_t] = 0 \end{aligned} \right\} \text{on } \Gamma_C,$$

with scalar normal and tangential stresses $\sigma_n := \mathbf{n} \cdot \sigma(\mathbf{u}) \cdot \mathbf{n}$, $\sigma_t := \mathbf{t} \cdot \sigma(\mathbf{u}) \cdot \mathbf{n}$. The given friction function \mathcal{F} defines pointwise the sticking threshold of the bodies; that is, as seen from (21), if the absolute value of the tangential stress does not reach the given friction $|\sigma_t| \leq \mathcal{F}$, then $[\mathbf{u}_t] = 0$, and $[\mathbf{u}_t] \neq 0$ is only possible if $[\sigma_t] = \mathcal{F}$.

As shown by Chernov, Maischak and Stephan [6, 7], the two body problem (21) is reduced to the following domain variation inequality for admissible functions belonging to the convex cone $K_\Omega := \{\mathbf{u} \in H_D^1(\Omega), [\mathbf{u}_n] \leq \mathbf{g} \text{ on } \Gamma_C\}$; that is, find $\mathbf{u} \in K_\Omega$ such that

$$\int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u}) \, dx + \int_{\Gamma_C} \mathcal{F}(|[\mathbf{v}_t]| - |[\mathbf{u}_t]|) \, ds \geq \int_{\Gamma_N} \hat{\mathbf{t}} \cdot (\mathbf{v} - \mathbf{u}) \, ds \quad (22)$$

for all $\mathbf{v} \in \mathcal{K}_\Omega$, where $:$ denotes the standard tensor product (as used by Wriggers [29]). Integration by parts gives

$$\int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u}) \, dx = \int_\Gamma \mathcal{T}(\mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}) \, ds - \int_\Omega \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}) \, dx \quad (23)$$

with $\operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{0}$ in Ω . On the other hand the DtN-map or Steklov–Poincaré operator $\mathbf{S} = \mathbf{W} + (\mathbf{I} + \mathbf{K}')\mathbf{V}^{-1}(\mathbf{I} + \mathbf{K})$ satisfies $\langle \mathcal{T}(\mathbf{u}), \mathbf{v} - \mathbf{u} \rangle = \langle \mathbf{S}\mathbf{u}, \mathbf{v} - \mathbf{u} \rangle$ where $\langle \cdot, \cdot \rangle$ means integration on Γ . Hence using (22) and (23) one reduces problem (21) to the boundary variational inequality: find $\mathbf{u} \in \mathcal{K} := \{\mathbf{u} \in \tilde{H}^{1/2}(\Gamma \setminus \Gamma_D), [\mathbf{u}_n] \leq \mathbf{g} \text{ on } \Gamma_C\}$ such that for all $\mathbf{v} \in \mathcal{K}$

$$\int_\Gamma (\mathbf{S}\mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}) \, ds + \int_{\Gamma_C} \mathcal{F}(|[\mathbf{v}_t]| - |[\mathbf{u}_t]|) \, ds \geq \int_{\Gamma_N} \hat{\mathbf{t}} \cdot (\mathbf{v} - \mathbf{u}) \, ds. \quad (24)$$

Applying penalty methods to (22) is a standard approach for FE-simulations of contact problems. As analyzed by Chernov et al. [5] we approximate the boundary variational inequality (24) by the penalty method; that is, we find $\mathbf{u}^\varepsilon \in \tilde{H}^{1/2}(\Gamma \setminus \Gamma_D)$ such that

$$\int_\Gamma \mathbf{S}\mathbf{u}^\varepsilon \cdot \boldsymbol{\phi} \, ds + \int_{\Gamma_C} \varepsilon_n^{-1} [\mathbf{u}_n^\varepsilon]^+ [\boldsymbol{\phi}_n] \, ds + \int_{\Gamma_C} \varepsilon_t^{-1} \mathcal{F}[\mathbf{u}_t^\varepsilon]^* [\boldsymbol{\phi}_t] \, ds = \int_{\Gamma_N} \hat{\mathbf{t}}_N \cdot \boldsymbol{\phi} \, ds$$

where $\varepsilon_n, \varepsilon_t > 0$ are penalty parameters, $\mathbf{a}^* := \operatorname{sign}(\mathbf{a}) \min(\varepsilon_t, |\mathbf{a}|)$, $\mathbf{a}^+ := \max(\mathbf{a}, \mathbf{0})$ for some $\mathbf{a} \in \mathbb{R}$. Note that the penalty method is formulated on the unconstrained space $\tilde{H}(\Gamma \setminus \Gamma_D)$ and with $\mathbf{g} \equiv \mathbf{0}$ for simplicity.

