Block monotone iterations for solving coupled systems of nonlinear parabolic equations

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Abstract

The article deals with numerical methods for solving a coupled system of nonlinear parabolic problems, where reaction functions are quasi-monotone nondecreasing. We employ block monotone iterative methods based on the Jacobi and Gauss–Seidel methods incorporated with the upper and lower solutions method. A convergence analysis and the theorem on uniqueness of a solution are discussed. Numerical experiments are presented.

Contents

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1 Introduction

Several problems in the chemical, physical and engineering sciences are characterized by coupled systems of nonlinear parabolic equations [5]. In this article, we construct block monotone iterative methods for solving the coupled system of nonlinear parabolic equations

$$\begin{aligned} & u_{\alpha,t} - \mathcal{A}_\alpha u_\alpha(x,y,t) + f_\alpha(x,y,t,u) = 0\,, \quad (x,y,t) \in Q_T = \omega \times (0,T]\,, \\ & \omega = \{(x,y) : 0 < x < l_1\,, \quad 0 < y < l_2\}\,, \\ & u_\alpha(x,y,t) = g_\alpha(x,y,t)\,, \quad (x,y,t) \in \partial Q_T = \partial\omega \times (0,T]\,, \\ & u_\alpha(x,y,0) = \psi_\alpha(x,y)\,, \quad (x,y) \in \bar{\omega}\,, \quad \alpha = 1,2\,, \end{aligned} \tag{1}$$

where $u = (u_1, u_2)$, $\partial\omega$ is the boundary of ω, and l_1 and l_2 are positive constants. The differential operators $\mathcal{A}_\alpha$, $\alpha = 1,2$, are defined by

$$\mathcal{A}_\alpha u(x,y,t) \equiv \varepsilon_\alpha(u_{\alpha,xx} + u_{\alpha,yy})\,,$$

where ε_α, $\alpha = 1,2$, are positive constants. It is assumed that the functions $f_\alpha(x,y,t,u)$, $g_\alpha(x,y,t)$ and $\psi_\alpha(x,y)$ are smooth in their respective domains.

Block monotone iterative methods, based on the method of upper and lower solutions, have been used to solve systems of nonlinear elliptic equations [1, 3], as well as systems of scalar [8] and nonlinear [6] parabolic problems. The basic idea of block monotone iterative methods is to decompose a two dimensional problem into a series of one dimensional two-point boundary value problems. Each of the one dimensional problems can be solved efficiently by a standard computational scheme such as the Thomas algorithm. Pao [6] and Zhao [8] did not consider two important points in investigating the block monotone iterative methods, namely, a stopping criterion on each time level and estimates of convergence rates, both of which are considered in this article.

In this article we construct and investigate block monotone iterative methods based on the Jacobi and Gauss–Seidel methods for solving the nonlinear system (1) with quasi-monotone nondecreasing reaction functions f_α, $\alpha = 1, 2$. We extend the block monotone iterative methods of Pao [6] to the case where on each time level, nonlinear difference schemes are solved inexactly and give an analysis of convergence rates of the block monotone iterative methods. In Section 2 we consider a nonlinear difference scheme which approximates the nonlinear parabolic problem (1). Constructions of the block monotone Jacobi and Gauss–Seidel iterative methods are presented. A convergence analysis of the block monotone methods is discussed. A theorem on uniqueness of a solution to the nonlinear difference scheme is given. Section 3 presents numerical experiments.

2 Block monotone iterative methods

On $\bar{\omega} = \omega \cup \partial\omega$ and $[0, T]$ we introduce a rectangular mesh $\bar{\omega}^h = \bar{\omega}^{hx} \times \bar{\omega}^{hy}$ and $\bar{\omega}^\tau$, such that

$$\begin{aligned}
\bar{\omega}^{hx} &= \{x_i, \quad i = 0, 1, \ldots, N_x; \quad x_0 = 0, \quad x_{N_x} = 1; \quad h_x = x_{i+1} - x_i\},\\
\bar{\omega}^{hy} &= \{y_j, \quad j = 0, 1, \ldots, N_y; \quad y_0 = 0, \quad y_{N_y} = 1; \quad h_y = y_{j+1} - y_j\},\\
\bar{\omega}^{\tau} &= \{t_m = m\tau, \quad m = 0, 1, \ldots, N_\tau; \quad N_\tau \tau = T\}.
\end{aligned}$$

For a vector mesh function $\mathbf{U}_{ij,m} = (U_{1,ij,m}, U_{2,ij,m})$, $(i, j) = (x_i, y_j) \in \bar{\omega}^h$, $m \geqslant 1$, we use the implicit difference scheme

$$\mathcal{L}_{\alpha,ij,m} U_{\alpha,ij,m} + \tfrac{1}{\tau}\left(U_{\alpha,ij,m} - U_{\alpha,ij,m-1}\right) + f_{\alpha,ij,m}(\mathbf{U}_{ij,m}) = 0, \quad (i, j) \in \omega^h; \quad (2)$$

$$U_{\alpha,ij,m} = g_{\alpha,ij,m}\,, \quad (i,j) \in \partial\omega^h, \quad m \geqslant 1;$$
$$U_{\alpha,ij,0} = \psi_{\alpha,ij}\,, \quad (i,j) \in \bar{\omega}^h; \quad \alpha = 1,2\,,$$

where $U_{\alpha,ij,m}$ are approximations of $u_\alpha(x,y,t,u)$, at mesh points x_i, y_j, t_m, and $\partial\omega^h$ is the boundary of the mesh $\omega^h = \{(x_i,y_j)\,, i = 1,2,\dots,N_x - 1\,, j = 1,2,\dots,N_y - 1\}$. The linear difference operators $\mathcal{L}_{\alpha,ij,m}$, $\alpha = 1,2$, are defined by

$$\mathcal{L}_{\alpha,ij,m}U_{\alpha,ij,m} = -\varepsilon_\alpha\left(D_x^2 U_{\alpha,ij,m} + D_y^2 U_{\alpha,ij,m}\right),$$

where $D_x^2 U_{\alpha,ij,m}$ and $D_y^2 U_{\alpha,ij,m}$ are the central difference approximations to the second derivatives

$$D_x^2 U_{\alpha,ij,m} = \frac{U_{\alpha,i-1,j,m} - 2U_{\alpha,ij,m} + U_{\alpha,i+1,j,m}}{h_x^2}\,,$$
$$D_y^2 U_{\alpha,ij,m} = \frac{U_{\alpha,i,j-1,m} - 2U_{\alpha,ij,m} + U_{\alpha,i,j+1,m}}{h_y^2}\,.$$

