On a non-standard two-species stochastic competing system and a related degenerate parabolic equation

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Abstract

We propose and analyse a new stochastic competing two-species population dynamics model. Competing algae population dynamics in river environments, an important engineering problem, motivates this model. The algae dynamics are described by a system of stochastic differential equations with the characteristic that the two populations are competing with each other through the environmental capacities. Unique existence of the uniformly bounded strong solution is proven and an attractor is identified. The Kolmogorov backward equation associated with the population dynamics is formulated and its unique solvability in a Banach space with a weighted norm is discussed. Our mathematical analysis results can be effectively utilized for a foundation

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of modelling, analysis, and control of the competing algae population dynamics.

Contents

1 Introduction

Competing population dynamics have long been a central topic in mathematical biology. In particular, stochastic process models based on stochastic differential equations (SDEs) have served as efficient mathematical tools for modelling and analysis of population dynamics [13]. Predator-prey systems under a stochastic environment have rich mathematical structures, and have been studied in detail [5]. Models with degenerate drift and/or diffusion coefficients are often realistic candidates for stochastic dynamical modelling, especially in ecology and epidemiology. Lv, Zou, and Tian [11] showed that solutions to a stochastic Lotka–Volterra model with degenerate diffusion coefficients are almost surely confined in a compact set. A stochastic epidemic model with a saturated contact rate has been studied with an SDE possessing degenerate drift and diffusion coefficients [9]. Grandits, Kovacevic, and Veliov [6] analysed the value of information in epidemiological dynamics using a system of SDEs with degenerate coefficients combined with a dynamic programming principle. Cai, Cai, and Mao ([2, 1]) considered a system of SDEs driven by independent and correlated Brownian noises multiplied by degenerate diffusion coefficients.

In this article, we propose a new stochastic two-species competing population dynamics model as a system of SDEs. The model is motivated by competing population dynamics of filamentous and non-filamentous algae attached on the riverbed in dam-downstream reaches [17]. The nuisance filamentous algae, such as the periphyton *Cladophora glomerata*, are weak against turbulent river flows found in natural rivers, but can persist under low-flow conditions occurring in rivers where humans regulate the flow conditions through dam operations. In addition, aquatic species like *Plecoglossus altivelis altivelis* eat but cannot digest them [18]. On the other hand, non-filamentous algae, such as diatoms which serve as staple food for fish, are more tolerant against turbulence but possibly have smaller intrinsic growth rates [16, for periphyton, 12, for diatoms]. These algae species compete on the riverbed, and tracking and predicting their population dynamics is an important industrial problem; however, least attention has been given to modelling and analysis of the population dynamics except via the engineering model [17].

Our first contribution in this article is a mathematical analysis of the population dynamics model. Our system of SDEs, which is new to the best of our knowledge, describes the species as interacting with each other through environmental capacities. We show that this formulation is well-posed and the system admits a unique bounded strong solution. This is a fundamentally important contribution because guaranteeing well-posedness of a mathematical model is essential in its analysis and computation. We also show that the solution eventually converges towards a part of the boundary of a compact set under certain conditions. Our another contribution is a unique solvability result of the Kolmogorov backward equation (KBE) associated with the population dynamics. The KBE is a fundamental equation when statistically assessing the population dynamics. This equation is of a degenerate parabolic type and does not always have classical solutions satisfying the equation in the point-wise sense [14]. With the help of function analysis techniques for degenerate parabolic problems [e.g., 8, 7], we show that the KBE has a variational weak solution in a Banach space with a weighted norm. Throughout this article, we set $\frac{0}{0} = 1$ for the sake of brevity.

