Optimal parameter for the stabilised five-field extended Hu–Washizu formulation

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Abstract

We present a mixed finite element method for the elasticity problem. We expand the standard Hu–Washizu formulation to include a pressure unknown and its Lagrange multiplier. By doing so, we derive a five-field formulation. We apply a biorthogonal system that leads to an efficient numerical formulation. We address the coercivity problem by adding a stabilisation term with a parameter. We also present an analysis of the optimal choices of parameter approximation.

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1 Introduction

The Hu–Washizu principle was originally formulated for linear elasticity theory. In our previous work [4, 6, 5], we applied the Hu–Washizu formulation to the Poisson problem. In this article, we consider our stabilised formulation for the linear elasticity problem. The linear elasticity problem is stated as follows. Let $\Omega \subseteq \mathbb{R}^d$, $d \in \{2,3\}$ be open and bounded domains. Given a prescribed body force $\mathbf{F} \in [L^2(\Omega)]^d$, the equilibrium equation is

$$-\nabla \cdot \boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \Omega\,, \tag{1}$$

where $\boldsymbol{\sigma}$ is the symmetric Cauchy stress tensor. The strain $\mathbf{d}$ is related to the displacement $\mathbf{u}$ through the relation

$$\mathbf{d} = \boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right]. \tag{2}$$

Here $\mathbf{f} : \Omega \to \mathbb{R}^d$ and $\mathbf{u} : \Omega \to \mathbb{R}^d$. The tensors $\boldsymbol{\sigma}$ and $\mathbf{d}$ are symmetric tensor functions of size $d \times d$ defined on Ω. We assume that $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$. The constitutive relation which relates the stress and strain is given by

$$\boldsymbol{\sigma} = \mathcal{C}\mathbf{d} = \lambda\,(\operatorname{tr}\mathbf{d})\,\mathbf{1} + 2\mu\mathbf{d}\,, \tag{3}$$

where $\mathcal{C}$ denotes the fourth-order elasticity tensor, $\mathbf{1}$ is the identity tensor, and λ and μ are the Lamé parameters. A nearly incompressible elasticity problem arises when λ is very large. In this article, our focus is not on the nearly incompressible material. We mainly focus on the choice of an optimal parameter in our parameterised formulation.

There are several ways to derive a mixed formulation for elasticity problems. Two possible methods are the Hu–Washizu formulation [2] and the displacement-pressure formulation [9]. The displacement, strain and stress are the unknowns in the Hu–Washizu formulation. While the displacement and strain take the role of two primal unknowns, stress takes the role of the Lagrange multiplier [2]. Biorthogonal systems were applied to the Hu–Washizu formulation by Lamichhane, McBride and Reddy [9]. They also introduced a parameterised stabilisation term to ensure that the conditions of well-posedness are satisfied. However, in this article, we construct a parameterised stabilisation that is different from that of Lamichhane, McBride and Reddy [9]. For the displacement-pressure formulation, Lamichhane and Stephan [10] proposed the pressure as the Lagrange multiplier and obtained a symmetric formulation for a biorthogonal system.

Djoko and Reddy [3] gave an extended four-field version of the Hu–Washizu formulation. The four-field formulation is obtained by adding the pressure variable as an extra unknown and introducing a stabilisation term to ensure that the conditions of well-posedness are satisfied. Recently, Zdunek, Rachowicz and Eriksson [12] developed a five-field Hu–Washizu for nearly inextensible and almost incompressible hyperelasticity. They implemented the formulation in an hp-adaptive setting and also included error estimation for an adaptive method.

In this article, we combine the Hu–Washizu and displacement-pressure formulations. In Section 2 we briefly describe the original Hu–Washizu formulation and our extended formulation. In our approach, stress takes the role of Lagrange multiplier in the displacement-strain equation and pressure takes the role of Lagrange multiplier in the displacement-pressure equation. By doing so, we arrive at the five-field formulation for elasticity. In Section 3, to prove the well-posedness condition, we adopt the stabilisation term that was used in our mixed formulation for the Poisson problem [4, 6, 5]. In Section 4 we introduce the finite element discretisation and algebraic formulation. Similarly to our previous work on the Poisson problem [6], in Section 5 we use the continuity and coercivity constants to approximate the optimal parameter for our stabilised form. In Section 6 we then show some numerical examples to verify the convergence rate of our approach.