The vector mesh functions $\widetilde{\mathbf{U}}$ and $\widehat{\mathbf{U}}$ are ordered upper and lower solutions of (2) and they satisfy the inequalities

$$\widehat{U}_{\alpha,ij,m} \leqslant \widetilde{U}_{\alpha,ij,m}\,, \quad (i,j) \in \bar{\omega}^h; \tag{3}$$
$$\left(\mathcal{L}_{\alpha,ij,m} + \tau^{-1}\right)\widetilde{U}_{\alpha,ij,m} + f_{\alpha,ij,m}(\widetilde{\mathbf{U}}_{ij,m}) - \tau^{-1}\widetilde{U}_{\alpha,ij,m-1} \geqslant 0\,, \quad (i,j) \in \omega^h;$$
$$\left(\mathcal{L}_{\alpha,ij,m} + \tau^{-1}\right)\widehat{U}_{\alpha,ij,m} + f_{\alpha,ij,m}(\widehat{\mathbf{U}}_{ij,m}) - \tau^{-1}\widehat{U}_{\alpha,ij,m-1} \leqslant 0\,, \quad (i,j) \in \omega^h;$$
$$\widehat{U}_{\alpha,ij,m} \leqslant g_{\alpha,ij,m} \leqslant \widetilde{U}_{\alpha,ij,m}\,, \quad (i,j) \in \partial\omega^h, \quad m \geqslant 1;$$
$$\widehat{U}_{\alpha,ij,0} \leqslant \psi_{\alpha,ij} \leqslant \widetilde{U}_{\alpha,ij,0}\,, \quad (i,j) \in \bar{\omega}^h.$$

For a given pair of ordered upper and lower solutions $\widetilde{\mathbf{U}}$ and $\widehat{\mathbf{U}}$, we define the sector

$$\langle \widehat{\mathbf{U}}_m, \widetilde{\mathbf{U}}_m \rangle = \left\{\mathbf{U}_{ij,m} : \widehat{\mathbf{U}}_{ij,m} \leqslant \mathbf{U}_{ij,m} \leqslant \widetilde{\mathbf{U}}_{ij,m}\,, \quad (i,j) \in \bar{\omega}^h, \quad m \geqslant 1\right\}.$$

We assume that on $\langle \widehat{\mathbf{U}}_m, \widetilde{\mathbf{U}}_m \rangle$ the vector function $\mathbf{f}_{ij,m}$ in (2) satisfies the constraints

$$\widehat{c}_{\alpha,ij,m} \leqslant [f_{\alpha,ij,m}(\mathbf{U}_{ij,m})]_{u_\alpha} \leqslant c_{\alpha,ij,m}\,, \quad \mathbf{U}_{ij,m} \in \langle \widehat{\mathbf{U}}_m\,, \widetilde{\mathbf{U}}_m \rangle, \quad (i,j) \in \bar{\omega}^h; \tag{4}$$

$$0 \leqslant -[f_{\alpha,ij,m}(\mathbf{U}_{ij,m})]_{u_{\alpha'}} \leqslant s_{\alpha,ij,m}\,, \quad \mathbf{U}_{ij,m} \in \langle \widehat{\mathbf{U}}_m\,, \widetilde{\mathbf{U}}_m \rangle\,, \quad (i,j) \in \bar{\omega}^h; \tag{5}$$

for $\alpha' \neq \alpha$ and $\alpha, \alpha' = 1,2$, and where $(f_\alpha)_{u_\alpha} \equiv \frac{\partial f_\alpha}{\partial u_\alpha}$, $(f_\alpha)_{u_{\alpha'}} \equiv \frac{\partial f_\alpha}{\partial u_{\alpha'}}$, functions $c_{\alpha,ij,m}$, $s_{\alpha,ij,m}$ are non-negative bounded functions and $\widehat{c}_{\alpha,ij,m}$ are bounded functions on $\bar{\omega}^h$. The vector function $\mathbf{f}_{ij,m}(\mathbf{U}_{ij,m})$ is quasi-monotone nondecreasing on $\langle \widehat{\mathbf{U}}, \widetilde{\mathbf{U}} \rangle$ if it satisfies the inequality $-(f_{\alpha,ij,m}(\mathbf{U}_{ij,m}))_{u_{\alpha'}} \leqslant s_{\alpha,ij,m}$ in (5).