Let $\boldsymbol{\psi}^\varepsilon = \mathbf{S}\mathbf{u}^\varepsilon$ be the traction, $\boldsymbol{\Psi}^\varepsilon$ its Galerkin approximation with exact \mathbf{u}^ε and its approximation \mathbf{U}^ε , and \mathbf{P}_n^ε and \mathbf{P}_t^ε be the Galerkin approximation of \mathbf{p}_n^ε and \mathbf{p}_t^ε respectively. We define the (triple bar) norm

$$\begin{aligned} \|\|\mathbf{u}^\varepsilon - \mathbf{U}^\varepsilon\|\| &:= (\|\mathbf{u}^\varepsilon - \mathbf{U}^\varepsilon\|_{\mathbf{H}^{1/2}(\Gamma)}^2 + \|\boldsymbol{\psi}^\varepsilon - \boldsymbol{\Psi}^\varepsilon\|_{\mathbf{H}^{-1/2}(\Gamma)}^2 \\ &+ \|\varepsilon_n^{1/2}(\mathbf{p}_n^\varepsilon - \mathbf{P}_n^\varepsilon)\|_{\mathbf{L}^2(\Gamma_C)}^2 + \|\varepsilon_t^{1/2}\mathcal{F}^{-1/2}(\mathbf{p}_t^\varepsilon - \mathbf{P}_t^\varepsilon)\|_{\mathbf{L}^2(\Gamma_C)}^2)^{1/2}. \end{aligned} \tag{25}$$

The following residual a posteriori error estimate is proven by Chernov [4] for the *h-version* of the FE/BE-coupling on quasiuniform meshes \mathcal{T}_h if $\varepsilon_n \geq \tilde{C}h$ and $\varepsilon_t \geq \tilde{C}\mathcal{F}h$ for some constant $\tilde{C} > 0$

$$c \sum_{I \in \mathcal{T}_h} \eta_h^2(I) \leq \|\|\mathbf{u}^\varepsilon - \mathbf{U}^\varepsilon\|\|^2 \leq C \sum_{I \in \mathcal{T}_h} \eta_h^2(I)$$

with the local error indicators

$$\begin{aligned} \eta_h^2(I) &:= h_I \|\hat{\mathbf{t}} - \hat{\mathbf{S}}\mathbf{U}^\varepsilon\|_{\mathbf{L}^2(I \cap \Gamma_N)}^2 + h_I \|\mathbf{P}_n^\varepsilon \mathbf{n} + \mathbf{P}_t^\varepsilon \mathbf{t} - \hat{\mathbf{S}}\mathbf{U}^\varepsilon\|_{\mathbf{L}^2(I \cap \Gamma_C)}^2 \\ &+ h_I \left\| \frac{\partial}{\partial \mathbf{S}} (\mathbf{V}\boldsymbol{\Psi}^\varepsilon - (\mathbf{K} + 1/2)\mathbf{U}^\varepsilon) \right\|_{\mathbf{L}^2(I)}^2. \end{aligned} \tag{26}$$

Here $\hat{\mathbf{S}}$ is the discretization of \mathbf{S} (as described by Chernov et al. [5]), $I \in \mathcal{T}_h$ is the mesh element and c and C are positive constants independent of the mesh size h .

Figure 2 shows a frictional contact problem (with Tresca friction) between an elastic body $\Omega = [-1, 1]^2$ and a rigid obstacle $\gamma := [-1, 1] \times \{-1, \mathbf{d}\}$ which is pushed upwards with $\mathbf{d} = 0.6 \cdot 10^{-4}$. Further details were given by Chernov and Stephan [7].