2 Formulations

2.1 Standard three-field formulation (Hu–Washizu)

In order to derive a mixed formulation for the linear elasticity problem, we start with the minimisation problem. Let $V = H_0^1(\Omega)$ and $\mathbf{V} = V^d$ for $d \in \{2, 3\}$. The Hilbert space H_0^1 is defined in the standard way [2, 1]. Let $\mathbf{S} = \left\{\mathbf{d} \in \left[L^2(\Omega)\right]^{d\times d} : \mathbf{d} \text{ is symmetric}\right\}$. Let $\mathbf{f} \in \left[L^2(\Omega)\right]^{d\times d}$. The variational formulation of the linear elasticity problem with homogenous Dirichlet boundary condition is

$$\min_{\substack{(\mathbf{u},\mathbf{d})\in\mathbf{V}\times\mathbf{S}\\ \mathbf{d}=\varepsilon(\mathbf{u})}} \frac{1}{2}\int_\Omega \mathcal{C}\mathbf{d} : \mathbf{d}\, dx - \ell(\mathbf{u}).$$

We write a weak variational formulation for the relation between the strain and the displacement in terms of the Lagrange multiplier space $\mathbf{T} = \mathbf{S}$ to obtain the saddle-point problem of the minimisation problem. Thus our problem is to find $(\mathbf{u}, \mathbf{d}, \boldsymbol{\sigma}) \in \mathbf{V} \times \mathbf{S} \times \mathbf{T}$ that satisfy

$$\begin{aligned} a[(\mathbf{u},\mathbf{d}),(\mathbf{v},\mathbf{e})] + b[(\mathbf{v},\mathbf{e}),\boldsymbol{\sigma}] &= \ell(\mathbf{v}), \quad (\mathbf{v},\mathbf{e}) \in \mathbf{V}\times\mathbf{S}, \\ b[(\mathbf{u},\mathbf{d}),\boldsymbol{\tau}] &= 0, \quad \boldsymbol{\tau} \in \mathbf{T}, \end{aligned} \tag{4}$$

where

$$a[(\mathbf{u},\mathbf{d}),(\mathbf{v},\mathbf{e})] = \int_\Omega \mathcal{C}\mathbf{d} : \mathbf{e}\, dx, \quad b[(\mathbf{u},\mathbf{d}),\boldsymbol{\tau}] = \int_\Omega [\mathbf{d} - \varepsilon(\mathbf{u})] : \boldsymbol{\tau}\, dx,$$

$$\ell(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx \, .$$

Here the Lagrange multiplier plays the role of stress,

$$\boldsymbol{\sigma} = \mathcal{C}\mathbf{d} = \mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u}) \, .$$

This is a three field formulation, popularly known as the Hu–Washizu formulation for linear elasticity, which is a mixed formulation based on displacement, strain and stress. Moreover, the existence, uniqueness and stability of the problem follows [2].

2.2 Extended Hu–Washizu formulation (Five-field formulation)

In this article, we are interested in an extended Hu–Washizu formulation that could be used for nearly incompressible elasticity by introducing a pressure-like variable $p = \sqrt{\lambda}\,(\nabla \cdot \mathbf{u})$ as an extra unknown [8, 10] defined on the space

$$Q = \left\{ p \in L^2(\Omega) : \int_\Omega p \, dx = 0 \right\} .$$

We write the standard weak formulation of the linear elasticity problem as a minimisation problem

$$\min_{\substack{(\mathbf{u},\mathbf{d},p)\in \mathbf{V}\times \mathbf{S}\times Q \\ \mathbf{d}=\boldsymbol{\varepsilon}(\mathbf{u}), \quad p=\sqrt{\lambda}(\nabla\cdot\mathbf{u})}} \frac{1}{2}\left(2\mu \int_\Omega |\mathbf{d}|^2 \, dx + \int_\Omega |p|^2 \, dx\right) - \ell(\mathbf{u}) \, ,$$

$$\text{where} \quad \ell(\mathbf{u}) = \int_\Omega \mathbf{f} \cdot \mathbf{u} \, dx \, .$$

To obtain the saddle point problem of the minimisation problem, we write a weak variational equation for both $\mathbf{d} = \boldsymbol{\varepsilon}(\mathbf{u})$ and $p = \sqrt{\lambda}\,(\nabla \cdot \mathbf{u})$ in terms of the Lagrange multiplier spaces $\mathbf{T} = \mathbf{S}$ and $M = Q$, respectively. This leads to a saddle point formulation of finding $(\mathbf{u}, \mathbf{d}, p, \boldsymbol{\sigma}, \xi) \in \mathbf{V} \times \mathbf{S} \times Q \times \mathbf{T} \times M$ such that