To construct block iterative methods, we write the difference scheme (2) at an interior mesh point $(i,j,m) \in \omega^{h\tau}$ in the form

$$\begin{aligned} & d^\tau_{\alpha,ij,m} U_{\alpha,ij,m} - l_{\alpha,ij,m} U_{\alpha,i-1,j,m} - r_{\alpha,ij,m} U_{\alpha,i+1,j,m} - b_{\alpha,ij,m} U_{\alpha,i,j-1,m} \\ & \quad - q_{\alpha,ij,m} U_{\alpha,i,j+1,m} = -f_{\alpha,ij,m}(U_{1,ij,m}, U_{2,ij,m}) + \tau^{-1} U_{\alpha,ij,m-1}\,, \quad (i,j) \in \omega^h; \\ & U_{\alpha,ij,m} = g_{\alpha,ij,m}\,, \quad (i,j) \in \partial\omega^h, \quad m \geqslant 1; \quad U_{\alpha,ij,0} = \psi_{\alpha,ij}\,, \quad (i,j) \in \bar{\omega}^h; \\ & l_{\alpha,ij,m} = r_{\alpha,ij,m} = \frac{\varepsilon_\alpha}{h_x^2}\,, \quad b_{\alpha,ij,m} = q_{\alpha,ij,m} = \frac{\varepsilon_\alpha}{h_y^2}\,, \\ & d^\tau_{\alpha,ij,m} = d_{\alpha,ij,m} + \tau^{-1}\,, \quad d_{\alpha,ij,m} = l_{\alpha,ij,m} + r_{\alpha,ij,m} + b_{\alpha,ij,m} + q_{\alpha,ij,m}\,. \end{aligned} \tag{6}$$

For $\alpha = 1,2$, $i \in \bar{\mathcal{I}} = \mathcal{I} \cup \partial\mathcal{I}$ with $\mathcal{I} = \{1, 2, \ldots, N_x - 1\}$ and $\partial\mathcal{I} \equiv \{0, N_x\}$, we define column vectors and diagonal matrices

$$\begin{aligned} & \mathbf{U}_{\alpha,i,m} = (U_{\alpha,i,1,m}, \ldots, U_{\alpha,i,N_y-1,m})^T \\ & F_{\alpha,i,m}(\mathbf{U}_{1,i,m}\,, \mathbf{U}_{2,i,m}) = \\ & \left(f_{\alpha,i,1,m}(U_{1,i,1,m}, U_{2,i,1,m})\,, \ldots, f_{\alpha,i,N_y-1,m}(U_{1,i,N_y-1,m}, U_{2,i,N_y-1,m})\right)^T, \\ & L_{\alpha,i,m} = \operatorname{diag}(l_{\alpha,i,1,m}, \ldots, l_{\alpha,i,N_y-1,m})\,, \quad R_{\alpha,i,m} = \operatorname{diag}(r_{\alpha,i,1,m}, \ldots, r_{\alpha,i,N_y-1,m})\,, \\ & \boldsymbol{\psi}_{\alpha,i} = (\psi_{\alpha,i,0}, \ldots, \psi_{\alpha,i,N_y})^T. \end{aligned}$$

Then the difference scheme (2) is written in the form

$$\begin{aligned} & A^\tau_{\alpha,i,m} \mathbf{U}_{\alpha,i,m} - L_{\alpha,i,m} \mathbf{U}_{\alpha,i-1,m} - R_{\alpha,i,m} \mathbf{U}_{\alpha,i+1,m} \\ & = -F_{\alpha,i,m}(\mathbf{U}_{1,i,m}, \mathbf{U}_{2,i,m}) + \tau^{-1} \mathbf{U}_{\alpha,i,m-1}\,, \quad i \in \mathcal{I}; \\ & \mathbf{U}_{\alpha,i,m} = \mathbf{g}_{\alpha,i,m}\,, \quad i \in \partial\mathcal{I}, \quad m \geqslant 1; \end{aligned} \tag{7}$$

$$u_{\alpha,i,0} = \psi_{\alpha,i}\,, \quad i \in \bar{\mathcal{I}}; \quad \alpha = 1,2\,,$$

where the tridiagonal matrices

$$A^{\tau}_{\alpha,i,m} = [-b_{\alpha,ij,m}, d^{\tau}_{\alpha,ij,m}, -q_{\alpha,ij,m}]\,, \quad j = 1,\ldots,N_y - 1\,,$$

for $i \in \mathcal{I}$, $\alpha = 1,2$, $m \geqslant 1$. The elements of the matrices $L_{\alpha,i,m}$ and $R_{\alpha,i,m}$ are the coupling coefficients of a mesh point to $U_{\alpha,i-1,j,m}$ and $U_{\alpha,i+1,j,m}$, respectively, for $j = 1,2,\ldots,N_y - 1$.

On each time level $m \geqslant 1$, the upper $\{\widetilde{U}^{(n)}_{\alpha,i,m}\}$ and lower $\{\widehat{U}^{(n)}_{\alpha,i,m}\}$, $i \in \bar{\mathcal{I}}$, $\alpha = 1,2$, $m \geqslant 1$, sequences of solutions, are calculated by the following block Jacobi and Gauss–Seidel iterative methods

$$(A^{\tau}_{\alpha,i,m} + C_{\alpha,i,m})Z^{(n)}_{\alpha,i,m} - \eta L_{\alpha,i,m} Z^{(n)}_{\alpha,i-1,m} = -K_{\alpha,i,m}\left(U^{(n-1)}_{\alpha,i,m}, U_{\alpha,i,m-1}, U^{(n-1)}_{\alpha',i,m}\right); \tag{8}$$

$$K_{\alpha,i,m}\left(U^{(n-1)}_{\alpha,i,m}, U_{\alpha,i,m-1}, U^{(n-1)}_{\alpha',i,m}\right) = A^{\tau}_{\alpha,i,m} U^{(n-1)}_{\alpha,i,m} - L_{\alpha,i,m} U^{(n-1)}_{\alpha,i-1,m}$$
$$- R_{\alpha,i} U^{(n-1)}_{\alpha,i+1,m} + F_{\alpha,i,m}\left(U^{(n-1)}_{1,i,m}, U^{(n-1)}_{2,i,m}\right) + \tau^{-1} U_{\alpha,i,m-1}\,, \quad i \in \mathcal{I};$$