An alternative technique for approximately solving the variational inequality (24) is the mortar method analyzed by Chernov et al. [6]. In contrast to the penalty method, no intermediate formulation is needed and the discrete version of the variational inequality (24) is solved. In order to obtain the *hp-version* of the BEM for the inequality (24) via the mortar method,

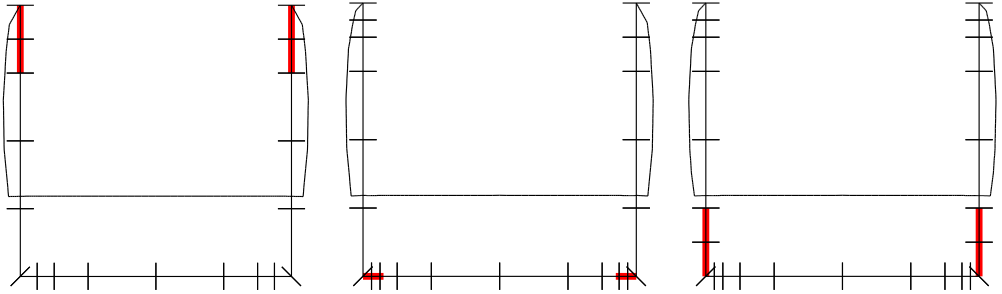


FIGURE 2: Sequence of the adaptively generated meshes and deformed geometries [7].

we decompose each Γ^i , $i = 1, 2$, into a finite family of straight line segments \mathcal{T}_h^i such that each segment belongs to Γ_D^i , or Γ_N^i , or Γ_C^i . Based on \mathcal{T}_h^i , $i = 1, 2$, we define the space of globally continuous piecewise polynomial functions \mathcal{V}_{hp}^i (discretization of the displacement) and the space of globally discontinuous piecewise polynomial functions \mathcal{W}_{hp}^i (discretization of the traction). Let $\mathcal{P}_{p_I}(I)$ be the space of polynomials on I of degree at most p_I . Then we demand $\mathbf{u}|_I \in \mathcal{P}_{p_I}(I)$ and $\Psi|_I \in \mathcal{P}_{p_I-1}(I)$ for arbitrary $\mathbf{u} \in \mathcal{V}_{hp}^i$ and $\Psi \in \mathcal{W}_{hp}^i$. Note that the mesh nodes and the polynomial degrees do not match in general across Γ_C , which is strongly desirable in many applications considered by Wriggers [29]. On the other hand, the non-matching property makes discretization of the convex set of admissible solutions \mathcal{K} in (24) more complex, since the condition $\mathbf{u}_n^1 - \mathbf{u}_n^2 \equiv [\mathbf{u}_n] \leq \mathbf{g}$ is difficult to realize in each $\mathbf{x} \in \Gamma_C$. We impose this condition in a weak sense. In order to define discrete contact conditions we introduce auxiliary spaces of normal traces on Γ_C^i by $\mathcal{N}_{hp}^i := \{W = \mathbf{u} \cdot \mathbf{n}^i|_{\Gamma_C} : \mathbf{u} \in \mathcal{V}_{hp}^i\}$, and the mortar space $\mathcal{M}_{hp}^1 := \{\Psi \in \mathcal{N}_{hp}^1 : \Psi \in \mathcal{P}_{p_I-1}(I), \text{ if } I \cap \partial\Gamma_C \neq \emptyset\}$, without loss of generality associated with \mathcal{T}_h^1 . We define the hp-mortar projection operator (as introduced by Chernov et al. [6] and by Seshaiyer and Suri [21])

$\pi_{\text{hp}}^1 : H^{1/2}(\Gamma_C) \rightarrow \mathcal{N}_{\text{hp}}^1$ by

$$\begin{aligned} \pi_{\text{hp}}^1 \phi &= \phi \quad \text{in } \partial\Gamma_C, \\ \int_{\Gamma_C} (\phi - \pi_{\text{hp}}^1 \phi) \Psi^1 \, ds &= 0 \quad \text{for all } \Psi^1 \in \mathcal{M}_{\text{hp}}^1. \end{aligned} \tag{27}$$