$$\begin{aligned} a\left[(\mathbf{u}, \mathbf{d}, p), (\mathbf{v}, \mathbf{e}, q)\right] + b\left[(\mathbf{v}, \mathbf{e}, q), (\boldsymbol{\sigma}, \xi)\right] &= \ell(\mathbf{v}), \quad (\mathbf{v}, \mathbf{e}, q) \in \mathbf{V} \times \mathbf{S} \times Q, \\ b\left[(\mathbf{u}, \mathbf{d}, p), (\boldsymbol{\tau}, \eta)\right] &= 0, \quad (\boldsymbol{\tau}, \eta) \in \mathbf{T} \times M, \end{aligned} \tag{5}$$

where

$$\begin{aligned} a\left[(\mathbf{u}, \mathbf{d}, p), (\mathbf{v}, \mathbf{e}, q)\right] &= 2\mu \int_\Omega \mathbf{d} : \mathbf{e}\, dx + \int_\Omega pq\, dx, \\ b\left[(\mathbf{u}, \mathbf{d}, p), (\boldsymbol{\tau}, \eta)\right] &= \int_\Omega \left[\mathbf{d} - \boldsymbol{\varepsilon}(\mathbf{u})\right] : \boldsymbol{\tau}\, dx + \int_\Omega \left(\frac{1}{\sqrt{\lambda}} p - \nabla \cdot \mathbf{u}\right) \eta\, dx. \end{aligned}$$

If we have $p = \sqrt{\lambda}\,(\nabla \cdot \mathbf{u})$, then equation (5) is equivalent to the standard Hu–Washizu formulation [2], and if we have $\mathbf{d} = \boldsymbol{\varepsilon}(\mathbf{u})$, then it is equivalent to the displacement-pressure three-field formulation [8, 10]. We define the norm on our product spaces as

$$\begin{aligned} \|\mathbf{u}, \mathbf{d}\|_{\mathbf{V}\times\mathbf{S}} &= \sqrt{\|\mathbf{u}\|_{1,\Omega}^2 + \|\mathbf{d}\|_{0,\Omega}^2} \quad \text{for } (\mathbf{u}, \mathbf{d}) \in \mathbf{V} \times \mathbf{S}, \\ \|\mathbf{u}, p\|_{\mathbf{V}\times Q} &= \sqrt{\|\mathbf{u}\|_{1,\Omega}^2 + \|p\|_{0,\Omega}^2} \quad \text{for } (\mathbf{u}, p) \in \mathbf{V} \times Q, \\ \|\mathbf{d}, p\|_{\mathbf{S}\times Q} &= \sqrt{\|\mathbf{d}\|_{0,\Omega}^2 + \|p\|_{0,\Omega}^2} \quad \text{for } (\mathbf{d}, p) \in \mathbf{S} \times Q, \\ \|\mathbf{u}, \mathbf{d}, p\|_{\mathbf{V}\times\mathbf{S}\times Q} &= \sqrt{\|\mathbf{u}\|_{1,\Omega}^2 + \|\mathbf{d}\|_{0,\Omega}^2 + \|p\|_{0,\Omega}^2} \quad \text{for } (\mathbf{u}, \mathbf{d}, p) \in \mathbf{V} \times \mathbf{S} \times Q. \end{aligned}$$

In order to show that the saddle-point has a unique solution, we need to show that the following three conditions of well-posedness are satisfied.

1. The linear form $\ell(\cdot)$ and the bilinear forms $a[\cdot,\cdot]$ and $b[\cdot,\cdot]$ are continuous on the spaces in which they are defined.

2. The bilinear form $a[\cdot,\cdot]$ is coercive on the kernel space defined as

$$\begin{aligned} K = \{(\mathbf{u}, \mathbf{d}, p) \in \mathbf{V} \times \mathbf{S} \times Q : b\left[(\mathbf{u}, \mathbf{d}, p), (\boldsymbol{\tau}, \eta)\right] = 0, \\ \text{for all } (\boldsymbol{\tau}, \eta) \in \mathbf{S} \times Q\}. \end{aligned}$$

3. The bilinear form $b[\cdot,\cdot]$ satisfies the *inf-sup* condition; that is, there exists $\gamma > 0$ so that

$$\inf_{(\sigma,\xi)\in \mathbf{S}\times Q} \sup_{(\mathbf{v},\mathbf{e},q)\in \mathbf{V}\times \mathbf{T}\times M} \frac{b\left[(\mathbf{v},\mathbf{e},q),(\boldsymbol{\sigma},\xi)\right]}{\|\mathbf{v},\mathbf{e},q\|_{\mathbf{V}\times\mathbf{S}\times Q}\,\|\boldsymbol{\sigma},\xi\|_{\mathbf{T}\times M}} \geqslant \gamma\,.$$

In general, the discrete kernel space is not a subset of a continuous kernel space. Thus we cannot guarantee that the discrete setting will have the coercivity property. However, if the bilinear form $a[\cdot,\cdot]$ is coercive on the whole space $\mathbf{V}\times\mathbf{S}\times Q$, then the discrete kernel space will be a subset of this space and hence the coercivity will be satisfied in the discrete setting.