$$Z^{(n)}_{\alpha,i,m} = \begin{cases} g_{\alpha,i,m} - U^{(0)}_{\alpha,i,m}\,, & n = 1\,, \\ 0\,, & n \geqslant 2\,, \end{cases} \quad i \in \partial\mathcal{I}\,, \quad m \geqslant 1\,;$$

$$U_{\alpha,i,0} = \psi_{\alpha,i}\,, \quad i \in \bar{\mathcal{I}}; \quad U_{\alpha,i,m} = U^{(n_m)}_{\alpha,i,m}\,, \quad \alpha' \neq \alpha\,, \quad \alpha,\alpha' = 1,2\,,$$

where $K_{\alpha,i,m}\left(U^{(n-1)}_{\alpha,i,m}, U_{\alpha,i,m-1}, U^{(n-1)}_{\alpha',i,m}\right)$, $\alpha' \neq \alpha$, $\alpha,\alpha' = 1,2$, $m \geqslant 1$, are the residuals of the block difference scheme (7) on $U^{(n-1)}_{\alpha,i,m}$, zero column vector 0 has $N_x - 1$ components, $U_{\alpha,i,m}$, $i \in \bar{\mathcal{I}}$, $\alpha = 1,2$, are the approximate solutions on time level $m \geqslant 1$, and n_m is a number of iterations on time level $m \geqslant 1$. For $\eta = 0$ and $\eta = 1$, respectively, we have the block Jacobi and block Gauss–Seidel methods.

2.1 Monotone convergence of iterative sequences

Introduce the notation

$$\Gamma_{\alpha,i,m}(\mathbf{U}_{1,i,m},\mathbf{U}_{2,i,m}) = C_{\alpha,i,m}\mathbf{U}_{\alpha,i,m} - F_{\alpha,i,m}(\mathbf{U}_{1,i,m},\mathbf{U}_{2,i,m}), \tag{9}$$
$$C_{\alpha,i,m} = \operatorname{diag}(c_{\alpha,i,1,m},\ldots,c_{\alpha,i,N_y-1,m}), \quad i\in\bar{\mathcal{I}}, \quad \alpha=1,2, \quad m\geqslant 1,$$

where $c_{\alpha,ij,m}$ are nonnegative bounded functions. Lemma 1 and Theorem 2 provide a monotone property of $\Gamma_{\alpha,i,m}(\mathbf{U}_{1,i,m},\mathbf{U}_{2,i,m})$.

Lemma 1. *Let $\mathbf{U}_{i,m} = (\mathbf{U}_{1,i,m},\mathbf{U}_{2,i,m})$ and $\mathbf{V}_{i,m} = (\mathbf{V}_{1,i,m},\mathbf{V}_{2,i,m})$, $i\in\bar{\mathcal{I}}$, $m\geqslant 1$, be vector mesh functions in the sector $\langle\widehat{\mathbf{U}}_m,\widetilde{\mathbf{U}}_m\rangle$, such that $\mathbf{U}_{\alpha,i,m}\geqslant \mathbf{V}_{\alpha,i,m}$, $i\in\bar{\mathcal{I}}$, $\alpha=1,2$, $m\geqslant 1$. Suppose that the right and left inequalities from (4) and (5), respectively, are satisfied. Then*

$$\Gamma_{\alpha,i,m}(\mathbf{U}_{i,m}) \geqslant \Gamma_{\alpha,i,m}(\mathbf{V}_{i,m}), \quad i\in\bar{\mathcal{I}}, \quad \alpha=1,2, \quad m\geqslant 1. \tag{10}$$

Al-Sultani [2] proves Lemma 1.

Theorem 2. *Let $(\widetilde{U}_{1,ij,m},\widetilde{U}_{2,ij,m})$ and $(\widehat{U}_{1,ij,m},\widehat{U}_{2,ij,m})$ be ordered upper and lower solutions (3). Assume that f_α, $\alpha=1,2$, satisfy (4) and (5). Then the upper $\{\widetilde{\mathbf{U}}^{(n)}_{\alpha,i,m}\}$ and lower $\{\widehat{\mathbf{U}}^{(n)}_{\alpha,i,m}\}$, $i\in\bar{\mathcal{I}}$, $\alpha=1,2$, $m\geqslant 1$, sequences generated by (8), with $\widetilde{\mathbf{U}}^{(0)}_{i,m}=\widetilde{\mathbf{U}}_{i,m}$ and $\widehat{\mathbf{U}}^{(0)}_{i,m}=\widehat{\mathbf{U}}_{i,m}$, converge monotonically such that*

$$\widehat{\mathbf{U}}^{(n-1)}_{\alpha,i,m} \leqslant \widehat{\mathbf{U}}^{(n)}_{\alpha,i,m} \leqslant \widetilde{\mathbf{U}}^{(n)}_{\alpha,i,m} \leqslant \widetilde{\mathbf{U}}^{(n-1)}_{\alpha,i,m}, \quad i\in\bar{\mathcal{I}}, \quad \alpha=1,2, \quad m\geqslant 1, \tag{11}$$

where the inequalities between vectors are in component-wise sense.

