Bounds on isolated scattering number

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Abstract

The isolated scattering number is a parameter that measures the vulnerability of networks. This measure is bounded by formulas depending on the independence number. We present new bounds on the isolated scattering number that can be calculated in polynomial time.

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1 Introduction

The isolated scattering number is a parameter that measures the vulnerability of networks [1]. The parameter was defined by Wang et al. [8]. They determined the isolated scattering number (e.g., for cycles, bipartite graphs, and the join of bipartite graphs) and they gave bounds on the isolated scattering number depending on the independence number. They also established the maximum and minimum isolated scattering numbers of trees with a given order and a maximum degree. Furthermore, Li et al. [6, 5] proved that for split and interval graphs the isolated scattering number can be computed in polynomial time. They also determined the isolated scattering number for some product graphs [6]. We present new bounds on the isolated scattering number that can be calculated in polynomial time.

2 Preliminaries

A *graph* is a finite set V of elements called *vertices* together with a set $E \subseteq [V]^2$ of elements called *edges*, where $[V]^2$ is the set of all two-element subsets of V. Let G be a graph. Let $u, v \in V(G)$ and $\{u, v\} \in E(G)$. The edge $\{u, v\}$ is said to be *incident* to the vertex u in G. The *open neighborhood* of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V(G) \colon \{u, v\} \in E(G)\}$, and its *closed neighborhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v, denoted by $d_G(v)$, is the cardinality of its open neighborhood. A vertex of degree zero is referred to as an *isolated vertex* and a vertex of degree one is a *leaf*. The *minimum degree* of G is the smallest degree among the vertices of G and is denoted by $\delta(G)$. If U is a subset of vertices of G, we write $G[U]$ and $G - U$ for $(U, E(G) \cap [U]^2)$ and $G[V(G) \setminus U]$, respectively.

A *independent vertex set* in a graph $G = (V, E)$ is a subset $V' \subseteq V$ such that no two vertices of V' are adjacent. The size of a largest independent vertex set in a graph G is called the *independence number* of G and is denoted by $\alpha(G)$. A graph G is *complete* if $\alpha(G) = 1$. A graph on more than one vertex is called a *nontrivial* graph.

A *matching* (respectively *fractional matching*) is a function f that assigns to each edge of a graph G a number in $W = \{0, 1\}$ (respectively $W = [0, 1]$), such that for each vertex v, we have $\sum f(e) \leqslant 1$, where the sum is taken over all edges e incident to v (i.e., over all edges e that contain v). The *matching number* $\mu(G)$ (respectively *fractional matching number* $\mu_f(G)$) is the supremum of $\sum_{e \in E(G)} f(e)$ over all matchings (respectively fractional matchings) f. A graph G is a *Kőnig–Egerváry* graph if $\alpha(G) + \mu(G) = |V(G)|$. A graph G has a *perfect matching* (respectively *fractional perfect matchings*) if $\mu(G) = |V(G)|/2$ (respectively $\mu_f(G) = |V(G)|/2$). Furthermore, we have

$$0 \leqslant \mu(G) \leqslant \mu_f(G) \leqslant \frac{|V(G)|}{2} \tag{1}$$

for every graph G [7].

A *cut set* of a noncomplete graph G is a set S of vertices of G such that $G - S$ is *disconnected*, which mean that there is no path between some two vertices in $G - S$. A cut set of minimum cardinality in G is called a *minimum cut set* of G and this cardinality is called the *connectivity* of G and is denoted by $\kappa(G)$. The set of all cut sets of G is denoted by $C(G)$. For $S \subseteq V(G)$, the value $i(G - S)$ denotes the number of all isolated vertices in $G - S$. The *isolated scattering number* of a noncomplete connected graph G is defined as

$$\operatorname{isc}(G) = \max_{S \in C(G)} \{i(G - S) - |S|\}.$$

We assume that the isolated scattering number of a complete graph on n vertices is equal to $2 - n$.

3 New upper bound on isolated scattering number

In this section, we present an upper bound on the isolated scattering number and we summarize classes of graphs for which $\operatorname{isc}(G)$ can be computed in polynomial time. We also establish the isolated scattering number of some coronas of a graph and we posed some conjectures for such graphs.