In our case, the bilinear form $a[\cdot,\cdot]$ on K is not coercive on the whole space $\mathbf{V}\times\mathbf{S}\times Q$. It is only coercive on the kernel subspace $K\subset\mathbf{V}\times\mathbf{S}\times Q$ which follows from Korn's inequality:

$$|a\left[(\mathbf{u},\mathbf{d},p),(\mathbf{u},\mathbf{d},p)\right]| = 2\mu\,\|\mathbf{d}\|^2_{0,\Omega} + \|p\|^2_{0,\Omega} \geqslant C\,\|\mathbf{u},\mathbf{d},p\|_{\mathbf{V}\times\mathbf{S}\times Q}\,,$$

as $\mathbf{d} = \boldsymbol{\varepsilon}(\mathbf{u})$ and $p = \sqrt{\lambda}\,(\nabla\cdot\mathbf{u})$ on K. Thus, in order to have the coercivity condition on the whole space $\mathbf{V}\times\mathbf{S}\times Q$, we now stabilise the bilinear form $a[\cdot,\cdot]$.

3 Stabilisation method and well-posedness

We define the stabilisation method for the bilinear form $a[\cdot,\cdot]$ so that it is coercive on the whole space $\mathbf{V}\times\mathbf{S}\times Q$ and prove that the conditions of well-posedness are satisfied. Furthermore, we introduce an additional parameter in our method so that we can approximate optimal parameters according to saddle-point theory.

We modify the bilinear form $a[\cdot,\cdot]$ by adding an extra term as follows:

$$\tilde{a}\left[(\mathbf{u},\mathbf{d},p),(\mathbf{v},\mathbf{e},q)\right] = 2\mu\left(r\int_\Omega \mathbf{d}:\mathbf{e}\,dx + (1-r)\int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}):\boldsymbol{\varepsilon}(\mathbf{v})\,dx\right)$$

$$+\int_\Omega pq\,dx\,.$$

Thus our problem is to find $(\mathbf{u},\mathbf{d},p,\boldsymbol{\sigma},\xi)\in\mathbf{V}\times\mathbf{S}\times Q\times\mathbf{T}\times M$ such that

$$\begin{aligned}\tilde{a}\left[(\mathbf{u},\mathbf{d},p),(\mathbf{v},\mathbf{e},q)\right]+b\left[(\mathbf{v},\mathbf{e},q),(\boldsymbol{\sigma},\xi)\right]&=\ell(\mathbf{v})\,,\quad (\mathbf{v},\mathbf{e},q)\in\mathbf{V}\times\mathbf{S}\times Q\,,\\ b\left[(\mathbf{u},\mathbf{d},p),(\boldsymbol{\tau},\eta)\right]&=0\,,\quad (\boldsymbol{\tau},\eta)\in\mathbf{T}\times M\,.\end{aligned}\tag{6}$$

Note that when $r=1$, the modified bilinear form $\tilde{a}[\cdot,\cdot]$ is equivalent to the standard Hu–Washizu formulation [2] and when $r=0$, the system is equivalent to the displacement-pressure three-field formulation [8, 10]. So we set the parameter $0<r<1$ to get the stabilised formulation.

In this setting, we need to show that the conditions of well-posedness are satisfied. The bilinear forms $\tilde{a}[\cdot,\cdot]$, $b[\cdot,\cdot]$ and linear form $\ell(\cdot)$ are continuous by the Cauchy–Schwarz inequality. The coercivity condition satisfied by the bilinear form $\tilde{a}[\cdot,\cdot]$ and the *inf-sup* condition satisfied by the bilinear form $b[\cdot,\cdot]$, follow from Korn's inequality.

The following lemmas only show the continuity and coercivity coefficient for the associated bilinear form $\tilde{a}[\cdot,\cdot]$. Ilyas [7] provides the complete proofs.