Let \mathbf{G}_{hp}^i be the set of Gauss–Lobatto nodes associated with the elements of \mathcal{T}_h^i . Let $\mathbf{V}_{\text{hp}} := \mathbf{V}_{\text{hp}}^1 \times \mathbf{V}_{\text{hp}}^2$. Now we give the discrete counterpart of (24) obtained by the hp-version BEM and the mortar projection: find $\mathbf{U} \in \mathcal{K}_{\text{hp}} := \{\mathbf{U} \in \mathbf{V}_{\text{hp}} : \pi_{\text{hp}}^1[\mathbf{U}_n](\mathbf{x}) \leq \mathbf{g}(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbf{G}_{\text{hp}}^1 \cap \Gamma_C\}$:

$$\begin{aligned} \int_{\Gamma} (\hat{\mathbf{S}}\mathbf{U}) \cdot (\mathbf{\Phi} - \mathbf{U}) \, ds + \int_{\Gamma_C} \mathcal{F}(|\pi_{\text{hp}}^1[\mathbf{\Phi}_t]| - |\pi_{\text{hp}}^1[\mathbf{U}_t]|) \, ds \\ \geq \int_{\Gamma_N} \hat{\mathbf{t}} \cdot (\mathbf{\Phi} - \mathbf{u}) \, ds \quad \text{for all } \mathbf{\Phi} \in \mathcal{K}_{\text{hp}}. \end{aligned} \tag{28}$$

Note that in general $\mathcal{K}_{\text{hp}} \not\subset \mathcal{K}$.

A heuristic local a posteriori error indicator for the variational inequality (28) was presented by Chernov and Stephan [7] together with numerical experiments (based on a three-step hp-adaptive algorithm introduced by Maischak and Stephan [17]). Our experiments [7] (compare Figures 3 and 4) show that this adaptive procedure leads to appropriate mesh refinement and distribution of polynomial degrees (given by the numbers in Figure 4) respecting the singular behavior of the solution at the contact zone and at the corners or where the boundary conditions change.

5 FE/BE for viscoplastic thermo-mechanical coupling

To compute the metal turning process, we use the following FE/BE-procedure for the velocity and temperature formulation of the viscoplastic thermo-

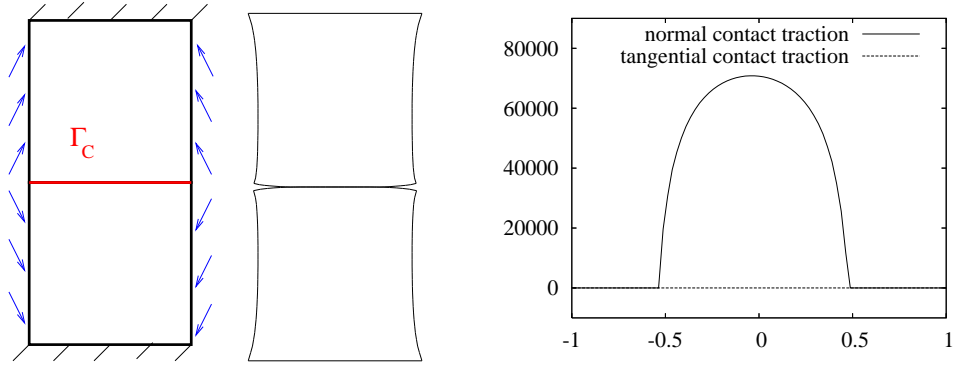


FIGURE 3: Model problem, deformed configuration and contact traction [7].