Proof: We consider the case of the Gauss–Seidel method with $\eta=1$ in (8). The case of the Jacobi method with $\eta=0$ in (8) can be proved by a similar manner. Since $\widetilde{\mathbf{U}}^{(0)}_{i,m}$, $i\in\bar{\mathcal{I}}$, $m\geqslant 1$, are initial upper solutions (3), from (8) we have

$$A^{\tau}_{\alpha,i,1}\widetilde{\mathbf{Z}}^{(1)}_{\alpha,i,1} - L_{\alpha,i,1}\widetilde{\mathbf{Z}}^{(1)}_{\alpha,i-1,1} + C_{\alpha,i,1}\widetilde{\mathbf{Z}}^{(1)}_{\alpha,i,1} \leqslant \mathbf{0}, \quad i\in\mathcal{I}; \tag{12}$$

$$\widetilde{Z}^{(1)}_{\alpha,i,1} \leqslant 0\,, \quad i \in \partial\mathcal{I}; \quad \alpha = 1,2\,.$$

Taking into account that $L_{\alpha,i,m} > O$, $(A^{\tau}_{\alpha,i,m} + C_{\alpha,i,m})^{-1} > O$, $i \in \mathcal{I}$, $m \geqslant 1$ [7, Corol. 3.20]), where O is the $(N_y - 1) \times (N_y - 1)$ null matrix, for $i = 1$ in (12) and $\widetilde{Z}^{(1)}_{\alpha,0,1} \leqslant 0$, we conclude that $\widetilde{Z}^{(1)}_{\alpha,1,1} \leqslant 0$, $\alpha = 1,2$. For $i = 2$ in (12), using $L_{\alpha,2,1} > O$ and $\widetilde{Z}^{(1)}_{\alpha,1,1} \leqslant 0$, we obtain $\widetilde{Z}^{(1)}_{\alpha,2,1} \leqslant 0$, $\alpha = 1,2$. Thus, by induction on i, we can prove that

$$\widetilde{Z}^{(1)}_{\alpha,i,1} \leqslant 0\,, \quad i \in \bar{\mathcal{I}}\,, \quad \alpha = 1,2\,. \tag{13}$$

Similarly, we can prove that

$$\widehat{Z}^{(1)}_{\alpha,i,1} \geqslant 0\,, \quad i \in \bar{\mathcal{I}}\,, \quad \alpha = 1,2\,. \tag{14}$$

We now prove that

$$\widehat{U}^{(1)}_{\alpha,i,1} \leqslant \widetilde{U}^{(1)}_{\alpha,i,1}\,, \quad i \in \bar{\mathcal{I}}\,, \quad \alpha = 1,2\,. \tag{15}$$

Let $W^{(n)}_{\alpha,i,1} = \widetilde{U}^{(n)}_{\alpha,i,1} - \widehat{U}^{(n)}_{\alpha,i,1}$, $i \in \bar{\mathcal{I}}$, $\alpha = 1,2$. For $\alpha = 1$, from (8) and using notation (9), we have

$$\begin{aligned}
&A^{\tau}_{1,i} W^{(1)}_{1,i,1} - L_{1,i,1} W^{(1)}_{1,i-1,1} + C_{1,i,1} W^{(1)}_{1,i,1} = \\
&\Gamma_{1,i,1}(\widetilde{U}^{(0)}_{1,i,1}, \widetilde{U}^{(0)}_{2,i,1}) - \Gamma_{1,i,1}(\widehat{U}^{(0)}_{1,i,1}, \widehat{U}^{(0)}_{2,i,1}) + R_{1,i,1} W^{(0)}_{1,i+1,1}\,, \quad i \in \mathcal{I}\,, \\
&W^{(1)}_{1,i,1} = 0\,, \quad i \in \partial\mathcal{I}\,.
\end{aligned} \tag{16}$$

From (10), $R_{\alpha,i,m} > O$ and $W^{(0)}_{\alpha,i,m} \geqslant 0$, we obtain

$$A^{\tau}_{1,i,1} W^{(1)}_{1,i,1} + C_{1,i,1} W^{(1)}_{1,i,1} \geqslant L_{1,i,1} W^{(1)}_{1,i-1,1}\,, \quad i \in \mathcal{I}\,, \quad W^{(1)}_{1,i,1} = 0\,, \quad i \in \partial\mathcal{I}\,. \tag{17}$$

Taking into account that $(A^{\tau}_{1,i,1} + C_{1,i,1})^{-1} > O$, $i \in \mathcal{I}$, for $i = 1$ in (17) and $W^{(1)}_{1,0,1} = 0$, we conclude that $W^{(1)}_{1,1,1} \geqslant 0$. For $i = 2$ in (17), using $L_{1,2,1} > 0$

and $\boldsymbol{W}^{(1)}_{1,1,1} \geqslant \mathbf{0}$, we obtain $\boldsymbol{W}^{(1)}_{1,2,1} \geqslant \mathbf{0}$. Thus, by induction on i, we can prove that

$$\boldsymbol{W}^{(1)}_{1,i,1} \geqslant \mathbf{0}, \quad i \in \bar{\mathcal{I}} .$$

By following a similar argument, we can prove (15) for $\alpha = 2$.

We now prove that $\widetilde{\mathbf{U}}^{(1)}_{\alpha,i,1}$ and $\widehat{\mathbf{U}}^{(1)}_{\alpha,i,1}$, $i \in \bar{\mathcal{I}}$, $\alpha = 1, 2$, are upper and lower solutions, respectively, for (3)–(8). From (8), by using notation (9), we conclude that

$$\begin{aligned} \mathbf{K}_{\alpha,i,1}\left(\widetilde{\mathbf{U}}^{(1)}_{\alpha,i,1}, \psi_{\alpha,i}, \widetilde{\mathbf{U}}^{(1)}_{\alpha',i,1})\right) = -R_{\alpha,i,1}\widetilde{\mathbf{Z}}^{(1)}_{\alpha,i+1,1} + \boldsymbol{\Gamma}_{\alpha,i,1}\left(\widetilde{\mathbf{U}}^{(0)}_{\alpha,i,1}, \widetilde{\mathbf{U}}^{(0)}_{\alpha',i}\right) \\ - \boldsymbol{\Gamma}_{\alpha,i,1}\left(\widetilde{\mathbf{U}}^{(1)}_{\alpha,i,1}, \widetilde{\mathbf{U}}^{(1)}_{\alpha',i,1}\right), \end{aligned}$$

for $i \in \mathcal{I}$, but $\alpha' \neq \alpha$, $\alpha, \alpha' = 1, 2$. From (13) and taking into account that $R_{1,i,1} > O$, $i \in \mathcal{I}$, by using (10), we conclude that

$$\mathbf{K}_{\alpha,i,1}\left(\widetilde{\mathbf{U}}^{(1)}_{\alpha,i,1}, \psi_{\alpha,i}, \widetilde{\mathbf{U}}^{(1)}_{\alpha',i,1}\right) \geqslant \mathbf{0}, \quad i \in \mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2. \tag{18}$$