Lemma 1. *The bilinear form $\tilde{a}[\cdot,\cdot]$ is continuous on $\mathbf{V}\times\mathbf{S}\times Q$; that is,*

$$\left|\tilde{a}\left[(\mathbf{u},\mathbf{d},p),(\mathbf{v},\mathbf{e},q)\right]\right|\leqslant C\,\|\mathbf{u},\mathbf{d},p\|_{\mathbf{V}\times\mathbf{S}\times Q}\,\|\mathbf{v},\mathbf{e},q\|_{\mathbf{V}\times\mathbf{S}\times Q}\,,$$

where $C=\sqrt{3}\max\{2\mu r,2\mu(1-r),1\}>0$.

Lemma 2. *The bilinear form $\tilde{a}[\cdot,\cdot]$ is coercive on $\mathbf{V}\times\mathbf{S}\times Q$; that is,*

$$\left|\tilde{a}\left[(\mathbf{u},\mathbf{d},p),(\mathbf{u},\mathbf{d},p)\right]\right|\geqslant\alpha\,\|\mathbf{u},\mathbf{d},p\|^2_{\mathbf{V}\times\mathbf{S}\times Q}\,,$$

where $\alpha=\min\{2C_K\mu r,2\mu(1-r),C_K\}>0$ *and* C_K *is the Korn's inequality constant with*

$$C_K\,\|\varepsilon(\mathbf{v})\|_{0,\Omega}\geqslant\|\mathbf{v}\|_{1,\Omega}\,.$$

Therefore, by saddle-point theory, there exists a unique solution of (6); that is, $(\mathbf{u}, \mathbf{d}, p, \boldsymbol{\sigma}, \xi) \in \mathbf{V} \times \mathbf{S} \times Q \times \mathbf{T} \times M$ and the solution is stable with respect to $\ell(\boldsymbol{v})$ on the right hand side of (6) such that:

$$\|\mathbf{u}\|_{1,\Omega} + \|\mathbf{d}\|_{0,\Omega} + \|p\|_{0,\Omega} + \|\boldsymbol{\sigma}\|_{0,\Omega} + \|\xi\|_{0,\Omega} \leqslant C \|\mathbf{f}\|_{\mathbf{V}'} ,$$

where $\mathbf{V}'$ is the dual space of $\mathbf{V}$.

4 Discrete system

We use the standard linear finite element space $V_h \subset H^1(\Omega)$ defined on the triangulation $\mathcal{T}_h$, where

$$V_h := \{v \in C^0(\Omega) : v|_T \in \mathcal{P}_1(T), T \in \mathcal{T}_h\}.$$

Let the finite element space for the pressure-like variables be $Q_h = \{q \in V_h : \int_\Omega q\, dx = 0\}$, for the displacement $\mathbf{V}_h = [V_h]^d \cap \mathbf{V}$ and the strain $\mathbf{S}_h = [V_h]^{d\times d}$.

Similarly to Lamichhane and Stephan [10], space M_h is the space of the Lagrange multiplier of the pressure-like variables. The basis functions of Q_h and M_h satisfy the biorthogonality condition

$$\int_\Omega \rho_i \mu_j \, dx = c_j \delta_{ij} , \quad c_j \neq 0 , \quad 1 \leqslant i, j \leqslant N ,$$

where $\{\rho_1, \rho_2, \ldots, \rho_N\}$ and $\{\mu_1, \mu_2, \ldots, \mu_N\}$ are the finite element bases for Q_h and M_h, respectively, δ_{ij} is the Kronecker symbol, and c_j a scaling factor. In a similar way, we also construct the space $\mathbf{T}_h$ which is the space of the Lagrange multiplier of the strain variables, where the basis functions of $\mathbf{S}_h$ and $\mathbf{T}_h$ satisfy the biorthogonality condition.

To present algebraic formulations of the problem, we use $(x_u, x_d, x_p, x_\sigma, x_\xi)$ for the vector representation of the solution and $(\mathbf{u}_h, \mathbf{d}_h, p_h, \boldsymbol{\sigma}_h, \xi_h)$ as elements in $\mathbf{V}_h \times \mathbf{S}_h \times Q_h \times \mathbf{T}_h \times M_h$. Using the representation of discrete functions in finite-dimensional spaces and performing integrations, all these bilinear forms result in matrix-vector multiplications with the vectors being the unknown variables in the equations. In this way, we define

$$\begin{aligned}
\mathbf{A} &= \int_\Omega \varepsilon\left(\mathbf{u}_h\right) : \varepsilon\left(\mathbf{v}_h\right) dx\,, \quad \mathbf{B}_1 = \int_\Omega \varepsilon\left(\mathbf{u}_h\right) : \boldsymbol{\tau}_h \, dx\,, \\
\mathbf{B}_2 &= \int_\Omega \left(\nabla \cdot \mathbf{u}_h\right) \eta_h \, dx\,, \quad \mathbf{D}_1 = \int_\Omega \mathbf{d}_h : \boldsymbol{\tau}_h \, dx\,, \\
\mathbf{D}_2 &= \int_\Omega p_h \eta_h \, dx\,, \quad \mathbf{M}_1 = \int_\Omega \mathbf{d}_h : \mathbf{e}_h \, dx\,, \quad \mathbf{M}_2 = \int_\Omega p_h q_h \, dx\,.
\end{aligned}$$

