Draining under gravity in steel galvanization

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Abstract

The problem of the coating of steel has been considered in several Mathematics in Industry study groups. In this process, after passing through a bath of molten alloy, steel sheeting is drawn upward to allow draining under gravity and stripping using an air knife, leaving a coating of desirable thickness. Here we discuss some aspects of the problem and in particular the gravity draining component. The problem is a very nice introduction to industrial modelling for students, but is also relevant for manufacturing.

Contents

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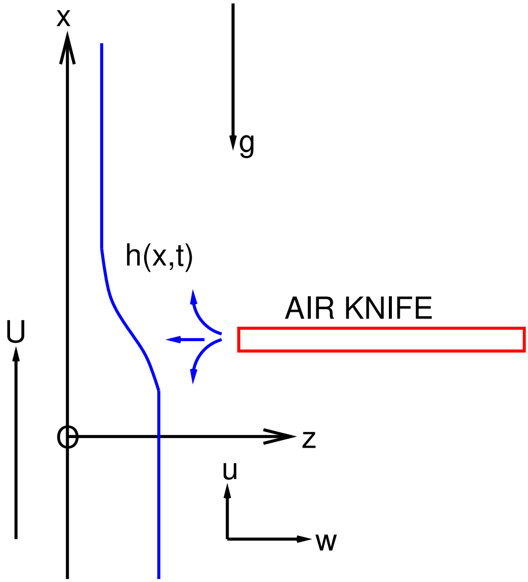
1 Introduction

The process of galvanizing steel has been considered by several Mathematics in Industry study groups [3, 4, 5]. The steel is passed through a bath of coating material and then drawn upward so that some of the liquid coating drains under gravity, and excess can be stripped using an air jet known as an air knife. Mathematical models provide an accurate calculation of the surface coating thickness, and so coatings can be pre-determined by the company, as required. The study groups were asked to investigate the formation of flaws such as pitting and gaps in the coating. However, in this article we discuss some of the basic aspects of the problem and in particular the gravity draining component for objects of different shape.

In the coating process, the sheet is first coated with aluminium (alloy won't 'stick' on steel) and then heated to several hundred degrees. The sheet is then passed through the alloy bath at about 1-2 m/s. A high velocity air jet (called an air knife) strips off excess liquid, which falls back into the bath as the sheet continues upward. As the sheet rises, it cools and eventually solidifies. The coating thickness is determined in the region just above the air knife, but before the solidification has occurred (see Figure 1).

The coating problems were brought to the study groups due to the formation of flaws in the coating as more extreme industrial conditions, such as thinner coatings and poorer quality steel, were employed. In particular, pitting on the surface [3, 4] and the formation of gaps in the coating on the edge of the sheet, called 'bananas" [5]. These flaws appear in different flow and configuration

Figure 1: Sketch of the air knife and coating configuration.



regimes. Studies of these flaws describe the effect of the air jet on a broad flat sheet, and so consider almost uni-directional flow to derive a partial differential equation to compute the coating thickness under different air knife conditions of pressure and shear stress [3, 4, 5]. However, in this work we start to consider the flow for objects of more general shape in the absence of the air knife, when the coating drains under gravity.

2 Unidirectional flow

In cases in which the coating drains under gravity, the so-called unidirectional flow assumption is appropriate. This is a good starting point for an analysis and is also an excellent example to introduce the ideas of industrial modelling to students in a fluid mechanics course!