2 Population dynamics

2.1 System dynamics

We consider continuous-time two-species population dynamics in a habitat. The time is denoted as $t \geqslant 0$. There are two species in the model and their populations are denoted as X_t and Y_t at time t. We assume that the populations are living in a habitat of unit area, and that the total populations represent the areas shared by the populations. Namely, we assume $0 \leqslant X_t, Y_t, X_t + Y_t \leqslant 1$. Accordingly, we set the compact triangular domain $A = \{(x, y); 0 \leqslant x, y, x + y \leqslant 1\}$ and its interior $\hat{A} = \{(x, y); 0 < x, y, x + y < 1\}$. The boundary of A is denoted as $\partial A = A - \hat{A}$. Later, we show that the dynamics can be completed in A. We do not consider spatially-distributed population dynamics, but this will be considered elsewhere. Hereafter, we use the elementary inequality

$$0 \leqslant \frac{x}{1-y}, \frac{y}{1-x} \leqslant 1 \quad \text{for all} \quad (x, y) \in A\,. \tag{1}$$

The system of SDEs governing the two-species population dynamics is proposed as

$$dX_t = X_t\left(1 - \frac{X_t}{1-Y_t}\right)(r\,dt + \sigma\,dB_{1,t}) \quad \text{for } t > 0\,, \tag{2}$$

and

$$dY_t = X_t\left(1 - \frac{Y_t}{1-X_t}\right)(R\,dt + \omega\,dB_{2,t}) \quad \text{for } t > 0\,, \tag{3}$$

subject to an initial condition (X_0, Y_0), where $r, R > 0$ are deterministic growth rates, $\sigma, \omega > 0$ are the stochastic growth rates, and the expectation

$\mathbb{E}\,[dB_{1,t}dB_{2,t}] = \rho\,dt$. We assume $|\rho| \leqslant 1$. In (2)–(3), the terms involving dt represent deterministic growth, while those involving $dB_{i,t}$ $(i = 1, 2)$ represent the stochastic fluctuations of the dynamics that are potentially correlated. In this system the unit area is shared by the two populations; increasing one population decreases the other's environmental capacity. The model can be formally considered as a competing population dynamics model having the stochastic growth rates $r + \sigma dB_{1,t}/dt$ and $R + \omega dB_{2,t}/dt$ with the state-dependent environmental capacities.

2.2 Mathematical analysis

A fundamental issue of the system (2)–(3) is its unique solvability. The following proposition resolves this issue, guaranteeing that the system is well-posed and the population is certainly confined in A, as desired. Hereafter, set

$$f(x, y) = x\left(1 - \frac{x}{1-y}\right) \quad \text{and} \quad g(x, y) = y\left(1 - \frac{y}{1-x}\right), \tag{4}$$

which are Lipschitz continuous in A.

Proposition 1. *We have $(X_t, Y_t) \in A$ for $t > 0$ almost surely, if $(X_0, Y_0) \in A$.*

Proof: The key of the proof is the Itô's formula [13]: for sufficiently smooth $h = h(X_t, Y_t)$, set $h_1 = \log X_t$, $h_2 = \log Y_t$, and $h_3 = \log(1 - X_t - Y_t)$. We have derivatives $h_{1,x} = 1/X_t$, $h_{1,xx} = -1/X_t^2$, $h_{1,y} = h_{1,yy} = h_{1,xy} = 0$, $h_{2,y} = 1/Y_t$, $h_{2,yy} = -1/Y_t^2$, $h_{2,x} = h_{2,xx} = h_{2,xy} = 0$, $h_{3,x} = h_{3,y} = -1/(1-X_t-Y_t)$, $h_{3,xx} = h_{3,yy} = h_{3,xy} = -1/(1 - X_t - Y_t)^2$. The stopping time, which is when either one of the populations vanishes or when the area A is full, is

$$\tau = \inf\{t > 0 \mid X_t = 0 \text{ or } Y_t = 0 \text{ or } 1 - X_t - Y_t = 0\}. \tag{5}$$

We firstly show $\tau \to +\infty$ almost surely if $(X_0, Y_0) \in \hat{A}$, from which we get $(X_t, Y_t) \in A$ almost surely for $t > 0$.