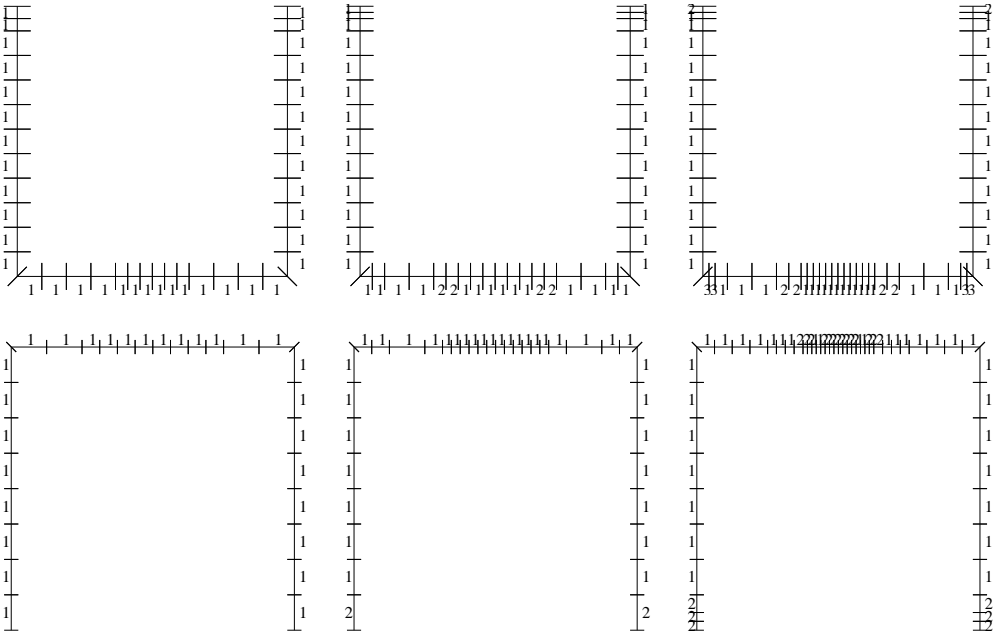


FIGURE 4: Adaptively generated meshes and polynomial degrees after three, six and nine refinement steps [7].

mechanical contact problem. For this purpose we choose Hart’s constitutive model to allow viscoplastic behaviour in the workpiece. Keeping track of two internal state variables, we only have to solve a series of linear elastic problems using the hypoelastic material law. These variables are the anelastic strain tensor $\hat{\varepsilon}^\alpha$ and a scalar $\sigma^{(*)}$, called hardness, which is similar to an isotropic strain hardening parameter or current yield stress. As opposed to other viscoplastic material models, Hart’s model enables us to use finite elements as well as boundary elements for a viscoplastic workpiece. Further details are investigated by Stephan et al. [25] and Mukherjee and Chandra [18].

Now, we consider the initial boundary value problem for velocity \mathbf{v} , temperature Θ and strain rates $\nabla \mathbf{v}$ as defined below.

Let $\mathbf{u}^i(\mathbf{x}, 0)$, $\mathbf{v}^i(\mathbf{x}, 0)$ and $\Theta^i(\mathbf{x}, 0)$ ($i = 1, 2$) denote the initial displacement, velocity and temperature, respectively. For given density ρ , rate of body force $\dot{\mathbf{f}}$ in $\Omega_t := (\Omega_t^1, \Omega_t^2)$ and boundary traction $\dot{\mathbf{t}}$ on the Neumann boundary Γ_t^N we seek $\mathbf{v}^1 \in \Omega_t^1$, \mathbf{v}^2 on $\Gamma_t^2 := \Gamma_{tN}^2 \cup \Gamma_{tC}^2$ and $\Theta := (\Theta^1, \Theta^2)$ in Ω_t , such that the following system of equations is satisfied