Assuming steady flow of a viscous, incompressible fluid, define a coordinate system such that x is directed vertically, and flow velocity in that direction is denoted u. Then, if the flow is unidirectional, the horizontal and lateral velocities are zero, that is $v = w = 0$. Implementing these assumptions in the continuity equation, we find (noting that all terms in red are zero)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0\,,$$

so that $u(y, z)$ is a function of y and z only. The momentum equations, with density ρ, pressure p, gravity g and kinematic viscosity ν, are

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} - g + \nu(u_{xx} + u_{yy} + u_{zz})\,,\\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v \quad \Rightarrow \quad p_y = 0\,,\\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w \quad \Rightarrow \quad p_z = 0\,,
\end{aligned}$$

meaning that $p(x)$ is a function of x only. The result is the Poisson equation

$$\nu\,(u_{yy} + u_{zz}) = g + p'(x)/\rho\,. \tag{1}$$

For a broad, flat sheet the flow can be considered as two dimensional, in which case $u(z)$ is a function only of z. The boundary conditions are: at the interface between the steel and the alloy the fluid sticks; on the free surface of the alloy coating $z = h(x)$; there must be no shear stress $u_z = 0$ (or there would be transient motions); and the pressure should be at the atmospheric value for all x so that $dp/dx = 0$, which means that h is a constant. The problem is now one dimensional and is written

$$\frac{d^2u}{dz^2} = g/\nu\,, \tag{2}$$

$$\text{with} \quad u = U \text{ at } z = 0\,, \quad u_z = 0 \text{ at } z = h\,, \tag{3}$$

$$p = 0 \text{ at } z = h\,,$$

where the upward speed of the sheet is U. Integrating (2) and applying the boundary conditions gives

$$u(z) = U + \frac{g}{2\nu} z(z - 2h)\,. \tag{4}$$

This gives the velocity profile, but we do not know the value of the coating thickness h, and so this solution is not complete. Earlier researchers [2, 8, 9] argued that the appropriate condition is the maximum flux criterion; that is, the amount of fluid carried upward is the maximum that can be sustained. The upward flux is

$$Q = \int_0^h u(z)\,dz \; = -\frac{g}{3\nu}h^3 + Uh\,,$$

and the maximum Q is found from

$$\frac{dQ}{dh} = 0 \quad \Rightarrow \quad -\frac{g}{\nu}h^2 + U = 0 \quad \Rightarrow \quad h^* = \sqrt{U\nu/g}\,. \tag{5}$$

Therefore, the thickness is determined by the upward speed U and the kinematic viscosity ν, and so this solution provides important information on the process. Importantly, this also gives the speed of the outside of the coating as $u(h^*) = U/2$, and the maximum flux as $Q_{\max} = \frac{2}{3}U^{2/3}(\nu/g)^{1/2}$.

In terms of the original industry problem, the next step is to consider a 'mainly' vertical flow. The details of this approximation can be found in the work of Tuck [9] and also the study group report [3], but in essence it assumes that although the flow depends on the x location, it is locally unidirectional, and a perturbation solution is obtained incorporating both the pressure and shear generated by the air knife. After some work, this approximation gives a first-order, nonlinear advection equation

$$h_t + c(h, p_a', \tau_a)h_x = A(h, p_a'', \tau_a')\,,$$

where $\tau_a(x)$ is the shear stress from the air knife and p_a is the pressure applied by the air knife. From the theory of first-order partial differential equations we know that $c(h, p'_a, \tau_a)$ is the speed and $A(h, p''_a, \tau'_a)$ is the amplification. Analysis of this equation is given by Tuck [9], and it was also considered in different ways in study groups [3, 4, 5], and in more detail by Hocking et al. [7]. In these works, possible steady state solutions were obtained, characteristics computed and small deviations to the surface were considered. However, our interest here is in the pure draining flow, and so we now consider draining flows for objects with other shapes.