In a similar manner, we obtain

$$\mathbf{K}_{\alpha,i,1}\left(\widehat{\mathbf{U}}^{(1)}_{\alpha,i,1}, \psi_{\alpha,i}, \widehat{\mathbf{U}}^{(1)}_{\alpha',i,1}\right) \leqslant \mathbf{0}, \quad i \in \mathcal{I}, \quad \alpha' \neq \alpha, \quad \alpha, \alpha' = 1, 2.$$

From here, (3), (15) and (18), we conclude that $(\widetilde{\mathbf{U}}^{(1)}_{1,i,1}, \widetilde{\mathbf{U}}^{(1)}_{2,i,1})$, $(\widehat{\mathbf{U}}^{(1)}_{1,i,1}, \widehat{\mathbf{U}}^{(1)}_{2,i,1})$, $i \in \bar{\mathcal{I}}$, are ordered upper and lower solutions of (3)–(7). By induction on n, we can prove that $\{\widetilde{\mathbf{U}}^{(n)}_{\alpha,i,1}\}$, $\{\widehat{\mathbf{U}}^{(n)}_{\alpha,i,1}\}$, $i \in \bar{\mathcal{I}}$, $\alpha = 1, 2$, are, respectively, monotone decreasing upper and monotone increasing lower sequences of solutions. Thus, (11) holds true on the first time level $m = 1$.

Since $\widetilde{\mathbf{U}}^{(0)}_{\alpha,i,2} = \widetilde{\mathbf{U}}_{\alpha,i,2}$, $i \in \bar{\mathcal{I}}$, $\alpha = 1, 2$, on the second time level $m = 2$, from (7), we obtain

$$\mathbf{K}_{\alpha,i,2}\left(\widetilde{\mathbf{U}}_{\alpha,i,2}, \widetilde{\mathbf{U}}^{(n_1)}_{\alpha,i,1}, \widetilde{\mathbf{U}}_{\alpha',i,2}\right) =$$

$$A^{\tau}_{\alpha,i,2}\widetilde{u}_{\alpha,i,2} - L_{\alpha,i,2}\widetilde{u}_{\alpha,i-1,2} - R_{\alpha,i,2}\widetilde{u}_{\alpha,i+1,2} + F_{\alpha,i,2}(\widetilde{u}_{\alpha,i,2}, \widetilde{u}_{\alpha',i,2}) - \tau^{-1}\widetilde{u}^{(n_1)}_{\alpha,i,1}\,,$$

for $i \in \mathfrak{I}$, $\alpha' \neq \alpha$, $\alpha, \alpha' = 1,2$, where $\widetilde{u}^{(n_1)}_{\alpha,i,1}$ are the approximate solutions on the first time level $m = 1$ defined in (8). From here and taking into account that from (11), $\widetilde{u}^{(n_1)}_{\alpha,i,1} \leqslant \widetilde{u}_{\alpha,i,1}$, $i \in \mathfrak{I}$, $\alpha = 1,2$, it follows that

$$K_{\alpha,i,2}\left(\widetilde{u}_{\alpha,i,2}, \widetilde{u}^{(n_1)}_{\alpha,i,1}, \widetilde{u}_{\alpha',i,2}\right) \geqslant K_{\alpha,i,2}\left(\widetilde{u}_{\alpha,i,2}, \widetilde{u}_{\alpha,i,1}, \widetilde{u}_{\alpha',i,2}\right) \geqslant 0\,, \tag{19}$$

which means that $\widetilde{u}^{(0)}_{\alpha,i,2} = \widetilde{u}_{\alpha,i,2}$, $i \in \bar{\mathfrak{I}}$, $\alpha = 1,2$, are upper solutions with respect to $\widetilde{u}^{(n_1)}_{\alpha,i,1}$, $i \in \bar{\mathfrak{I}}$, $\alpha = 1,2$. Similarly, we obtain

$$K_{\alpha,i,2}\left(\widehat{u}_{\alpha,i,2}, \widehat{u}^{(n_1)}_{\alpha,i,1}, \widehat{u}_{\alpha',i,2}\right) \leqslant 0\,, \quad i \in \mathfrak{I}\,, \quad \alpha' \neq \alpha\,, \quad \alpha, \alpha' = 1,2\,,$$

which means that $\widehat{u}^{(0)}_{\alpha,i,2} = \widehat{u}_{\alpha,i,2}$, $i \in \bar{\mathfrak{I}}$, $\alpha = 1,2$, are lower solutions with respect to $\widehat{u}^{(n_1)}_{\alpha,i,1}$, $i \in \bar{\mathfrak{I}}$, $\alpha = 1,2$. From here, (8) and (19), on the second time level $m = 2$, we have

$$(A^{\tau}_{\alpha,i,2} + C_{\alpha,i,2})\,\widetilde{Z}^{(1)}_{\alpha,i,2} \leqslant L_{\alpha,i,2}\widetilde{Z}^{(1)}_{\alpha,i-1,2}\,, \quad i \in \mathfrak{I}\,, \quad \alpha = 1,2\,. \tag{20}$$