Let $\vec{\boldsymbol{\ell}}$ be the vector denoting the discrete form of the linear functional $\ell(\cdot)$. After statically condensing out degrees of freedom corresponding to $x_{\mathbf{d}}$, x_p, $x_{\boldsymbol{\sigma}}$ and x_ξ, we arrive at the reduced system of displacement

$$\left[2\mu\left(1-r\right)\mathbf{A} + 2\mu r \mathbf{B}_2^T \mathbf{D}_2^{-1} \mathbf{M}_2 \mathbf{D}_2^{-1} \mathbf{B}_2 + \lambda \mathbf{B}_1^T \mathbf{D}_1^{-1} \mathbf{M}_1 \mathbf{D}_1^{-1} \mathbf{B}_1\right] x_{\mathbf{u}} = \vec{\boldsymbol{\ell}}\,.$$

We approximate the other variables by the following relations:

$$\begin{aligned}
x_{\mathbf{d}} &= \mathbf{D}_2^{-1} \mathbf{B}_2 x_{\mathbf{u}}\,, & x_p &= \sqrt{\lambda}\mathbf{D}_1^{-1} \mathbf{B}_1 x_{\mathbf{u}}\,, \\
x_{\boldsymbol{\sigma}} &= -2\mu r \mathbf{D}_2^{-1} \mathbf{M}_2 \mathbf{D}_2^{-1} \mathbf{B}_2 x_{\mathbf{u}}\,, & x_\xi &= \lambda \mathbf{D}_1^{-1} \mathbf{M}_1 \mathbf{D}_1^{-1} \mathbf{B}_1 x_{\mathbf{u}}\,.
\end{aligned}$$

5 Optimal parameter approximation

We consider continuity and coercivity constant to approximate optimal parameter r that minimises the error in Céa's Lemma [2]. Our bilinear form $\tilde{a}\left[\cdot,\cdot\right]$ has continuity coefficient $C = \sqrt{2}\max\{2\mu r, 2\mu\left(1-r\right), 1\}$ and coercivity coefficient $\alpha = \min\{2C_K\mu r, 2\mu(1-r), c_K\}$ and C_K is the constant in Korn's inequality. Thus the smallest possible value of constant in Céa's Lemma is obtained when we choose r such that

$$\operatorname*{argmin}_r \frac{C}{\alpha} = \operatorname*{argmin}_r \left\{ \frac{\sqrt{2}\max\{2\mu r, 2\mu\left(1-r\right), 1\}}{\min\{2C_K\mu r, 2\mu(1-r), C_K\}} \right\}.$$

The detailed calculation of r as a function of μ and C_K is omitted. In Section 6, given the value of μ, which depends on the material, and C_K, which depends on the domain size, we calculate and use the optimal parameter as part of the numerical calculation. However, by some scaling we assume $C_K = 1$ for simplicity.

6 Numerical examples

In this section we show two numerical examples to verify the convergence rate of our approach. Since we have uniform refinement, the number of elements in the error approximation N (Table 1 and 2) corresponds to the uniform mesh-size $1/N$.

In the first example, the right hand side vector $\mathbf{f}$ of equation (1) is derived from the following exact solution for displacement $\mathbf{u} = (u_1, u_2)$:

$$\begin{aligned} u_1 &= \sin(2\pi y)[-1 + \cos(2\pi x)] + \frac{1}{1+\lambda}\sin(\pi x)\sin(\pi y)\,, \\ u_2 &= \sin(2\pi x)[1 - \cos(2\pi y)] + \frac{1}{1+\lambda}\sin(\pi x)\sin(\pi y)\,. \end{aligned} \tag{7}$$