3 Other solutions—Axisymmetric flow

Let us consider the draining problem under the unidirectional flow assumption in cylindrical coordinates. This is a good example for students to try (after they have seen the two dimensional example). In cylindrical, polar coordinates, the resulting problem is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{g}{\nu}, \tag{6}$$

$$\text{subject to} \quad u = U \text{ on } r = R, \quad u'(r) = 0 \text{ on } r = h.$$

Solving for the velocity, we find

$$\begin{aligned} u(r) &= \frac{gr^2}{4\nu} + A\log r + B, \\ &= U + \frac{g}{4\nu}\left(r^2 - R^2\right) + \frac{gh^2}{2\nu}\log(R/r), \end{aligned} \tag{7}$$

but, as before, h is unknown. Here, the upward flux of the coating is

$$\begin{aligned} Q/2\pi &= \int_R^h u(r) r\,dr \\ &= U\frac{(h^2 - R^2)}{2} + \frac{g}{4\nu}\left[(h^4 - R^4)/4 - (h^2 - R^2)R^2/2\right] \\ &\quad + \frac{gh^2}{8\nu}\left[(h^2 - R^2) + 2h^2\log(R/h)\right], \end{aligned} \tag{8}$$

and to find the maximum we require

$$\begin{aligned} \frac{dQ}{dh} &= 2\pi h\left[U + \frac{g}{4\nu}(h^2 - R^2) + \frac{g}{4\nu}\left(h^2 - R^2 - 4h^2\log(h/R)\right)\right] = 0\,, \\ \Rightarrow \quad & (h^2 - R^2) - 2h^2\log(h/R) + 2\nu U/g = 0\,. \end{aligned} \tag{9}$$

This transcendental equation can be solved iteratively for any parameter values, but if we define $\varepsilon = h - R$, then

$$\begin{aligned} 2R\varepsilon + \varepsilon^2 - (2R^2 + 4R\varepsilon + 2\varepsilon^2)\log(1 + \varepsilon/R) + 2\nu U/g = 0\,, \\ -2\varepsilon^2 + \cdots + 2\nu U/g = 0 \quad \Rightarrow \quad \varepsilon \approx \sqrt{\nu U/g}\,, \end{aligned} \tag{10}$$

so that in the limit as $R \to \infty$ the coating thickness is the same as that for a broad sheet.

At the outer edge of the coating, substitute (7) into (9) to obtain the surface velocity

$$\begin{aligned} u(h) &= U + \frac{g}{4\nu}\left(h^2 - R^2\right) - \frac{gh^2}{2\nu}\log(h/R) \\ &= U - U/2 \quad \Rightarrow \quad u = U/2\,. \end{aligned} \tag{11}$$

This is the same relative velocity as for the broad sheet and so the question is whether this is the case for all shapes?

4 General shapes

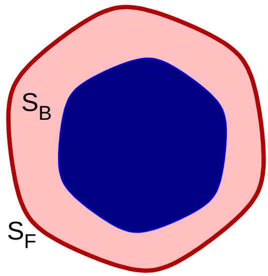
Consider the draining flow problem more generally [10] and recall that

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = g/\nu\,, \tag{12}$$

$$\text{subject to} \quad u = U \quad \text{when } (y,z) \in S_B\,, \tag{13}$$

$$u_n = 0 \quad \text{when } (y,z) \in S_F\,, \tag{14}$$

Figure 2: General shape for draining flow with substrate surface S_B and coating surface S_F.



for flow region S_B and surface S_F (Figure 2), and where u_n is the normal derivative at the outer surface.

This problem has a unique classical solution for *any* S_F and S_B. Therefore, again it is insufficient to solve our problem. It turns out, as before, that there are two extra conditions required:

$$\text{Maximize } Q = \int\!\!\int_R u(y,z)\,dy\,dz\,,$$
$$\text{and} \quad u = U_F \quad \text{when } (y,z) \in S_F\,.$$

Therefore we must optimize the flux upward, as before. This is a classical calculus of variations problem [1]. In both earlier examples it was found that the outer surface moved with velocity one half of the object speed. Tuck et al. [10] showed that this is indeed the case for any shape, so that the final condition that maximizes the flux Q is

$$u = U/2 \quad \text{when } (y,z) \in S_F\,. \tag{15}$$

In principle, we can now solve for any shape. Tuck et al. [10] and Howison and King [6] computed coatings for special cases using numerical and conformal mappings, respectively. Here we demonstrate a simple numerical scheme to compute draining-coat thicknesses for any shape.