Taking into account that $L_{\alpha,i,2} > O$, $(A^{\tau}_{\alpha,i,2} + C_{\alpha,i,2})^{-1} > O$, $i \in \mathfrak{I}$, $\alpha = 1,2$, and $\widetilde{Z}^{(1)}_{\alpha,0,2} \leqslant 0$, for $i = 1$ in (20), it follows that $\widetilde{Z}^{(1)}_{\alpha,1,2} \leqslant 0$, $\alpha = 1,2$. From here and (20) with $i = 2$, we conclude that $\widetilde{Z}^{(1)}_{\alpha,2,2} \leqslant 0$, $\alpha = 1,2$. By induction on i, we can prove that

$$\widetilde{Z}^{(1)}_{\alpha,i,2} \leqslant 0\,, \quad i \in \bar{\mathfrak{I}}\,, \quad \alpha = 1,2\,. \tag{21}$$

Similarly, for initial lower solutions $\widehat{u}_{\alpha,i,2}$, $i \in \bar{\mathfrak{I}}$, $\alpha = 1,2$, we can prove that

$$\widehat{Z}^{(1)}_{\alpha,i,2} \geqslant 0\,, \quad i \in \bar{\mathfrak{I}}\,, \quad \alpha = 1,2\,. \tag{22}$$

The proof that $\widetilde{u}^{(1)}_{\alpha,i,2}$ and $\widehat{u}^{(1)}_{\alpha,i,2}$, $i \in \bar{\mathfrak{I}}$, $\alpha = 1,2$, are ordered upper and lower solutions (3) repeats the proof on the first time level $m = 1$. By induction on n, we can prove that $\{\widetilde{\mathbf{U}}^{(n)}_{\alpha,i,2}\}$, $\{\widehat{\mathbf{U}}^{(n)}_{\alpha,i,2}\}$, $i \in \bar{\mathcal{I}}$, $\alpha = 1,2$, are, respectively, monotone decreasing upper and monotone increasing lower sequences of solutions. Thus, (11) holds true on the second time level $m = 2$. By induction on m, we can prove (11) for $m \geqslant 1$. ♠

2.2 Existence and uniqueness of a solution to the nonlinear difference scheme

Assume that the time step τ satisfies the inequality

$$\tau < \max_{m\geqslant 1} \frac{1}{\beta_m}, \quad \beta_m = \max(0, s_m - \underline{c}_m), \tag{23}$$
$$\widehat{c}_m = \min_{\alpha=1,2}\left[\min_{(i,j)\in\bar{\omega}^h} \underline{c}_{\alpha,ij,m}\right], \quad s_m = \max_{\alpha=1,2}\max_{(i,j)\in\bar{\omega}^h} |s_{\alpha,ij,m}|,$$

where $\underline{c}_{\alpha,ij,m}$ and $s_{\alpha,ij,m}$, $(i,j) \in \bar{\omega}^h$, $\alpha = 1,2$, $m \geqslant 1$, are defined in (4) and (5), respectively. When $\beta_m = 0$, $m \geqslant 1$ there is no restriction on τ.

Theorem 3. *Let $(\widetilde{\mathbf{U}}_{1,ij,m}, \widetilde{\mathbf{U}}_{2,ij,m})$ and $(\widehat{\mathbf{U}}_{1,ij,m}, \widehat{\mathbf{U}}_{2,ij,m})$, $(i,j) \in \bar{\omega}^h$, $m \geqslant 1$, be ordered upper and lower solutions, respectively, of (3)–(7). Suppose that the functions f_α, $\alpha = 1,2$, in (1) satisfy (4), (5), and assumption (23) on the time step τ is satisfied. Then the nonlinear difference scheme (7) has a unique solution.*

Al-Sultani [2] proved Theorem 3.

Remark 4. Al-Sultani [2] proved the existence of a solution to the nonlinear difference scheme (2) under the inequalities $(f_{\alpha,ij,m}(\mathbf{U}_{ij,m}))_{u_\alpha} \leqslant c_{\alpha,ij,m}$ and $0 \leqslant -(f_{\alpha,ij,m}(\mathbf{U}_{ij,m}))_{u_{\alpha'}}$ from (4) and (5), respectively.

2.3 Convergent analysis

Instead of (4), we now assume that

$$s_m \leqslant (f_{\alpha,ij,m}(\mathbf{U}_{ij,m}))_{u_\alpha} \leqslant c_{\alpha,ij,m}\,, \quad \mathbf{U}_{ij,m} \in \langle \widehat{\mathbf{U}}_m\,, \widetilde{\mathbf{U}}_m \rangle\,, \quad (i,j) \in \bar{\omega}^h\,, \tag{24}$$

where $\alpha = 1, 2$, and s_m is defined in (23).

Remark 5. The inequality $s_m \leqslant (f_{\alpha,ij,m}(\mathbf{U}_{ij,m}))_{u_\alpha}$ in (24) can always be obtained by a change of variables. Boglaev [3] provides more details.

A stopping test for the block monotone iterative methods (8) is chosen to be

$$\max_{\alpha=1,2}\left[\max_{(i,j)\in\omega^h}\left|\mathcal{K}_{\alpha,ij,m}(U^{(n)}_{\alpha,ij,m}, U_{\alpha,ij,m-1}, U^{(n)}_{\alpha',ij,m})\right|\right] \leqslant \delta\,, \tag{25}$$

where $\mathcal{K}_{\alpha,ij,m}(U^{(n)}_{\alpha,ij,m}, U_{\alpha,ij,m-1}, U^{(n)}_{\alpha',ij,m})$, $(i,j) \in \omega^h$, $\alpha' \neq \alpha$, $\alpha, \alpha' = 1, 2$, are residuals of the nonlinear difference scheme (7), $\{U^{(n)}_{\alpha,ij,m},\ (i,j) \in \bar{\omega}^h, \alpha = 1, 2,\ m \geqslant 1\}$, are generated by (8), and δ is a prescribed accuracy. On each time level $m \geqslant 1$, we set up $U_{\alpha,ij,m} = U^{(n_m)}_{\alpha,ij,m}$, $(i,j) \in \bar{\omega}^h$, $\alpha = 1, 2$, such that n_m is the minimal number of iterations subject to (25).