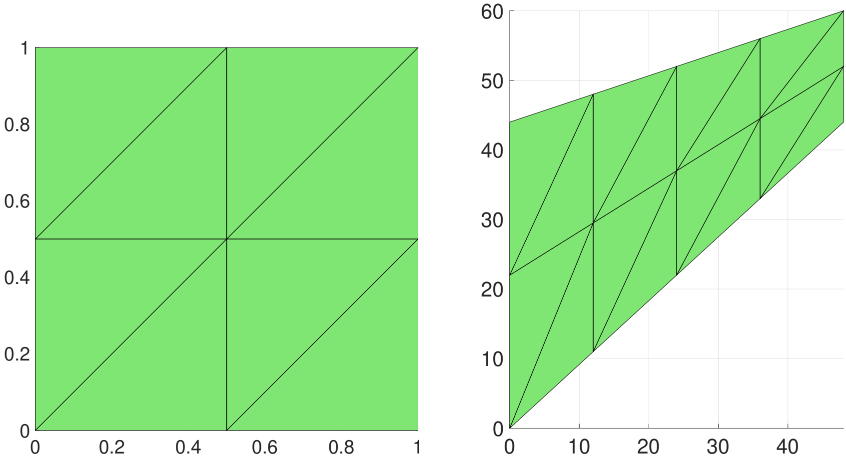
We compute the errors in L^2-norm and the H^1-norm and the rates of convergence for $\mathbf{u}$.

This example has Dirichlet boundary conditions on $\partial\Omega$ with $\Omega = [0, 1]^2$. This example is a well behaved problem with a smooth solution that has no trouble spots. The mesh initialisation is given in the left image of Figure 1. In this example, we set the Young's modulus of elasticity $E = 1500\,\text{MPa}$ and Poisson's ratio $\nu = 0.4999$ so that a nearly incompressible response is obtained. The Lamé parameters are

$$\mu = \frac{E}{2(1+\nu)}\,, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}\,.$$

We calculate the optimal parameter based on the continuity and coercivity

Figure 1: Mesh initialisation.



constant using the Lamé parameter μ and Korn's constant C_K. According to Céa's Lemma, $r = 1/2$ is the optimal value for our problem.

Tables 1 and 2 give the errors and the rates of convergence for $\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$ and $\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}$, respectively, for $r = 1/3$, $r = 1/2$ and $r = 2/3$. These tables show that the error rate for $\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$ and $\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}$ converges with higher order than predicted by the theory. However, when we increase the refinement, both convergence rates decrease to order two in the L^2-norm and order one in the H^1-norm. We conclude that the asymptotic rates are not being achieved at the earlier steps of refinement. The tables also show that our choice of optimal parameter gives better rates of convergence compared to other choices of parameter.

In the second example, we use the popular benchmark called Cook's membrane problem [11]. It was observed that lower order elements with a pure displacement formulation suffer from a severe locking problem. Cook's membrane problem is defined on the two-dimensional tapered panel

$$\Omega = \operatorname{conv}\{(0,0), (48,44), (48,60), (0,44)\}$$

Table 1: Discretisation error $\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}$ for equation (7).

N	$r=1/3$ error	rate	$r=1/2$ error	rate	$r=2/3$ error	rate
8	1.22		1.22		1.22	
32	$7.75\cdot10^{-1}$	0.66	$7.12\cdot10^{-1}$	0.78	$6.44\cdot10^{-1}$	0.92
128	$1.78\cdot10^{-1}$	2.13	$1.20\cdot10^{-1}$	2.57	$8.86\cdot10^{-2}$	2.86
512	$3.65\cdot10^{-2}$	2.28	$1.87\cdot10^{-2}$	2.69	$2.14\cdot10^{-2}$	2.05
2048	$7.69\cdot10^{-3}$	2.25	$3.08\cdot10^{-3}$	2.60	$5.75\cdot10^{-3}$	1.90
8192	$1.69\cdot10^{-3}$	2.19	$5.11\cdot10^{-4}$	2.59	$1.50\cdot10^{-3}$	1.94