There exist the six possible cases:

1. $X_\tau > 0\,,\ Y_\tau = 0\,,\ 1 - X_\tau - Y_\tau = 0\,;$
2. $X_\tau = 0\,,\ Y_\tau > 0\,,\ 1 - X_\tau - Y_\tau = 0\,;$
3. $X_\tau = 0\,,\ Y_\tau = 0\,,\ 1 - X_\tau - Y_\tau > 0\,;$
4. $X_\tau > 0\,,\ Y_\tau > 0\,,\ 1 - X_\tau - Y_\tau = 0\,;$
5. $X_\tau = 0\,,\ Y_\tau > 0\,,\ 1 - X_\tau - Y_\tau > 0\,;$
6. $X_\tau > 0\,,\ Y_\tau = 0\,,\ 1 - X_\tau - Y_\tau > 0\,.$

Assume $(X_0, Y_0) \in \hat{A}$. By symmetry, it is enough to show that the cases 1, 3, 4, 5 do not occur. Now, we have

$$dh_1 = r\left(1 - \frac{X_t}{1-Y_t}\right)dt - \frac{\sigma^2}{2}\left(1 - \frac{X_t}{1-Y_t}\right)^2 dt + \sigma\left(1 - \frac{X_t}{1-Y_t}\right)dB_{1,t}\,, \tag{6}$$

$$dh_2 = R\left(1 - \frac{Y_t}{1-X_t}\right)dt - \frac{\omega^2}{2}\left(1 - \frac{Y_t}{1-X_t}\right)^2 dt + \omega\left(1 - \frac{Y_t}{1-X_t}\right)dB_{2,t}\,, \tag{7}$$

$$\begin{aligned} dh_3 = &-\left(r\frac{X_t}{1-Y_t} + R\frac{Y_t}{1-X_t}\right)dt \\ &- \frac{1}{2}\left(\sigma^2\frac{X_t}{1-Y_t}\frac{X_t}{1-Y_t} + \omega^2\frac{Y_t}{1-X_t}\frac{Y_t}{1-X_t} + 2\rho\sigma\omega\frac{Y_t}{1-X_t}\frac{X_t}{1-Y_t}\right)dt \\ &- \sigma\frac{X_t}{1-Y_t}dB_{1,t} - \omega\frac{Y_t}{1-X_t}dB_{2,t}\,. \end{aligned} \tag{8}$$

All the coefficients multiplies of dt, $dB_{1,t}$, $dB_{2,t}$ are bounded for $0 < t < \tau$ by (1) and (5), while at least one of the left-hand sides of (6)–(8) diverges as $t \to \tau$ by (5). This is a contradiction if $\tau < +\infty$, meaning that $\tau = +\infty$. Notice that the drift and diffusion coefficients can be extended to globally Lipschitz continuous functions over $\mathbb{R}^2$.

If $(X_0, Y_0) \in \partial A$, then we get a unique strong solution with the extended coefficients. This completes the proof since $(X_t, Y_t) = (X_0, Y_0)$ almost surely for $t > 0$. ♠

The next proposition is on a long-time limit of the population dynamics. Set $L = \{(x, y)\,; x, y \geqslant 0\,, x + y = 1\}$, which is a subset of ∂A.

Proposition 2. *If* $(X_0, Y_0) \in \hat{A}$ *and* $r + R > \frac{1}{2}\sigma + \frac{1}{2}\omega^2 + \max\{-\rho, 0\}\,\sigma\omega$, *then almost surely* $\lim_{t \to +\infty} (X_t, Y_t) \in L$.

Proof: From the conventional iterated logarithm estimate for a 1D standard Brownian motion B_t [9, Th. 5.1.2], we have, almost surely

$$\limsup_{t \to +\infty} \frac{B_t}{\sqrt{2t \log(\log t)}} = 1\,. \tag{9}$$

Assume $(X_0, Y_0) \in \hat{A}$. Then, by Ito's formula [13], we get

$$dh_1 + dh_2 - dh_3 = \left(r + R - \frac{\sigma^2}{2} - \frac{\omega^2}{2}\right) dt + \omega^2 \frac{Y_t}{1 - X_t} dt + \sigma^2 \frac{X_t}{1 - Y_t} dt \\ + \rho\sigma\omega \frac{Y_t}{1 - X_t} \frac{X_t}{1 - Y_t} dt + \sigma\, dB_{1,t} + \omega\, dB_{2,t}\,. \tag{10}$$