Non-dimensionalizing with respect to velocity U and length $L = \sqrt{U\nu/g}$, the problem becomes

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 1 \,, \tag{16}$$

$$\text{subject to} \quad u = 1 \quad \text{when } (y,z) \in S_B \,, \tag{17}$$

$$u_n = 0 \,, \quad u = 1/2 \quad \text{when } (y,z) \in S_F \,, \tag{18}$$

for flow region S_B and surface S_F. Define a new function ϕ such that in non-dimensional form

$$u(y,z) = \tfrac{1}{4}(y^2+z^2) + \phi(z,y) \,, \tag{19}$$

so that the new problem is

$$\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \nabla^2\phi = 0 \,, \tag{20}$$

$$\begin{aligned}
\text{subject to} \quad & \phi = 1 - \tfrac{1}{4}(y^2+z^2) \,, \quad (y,z) \in S_B \,, \\
\text{with} \quad & \nabla\left[\tfrac{1}{4}(y^2+z^2) + \phi(x,y)\right] \cdot \mathbf{n} = 0 \quad \text{when } (y,z) \in S_F \,, \\
\text{and} \quad & \phi = \tfrac{1}{2} - \tfrac{1}{4}(y^2+z^2) \quad \text{when } (y,z) \in S_F \,,
\end{aligned}$$

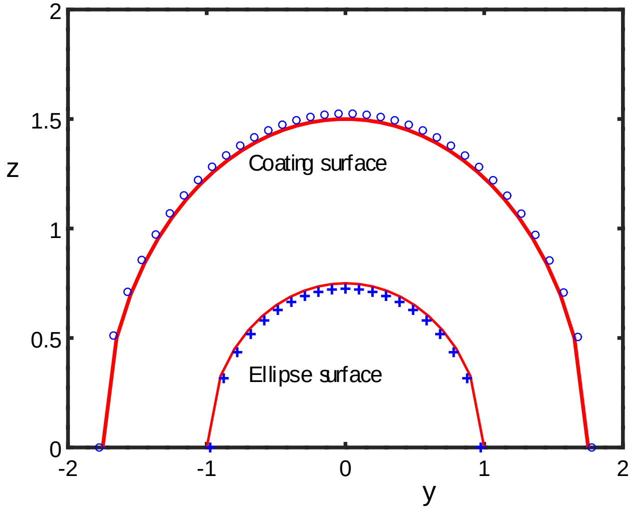
where $\mathbf{n}$ is the outward normal to the surface S_F.

The method is to use a series of fundamental singular solutions to Laplace's equation located outside of the flow region and compute the coefficients to satisfy the boundary conditions. A set of M points are placed inside the body and N points are placed outside of the body and coating (see Figure 3). Let

$$\phi(y,z) = \tfrac{3}{4} + \log(y^2+z^2) + \sum_{k=1}^{M+N} \gamma_k \log\left[(y-Y_k)^2 + (z-Z_k)^2\right] \,, \tag{21}$$

so that $\nabla^2\phi = 0$ is immediately satisfied. The γ_k, $k = 1,2,\ldots,M+N$, are unknown and will need to be determined to satisfy the boundary conditions. The constant term and the first log term provide a solution that 'almost'

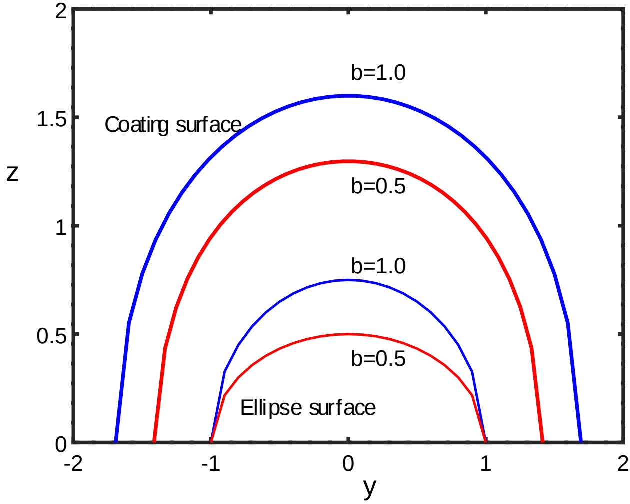
Figure 3: Computed surface shape for an ellipse of unit length with minor axis $b = 0.9$, showing the location of singular points inside the body and outside the surface.