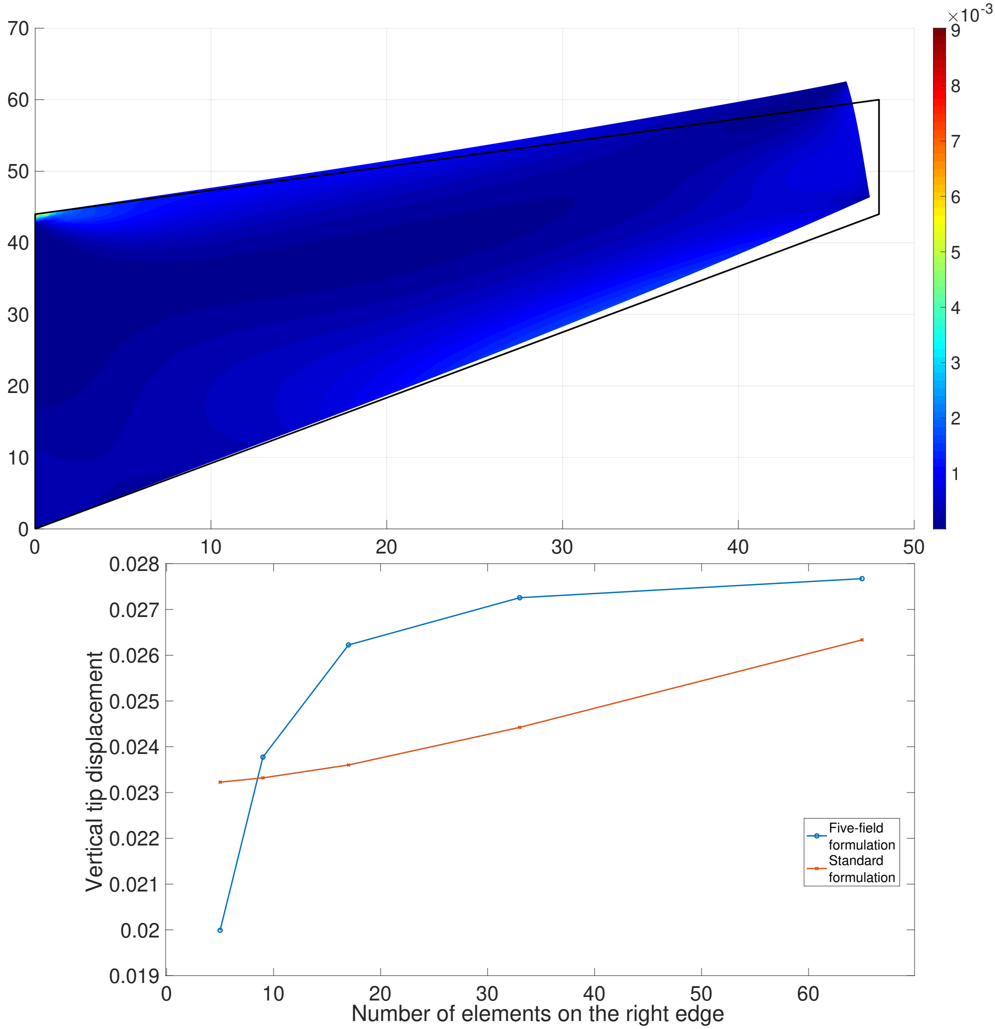
Table 2: Discretisation error $\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}$ for equation (7).

N	$r=1/3$ error	rate	$r=1/2$ error	rate	$r=2/3$ error	rate
8	8.59		8.59		8.59	
32	6.77	0.34	6.87	0.32	7.36	0.22
128	3.15	1.10	3.22	1.09	3.45	1.09
512	1.53	1.04	1.56	1.04	1.62	1.09
2048	$7.41\cdot10^{-1}$	1.05	$7.49\cdot10^{-1}$	1.0590	$7.65\cdot10^{-1}$	1.08
8192	$3.61\cdot10^{-1}$	1.04	$3.63\cdot10^{-1}$	1.05	$3.66\cdot10^{-1}$	1.06

of Plexiglass (with $E = 2900$ MPa and $\nu = 0.4$). This problem has a bending dominated elastic response. The panel is clamped at one end ($x = 0$) and subjected to a shearing load $g = (0, 10)$ on the other end ($x = 48$) with zero volume force. In this example, we use the optimal parameter $r = 1/2$. The mesh initialisation is given in the right image of Figure 1. Figure 2 shows the von Mises stress, representing the three-dimensional stress of the deformed mesh, and the vertical tip displacement comparison.

Our result show von Mises peak stresses only at the corner $(0, 44)$, as expected. The vertical tip displacement from our approach also shows good convergence behaviour without any locking effect. As a comparison, the standard formulation exhibits the locking phenomenon as the vertical displacement blows up.

Figure 2: Deformed mesh for Cook's membrane for five field formulation (top), and vertical tip displacement at $(48, 60)$ versus number of elements on the right edge (bottom).



7 Conclusion

In this article, we describe a mixed finite element method to solve elasticity problems based on the Hu–Washizu formulation. We add a stabilisation term so that our bilinear form is coercive on the whole space. We calculate the optimal parameter based on an extension of Céa's Lemma for mixed finite element problems, approximated by the continuity and coercivity conditions of the associated bilinear form. Numerical examples show that our approach gives the expected results.

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