By the assumption of the proposition, if $\rho < 0$, then

$$\omega^2 \frac{Y_t}{1 - X_t} + \sigma^2 \frac{X_t}{1 - Y_t} + \rho\sigma\omega \frac{Y_t}{1 - X_t} \frac{X_t}{1 - Y_t} \geqslant \rho\sigma\omega \frac{Y_t}{1 - X_t} \frac{X_t}{1 - Y_t} \geqslant \rho\sigma\omega\,, \tag{11}$$

and thus

$$\liminf_{t \to +\infty} \int_0^t \left[\left(r + R - \frac{\sigma^2}{2} - \frac{\omega^2}{2}\right) \right. \\ \left. + \omega^2 \frac{Y_t}{1 - X_t} + \sigma^2 \frac{X_t}{1 - Y_t} + \rho\sigma\omega \frac{Y_t}{1 - X_t} \frac{X_t}{1 - Y_t}\right] dt = +\infty\,. \tag{12}$$

Integrating (6) in time over $(0,t)$ and taking $\liminf_{t\to+\infty}$ of both sides gives $\liminf_{t\to+\infty}[h_1(X_t,Y_t)+h_2(X_t,Y_t)-h_3(X_t,Y_t)]=+\infty$. By Proposition 1, this means $1-X_t+Y_t\to 0$ as $t\to+\infty$. The case $\rho\geqslant 0$ is essentially the same, and is omitted. ♠

An implication of Proposition 2 is that both populations do not become extinct if the sum of their growth rates is sufficiently large. But a large negative correlation may prevent the dynamics from converging to L.

3 Kolmogorov backward equation

3.1 Formulation

For each $(t,x,y)\in\Omega=[0,T]\times A$ with $T>0$, the conditional expectation

$$\phi(t,x,y)=\mathbb{E}\left[\int_0^t e^{-\delta s}F(s,X_s,Y_s)\,ds+e^{-\delta t}G(X_t,Y_t)\mid(X_0,Y_0)=(x,y)\right], \tag{13}$$

with the discount rate $\delta>0$ and sufficiently regular functions F, G (bounded and continuously differentiable in all the arguments). This ϕ statistically evaluates the population dynamics [13]. The formal governing equation of ϕ is the KBE

$$\frac{\partial\phi}{\partial t}-\mathfrak{A}\phi=F, \tag{14}$$

subject to the initial condition $\phi(t=0)=G$ in A, where

$$\mathfrak{A}\phi=rf\frac{\partial\phi}{\partial x}+Rg\frac{\partial\phi}{\partial y}+\frac{\sigma^2}{2}f^2\frac{\partial^2\phi}{\partial x^2}+\rho\sigma\omega fg\frac{\partial^2\phi}{\partial x\partial y}+\frac{\omega^2}{2}g^2\frac{\partial^2\phi}{\partial y^2}-\delta\phi. \tag{15}$$

Due to the degenerate coefficients, no boundary condition is necessary along ∂A. The KBE is degenerate parabolic, meaning that its unique solvability is not a trivial issue, and that there is no guarantee to have solutions satisfying the equation in a classical point-wise manner. This issue is briefly analysed from a variational viewpoint. Hereafter, we assume $F = G = 0$ on ∂A without significant loss of generality because of the linearity of the KBE.