satisfies the case of a circular geometry and improves the convergence of the numerical scheme. Defining the surface of the (object) substrate to be $z = S(y)$, and the surface of the coating to be $z = \eta(y)$, we discretize the surface as $z_j = S(y_j)$, $j = 1, 2, \ldots, M$, and the unknown coating surface as $z_j = \eta(y_j) = \eta_j$, $j = 1, 2, \ldots, N$. Substituting ϕ (21) into equations (12–15) and applying at a set of points on the two surfaces provides $M+2N$ equations for the $M+N$ values of γ_k and the N values of $z_j = \eta_j$.

Figure 3 shows a distribution of the singular points for an elliptical object with a surface coating. An ellipse is a good starting shape since, by changing the axis ratio, you can go from a circular object to a long thin object like a broad sheet. The resulting set of equations need to be solved for the unknown values of the surface $z_j = \eta(y_j)$, $j = 1, 2, \ldots, N$, and strength of the singular points γ_j, $j = 1, 2, \ldots, N+M$. The location of the singular points is determined to ensure that the resulting system of nonlinear equations have a

Figure 4: Surface shapes for ellipses of unit length with minor axis radii $b = 0.75, 0.5$. The upper two curves are the surfaces and the lower two are the substrate shapes. Matching surfaces correspond in colour.

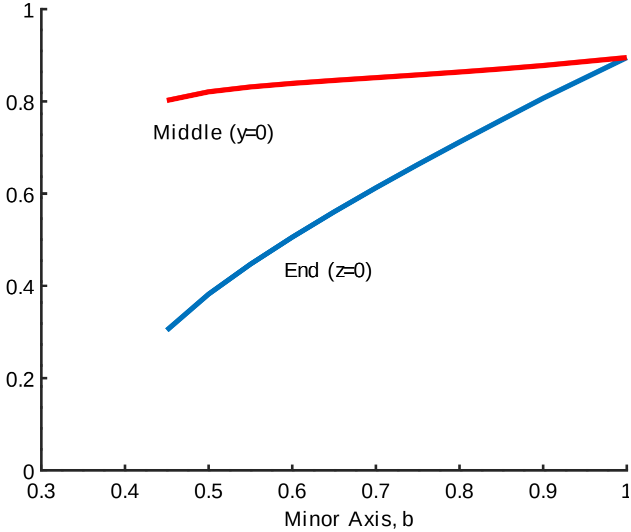


diagonally dominant Jacobian matrix. Each singular point is chosen near one of the surface points, but outside of the flow domain. The normal vector from the surface point is calculated and the distance to the singular point is chosen to be some proportion of the point spacing dy. Typically, this distance was chosen to be $dy/4$, although other values were chosen to test the convergence of the solutions.

Once the equations were set up, an initial guess was made for η_j, $j = 1, 2, \ldots, N$, and γ_j, $j = 1, 2, \ldots, M + N$, and a Newton's method was used to iterate to a solution. A guess that is a surface parallel to the substrate surface with thickness given by (5) was found to be adequate in all cases. Solutions were computed for ellipses with major axes of unit radius and minor axes of radius b.