Theorem 6. *Let* $(\widetilde{U}_{1,ij,m}, \widetilde{U}_{2,ij,m})$ *and* $(\widehat{U}_{1,ij,m}, \widehat{U}_{2,ij,m})$, $(i,j) \in \bar{\omega}^h$, $\alpha = 1, 2$, $m \geqslant 1$, *be, respectively, ordered upper and lower solutions of (3)–(7). Suppose that the functions* f_α, $\alpha = 1, 2$, *in (1) satisfy (4), (5), and the time step* τ *satisfies (23). Then, for the sequence of solutions* $\{\mathbf{U}^{(n)}_{\alpha,i,m},\ i \in \bar{\mathcal{I}}, \alpha = 1, 2,\ m \geqslant 1\}$ *generated by (8) and (25), we have the estimate*

$$\max_{m\geqslant 1}\max_{\alpha=1,2}\|\mathbf{U}_{\alpha,m} - \mathbf{U}^*_{\alpha,m}\|_{\bar{\omega}^h} \leqslant T\delta\,, \tag{26}$$

where $\mathbf{U}^*_{\alpha,i,m}$, $i \in \bar{\mathcal{I}}$, $\alpha = 1, 2$, $m \geqslant 1$, *are the unique solutions to the nonlinear difference scheme (7).*

Al-Sultani [2] proved Theorem 6.

3 Numerical experiments

As a test problem we consider the Volterra–Lotka cooperating model [4], which is governed by (1) with the reaction functions

$$f_1(u_1,u_2) = -u_1(1-u_1+a_1u_2)\,, \quad f_2(u_1,u_2) = -u_2(1+a_2u_1-u_2)\,, \tag{27}$$

where $u_\alpha \geqslant 0$, $\alpha = 1,2$, are the populations of two species with a symbiotic relationship, and a_α, $\alpha = 1,2$, are positive constants which describe the interaction of the two species. The pairs $(\widetilde{U}_{1,ij,m}, \widetilde{U}_{2,ij,m}) = (K_1, K_2)$ and $(\widehat{U}_{1,ij,m}, \widehat{U}_{2,ij,m}) = (0,0)$ are ordered upper and lower solutions, respectively, on each time level $m \geqslant 1$, with

$$K_1 = a_1K_2 + 1\,, \quad K_2 \geqslant \max\left(\frac{1}{a_1}(\lambda_1 - 1);\, \lambda_2;\, \frac{1}{a_1}(\zeta_1 - 1);\, \zeta_2;\, \frac{a_1+1}{1-a_1a_2}\right),$$

where $\lambda_\alpha = \max_{(x,y)\in\bar\omega} |g_\alpha(x,y)|$, $\zeta_\alpha = \max_{(x,y)\in\bar\omega} |\psi_\alpha(x,y)|$, $\alpha = 1,2$, under the assumption $a_1a_2 < 1$. From here and (27) in the sector $\langle \mathbf{0}, K\rangle$, $K = (K_1,K_2)$, we conclude that f_α, $\alpha = 1,2$, satisfies (4) and (5) with $c_\alpha = K_\alpha$, $\alpha = 1,2$.

Since the exact solution of the test problem is unavailable, we define the numerical error and the order of convergence of the numerical solution as, respectively,

$$E(N) = \max_{\alpha=1,2}\left[\max_{(i,j)\in\bar\omega}\left|U^{(n_\delta)}_{\alpha,ij,m}(N) - U^{(n_\delta)}_{\alpha,ij,m}(N_r)\right|\right], \quad \gamma(N) = \log_2\left(\frac{E(N)}{E(2N)}\right),$$

where $U^{(n_\delta)}_{\alpha,ij,m}(N)$, $\alpha = 1,2$, are the approximate solutions generated by (8) with number of mesh points N, n_δ is the minimal number of iterations subject to (25), and $U^{(n_\delta)}_{\alpha,ij,m}(N_r)$, $\alpha = 1,2$, are reference solutions with number of mesh points N_r.

We choose $\varepsilon_1 = 0.7$, $\varepsilon_2 = 1$, $a_1 = 0.5$, $a_2 = 1$, $g_\alpha(x,y,t) = 0$, $(x,y,t) \in \partial Q_T$, $\alpha = 1,2$, and $\psi_\alpha(x,y) = 1$, $(x,y) \in \bar\omega$, $\alpha = 1,2$, in (1). We take $\delta = 10^{-5}$ in (25) and $N_r = 256$ in the reference solutions.

Table 1: Error and order of convergence of the nonlinear scheme (6).

N	8	16	32	64	128
E	3.46×10^{-1}	9.02×10^{-2}	2.73×10^{-2}	5.65×10^{-3}	1.36×10^{-3}
γ	1.94	1.72	2.27	2.06	

Table 2: Number of iterations n_δ per time step and CPU times for the block methods.

N	8	16	32	64	128
the block Jacobi method					
n_δ	7.23	10.13	17.14	29.47	41.99
CPU (s)	0.02	0.11	0.91	14.17	225.99
the block Gauss–Seidel method					
n_δ	4.00	5.14	8.20	16.48	23.615
CPU (s)	0.01	0.06	0.47	7.34	117.62

In Table 1, for different values of $N = N_x, N_y$, we present the error $E(N)$ and order of convergence $\gamma(N)$. The data in the table indicate that the numerical solution of the nonlinear difference scheme (2) converges to the reference solution with second-order accuracy in the space variables. In Table 2, for different values of N, $T = 0.5$ and $\tau = 0.01$, we present average numbers of iterations n_δ per time step and corresponding CPU times for the block monotone methods. The data show that the block monotone Gauss–Seidel method is approximately twice as fast as the block monotone Jacobi method.

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