3.2 Mathematical analysis

We introduce a weighted Sobolev space specialized for our problem:

$$V = \left\{\phi : A \to \mathbb{R}, \|\phi\|_V^2 < +\infty, \phi = 0 \text{ on } \partial A\right\}, \tag{16}$$

with the norm

$$\|\phi\|_V^2 = \|\phi\|_{L^2}^2 + \left\|f\frac{\partial \phi}{\partial x}\right\|_{L^2}^2 + \left\|g\frac{\partial \phi}{\partial y}\right\|_{L^2}^2 . \tag{17}$$

The dual V^* of V is equipped with the dual norm

$$\|v\|_{V^*} = \sup_{u \in V} \frac{(v,u)_{V^* \times V}}{\|u\|_V} . \tag{18}$$

Set the bilinear form $a : V \times V \to \mathbb{R}$ corresponding to the KBE as

$$\begin{aligned} a\,(u,v) = {}& \int_A \left[\frac{\sigma^2}{2} f^2 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\rho\sigma\omega}{2} fg\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right) + \frac{\omega^2}{2} g^2 \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right] dx dy \\ & - \int_A \left(b_0 \frac{\partial u}{\partial x} v + b_1 \frac{\partial u}{\partial y} v\right) dx dy + \int_A \delta u v \, dx dy , \end{aligned} \tag{19}$$

for $u, v \in V$, where

$$b = (b_0, b_1) = \left[rf - \left(\frac{\sigma^2}{2} f^2\right)_x - \left(\frac{\rho\sigma\omega}{2} fg\right)_y, Rg - \left(\frac{\omega^2}{2} g^2\right)_y - \left(\frac{\rho\sigma\omega}{2} fg\right)_x \right]. \tag{20}$$

Notice that $b \perp n$ almost everywhere on ∂A, where n is the outer unit normal on ∂A.

The next proposition is our main result for the KBE.

Proposition 3. *If $|\rho| < 1$, then the* KBE *is uniquely solvable in*

$$L^2((0,T);V) \cap H^1((0,T);V^*). \tag{21}$$

Proof: We only present a sketch of the proof because it is sufficient to check the continuity of a and the Garding inequality

$$a(u,u) \geqslant c\,\|u\|_V^2 - \lambda\,\|u\|_{L^2}^2 \quad \text{for } u \in V, \tag{22}$$

with some constants $c, \lambda > 0$. For example, see the discussion by Lions and Magenes [10, Chap. 3, Sec. 4.4, Th. 4.1]. It is sufficient to check these conditions for every function in $C_0^\infty(A)$ because it is dense in V. A lengthy but straightforward calculation with integration by parts gives that the Garding inequality is satisfied with the constants $c = \min\left\{\frac{1-|\rho|}{2}\sigma^2, \frac{1-|\rho|}{2}\omega^2, \delta\right\}$ and $\lambda = \frac{1}{2}\|\operatorname{div} b\|_{L^\infty} < +\infty$. Checking the continuity of a is rather straightforward with the Cauchy–Schwartz inequality and is omitted here. ♠

As implied in the above proof, the case $|\rho| = 1$ is not covered by the Garding inequality. This case may be handled in the framework of viscosity solutions [4]. It should be noted that using the weighted Sobolev space is essential because of the degenerate diffusion coefficients.

4 Conclusions

We proposed a new stochastic population dynamics model and analysed its unique solvability and boundedness. Unique solvability of the associated KBE was proven in a variational sense. The resulting analysis suggests that the model could potentially serve as a foundation for modelling, analysing and controlling the algae population dynamics.

The stochastic model might assist us in understanding the role of water quality dynamics. For example, the role of dissolved silica and dissolved oxygen can significantly affect the population dynamics [3]. A key practical consideration is that it is often difficult to accurately identify the model parameters through field surveys. Therefore, the model uncertainty should be considered when using the model. The issue of uncertainty can be handled in the framework of non-linear expectation [15], which enable us to analyse SDEs with random coefficients in a rigorous manner. For example, we can allow for uncertainties in the growth rates and/or correlation coefficient. The presented model does not consider catastrophic flood events that suddenly remove the algae population from the riverbed. Such events can be considered to follow some jump process. Studying a discrete-time counterpart of the population dynamics is an interesting topic as well.

There are several issues that need to be addressed in the future. A thorough investigation of the validity of the assumptions in the mathematical analysis needs to be carried out. Determining sufficient conditions leading to species extinction is important as it may lead to a better understand of the singular coefficients in the model. We are currently trying to identify model parameter values through field observations.

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