Figure 4 shows surfaces computed for ellipses of (non-dimensional) unit length, with minor axes $b = 0.5, 0.75$. Only the top half is shown. Figure 5 shows

Figure 5: Coating thickness at the end point ($z = 0$) and middle points ($y = 0$) on the ellipse. When $b = 1$ the object is a circle, so the two values are identical.



the coating thickness on the outer radius and at the narrowest point. As the ellipse becomes thinner the coating all around the object also gets thinner, and the thinning is most pronounced at the ends of the object where the curvature is highest, that is at $z = 0$, reducing to around 30% of the substrate thickness, while the main part of the object reduces by only about 20%. The greater reduction in thickness of the ends, so the coating is thinnest at the points of highest curvature, is consistent with observations in the factory and indicates why many of the flaws observed in practice occur near the edges.

5 Final remarks

Making the assumption of unidirectional flow, a set of equations is derived to model the fundamentals of draining under gravity for an object of any shape. In this article, a simple numerical method has been set up to compute the coating thickness for elliptical shaped objects. These calculations assist in better understanding the galvanization process for objects that contain corners or regions with non-zero curvature. Results indicate that the coating thickness is thinnest in regions of highest curvature, and so it is likely it will be thinnest at edges. The method described herein can be used for any shape.

References

[1] Craggs, J. W. *Calculus of variations*. Allen and Unwin, London, 1973. C38

[2] Elsaadawy, E. A., Hanumanth, G. S., Balthazaar, A. K. S., McDermid, J. R., Hrymak, A. N. and Forbes, J.F. "Coating weight model for the continuous hot-dip galvanizing process", *Metal. Mat. Trans. B*, 38:413–424, 2007. doi:10.1007/s11663-007-9037-2 C35

[3] Hocking, G. C., Sweatman, W. L., Fitt, A. D., and Roberts M. "Coating Deformation in the jet stripping process" in *Proceedings of the 2009 Mathematics and Statistics in Industry Study Group*, Eds. T. Marchant, M. Edwards, G. Mercer. Wollongong, Austealia, 2010. https://documents.uow.edu.au/content/groups/public/@web/@inf/@math/documents/doc/uow073330.pdf C32, C33, C35, C36

[4] Hocking, G. C., Sweatman, W. L., Fitt, A. D., and Breward, C. "Deformations arising during air-knife stripping in the galvanization of steel", in *Progress in Industrial Mathematics at ECMI 2010*, Eds. M. Günther, A. Bartel, M. Brunk, S. Schöps, M. Striebel. Mathematics in Industry 17, pp. 311-317. Springer, Berlin Heidelberg, 2011. doi:10.1007/978-3-642-25100-9_36 C32, C33, C36

[5] Hocking, G. C., Lavalle, G., Novakovic, R., O'Kiely, D., Thomson, S., Mitchell, S. J., Herterich, R. "Bananas—defects in the jet stripping process". *Proceedings of the European Study Group with Industry* in Mathematics and Statistics Research Collection. Rome Italy, 2016.

https://researchrepository.ucd.ie/handle/10197/10215 C32, C33, C36

[6] Howison, S. D. and King, J. R. "Explicit solutions to six free-boundary problems to fluid flow and diffusion". *IMA J. Appl. Math.* 42:155–175, 1989. doi:10.1093/imamat/42.2.155 C38

[7] Hocking, G. C., Sweatman, W., Fitt, A. D. and Breward, C. "Deformations during jet-stripping in the galvanizing process". *J. Eng. Math. Tuck Special Issue,* 70:297–306, 2011. doi:10.1007/s10665-010-9394-8 C36

[8] Thornton, J. A. and Graff, H. F. "An analytical description of the jet-finishing process for hot-dip metallic coatings on strip". *Metal. Mat. Trans. B*, 7:607–618, 1976. doi:10.1007/BF02698594 C35

[9] Tuck, E. O. "Continuous coating with gravity and jet stripping". Phys. Fluids, 26(9):2352–2358, 1983. doi:10.1063/1.864438 C35, C36

[10] Tuck, E. O., Bentwich, M., and van der Hoek, J. "The free boundary problem for gravity-driven unidirectional viscous flows". *IMA J. Appl. Math.* 30:191–208, 1983. doi:10.1093/imamat/30.2.191 C37, C38

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