$$\begin{aligned} & \int_{\Omega_t^1} (\nabla \mathbf{v}^1) : \mathbb{C} : (\nabla \tilde{\mathbf{v}}^1) - \int_{\Omega_t^1} (\nabla \mathbf{v}^1) : \mathring{\mathbf{G}}^T(\sigma) : (\nabla \tilde{\mathbf{v}}^1) + \langle \mathbf{S} \mathbf{v}^2, \tilde{\mathbf{v}}^2 \rangle_{\Gamma_t^2} \\ & + \langle \dot{\mathbf{P}}(\mathbf{v}^1, \mathbf{v}^2), [\tilde{\mathbf{v}}] \rangle_{\Gamma_{tC}^1} - \int_{\Omega_t^1} \mathbf{d}^n : \mathbb{C} : \nabla \tilde{\mathbf{v}}^1 = \int_{\Omega_t^1} \rho \dot{\mathbf{f}} \cdot \tilde{\mathbf{v}}^1 + \int_{\Gamma_t^N} \dot{\mathbf{t}} \cdot \tilde{\mathbf{v}}^1 \\ & \int_{\Omega_t} \left[\frac{\partial \Theta}{\partial t} \tilde{\Theta} + \varkappa \nabla \Theta \nabla \tilde{\Theta} \right] - \gamma_{12} \int_{\Gamma_{tC}^1} \mathbf{P} \cdot \tilde{\mathbf{n}}[\Theta][\tilde{\Theta}] \\ & - \int_{\Gamma_{tC}^1} \mu_f \mathbf{P} \cdot \tilde{\mathbf{n}} |[\mathbf{v}]_{\tau}| \left(\gamma_1 \tilde{\Theta}^1 + \gamma_2 \tilde{\Theta}^2 \right) = 0 \end{aligned}$$

for $\tilde{\mathbf{v}}^1$ in Ω_t^1 , $\tilde{\mathbf{v}}^2$ on Γ_t^2 , $\tilde{\Theta}$ in Ω_t and $0 \leq t \leq T$.

On the contact boundary Γ_{tC}^1 the term $\dot{\mathbf{P}}$ denotes the rate of the boundary traction. In the model introduced by Hart [11], the nonelastic part \mathbf{d}^n of the deformation rate \mathbf{d} describes the viscoplasticity together with the fourth

order tensor $\overset{\circ}{\mathbb{G}}$. \mathbb{C} denotes the Hooke's tensor, μ_f the friction coefficient. \mathbb{S} is the Steklov–Poincaré boundary integral operator of linear elasticity (DtN mapping) and γ_1 , γ_2 , γ_{12} and \varkappa are suitable constants. $[\Theta]$ and $[\tilde{\mathbf{v}}]$ denote the jump of the temperature and displacement between the two bodies, respectively. $\vec{\mathbf{n}}$ is the exterior normal on Ω_f^2 .

As described by Stephan et al. [25], the heat generation during the process is incorporated by applying a fixed point (staggered) iteration between the equations of mechanical equilibrium and heat conduction. We compute the velocity with finite elements in the work piece and with boundary elements on the boundary of the work tool. The temperature is computed using finite elements in both bodies. The heat conduction equation is discretized with backward Euler in time whereas the viscoplastic material in the work piece is discretized with an explicit Euler time stepping procedure. Finally, the friction at the contact edge between workpiece and tool is computed by a penalty method as performed by Stephan et al. [25].

6 Conclusion

As demonstrated the symmetric coupling of finite elements and boundary elements is a powerful approach for solving nonlinear transmission problems (with linear operators in the unbounded region). Optimal convergence rates are obtained when the hp-version together with geometric mesh refinement is used. For the h-version, hierarchical error estimators are cheaper than residual-type error estimators, whereas both estimators lead to similar adaptive refinements. Since the large discrete systems resulting from the FE/BE-coupling are ill-conditioned and dense, preconditioners are crucial for the applicability of iterative solvers. As shown, efficient preconditioners for the h-, p- and hp-versions can be constructed by the Schwarz method. Since in case of linear elasticity, contact problems can be reduced to variational inequalities on the domain boundary, boundary elements can be applied very successfully. Here one can either perform the traditional penalty method or

implement the more powerful **hp**-version of the BEM together with the mortar projection. Presented numerical results for adaptive refinements show the strength of the methods. Especially for nonlinear problems like the metal turning process with a viscoplastic working piece and a linear elastic working tool, the FE/BE-coupling gives an efficient solution procedure as described by Denkena et al. [9]. Here, of course, the FEM-part takes care of the viscoplastic material, whereas the linear elastic tool is modelled via boundary elements.

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