Wang et al. [8] established the following lower and upper bounds on the isolated scattering number.

Theorem 1 (Wang et al. [8]). *Let* G *be a noncomplete connected graph. Then*

$$2\alpha(G) - |V(G)| \leqslant \mathrm{isc}(G) \leqslant \alpha(G) - \kappa(G)\,. \tag{2}$$

We propose the following upper bound.

Theorem 2. *Let* G *be a connected graph. Then*

$$\mathrm{isc}(G) \leqslant |V(G)| - 2\mu_f(G)\,. \tag{3}$$

Furthermore, the equality holds if G *is a Kőnig–Egerváry graph.*

Proof: Scheinerman and Ullman [7] showed that

$$\mu_f(G) = \frac{1}{2}\Big(|V(G)| - \max_{S \in 2^{V(G)}} \{i(G-S) - |S|\}\Big),$$

where $2^{V(G)}$ is the set of all subsets of $V(G)$.

If G is trivial, that is, $|V(G)| = 1$, then $\mu_f(G) = 0$ and

$$\mathrm{isc}(G) = 2 - |V(G)| = 1 \leqslant |V(G)| - 2\mu_f(G) = 1\,.$$

If G is a nontrivial complete graph, then

$$\mathrm{isc}(G) = 2 - |V(G)| \leqslant |V(G)| - 2 \cdot (|V(G)|/2) = |V(G)| - 2\mu_f(G)\,.$$

Now let G be a nontrivial, noncomplete connected graph. Since $C(G) \subseteq 2^{V(G)}$, it follows that

$$\mathrm{isc}(G) = \max_{S \in C(G)} \{i(G-S) - |S|\} \leqslant \max_{S \in 2^{V(G)}} \{i(G-S) - |S|\} = |V(G)| - 2\mu_f(G)\,.$$

From Theorem 1 and the first part of Theorem 2, the equality holds in (3) if $2\alpha(G) - |V(G)| = |V(G)| - 2\mu_f(G)$, that is, if $\mu_f(G) + \alpha(G) = |V(G)|$. For every connected graph G, we have $\mu(G) \leqslant \mu_f(G) \leqslant |V(G)| - \alpha(G)$ [7]. Hence $\{G\colon \mu(G) + \alpha(G) = |V(G)|\} \subseteq \{G\colon \mu_f(G) + \alpha(G) = |V(G)|\}$ and finally $\mathrm{isc}(G) = |V(G)| - 2\mu_f(G)$ for every Kőnig–Egerváry graph. ♠

It turns out that $\mu_f(G)$, used in (3), can be computed in $O(|V| \cdot |E|)$ time (i.e., polynomial time) in contrast to $\alpha(G)$ (used in (2)), which can be computed in $O(1.1996^{|V|})$ time [7, 9]. Furthermore, the next results show that the new upper bound is better than the old one for some graphs.

Lemma 3. *Let G be a graph. Then*

$$|V(G)| - 2\mu_f(G) \leqslant \alpha(G) - \kappa(G) \tag{4}$$

if and only if

(i) $\mu_f(G) \leqslant \frac{|V(G)|}{2} - 1$, *or*

(ii) $\mu_f(G) = \frac{|V(G)|}{2}$ *and* $\kappa(G) \leqslant \alpha(G)$, *or*

(iii) $\mu_f(G) = \frac{|V(G)|-1}{2}$ *and* $\kappa(G) + 1 \leqslant \alpha(G)$.

Furthermore, the inequality (4) is strict if (i) and $\mu(G) < \mu_f(G)$, *or* $\mu_f(G) = |V(G)|/2$ *and* $\kappa(G) < \alpha(G)$, *or* $\mu_f(G) = (|V(G)|-1)/2$ *and* $\kappa(G)+1 < \alpha(G)$ *holds.*

Proof: We first prove that either (i), (ii) or (iii) implies (4). Let G be a graph and $\mu(G) \leqslant |V(G)|/2 - 1$. The following formula is a conclusion of the Gallai–Edmonds Structure Theorem [2]:

$$|V(G)| - 2\mu(G) \leqslant \alpha(G) - \kappa(G). \tag{5}$$

Hence, from (1) we have $|V(G)| - 2\mu_f(G) \leqslant |V(G)| - 2\mu(G) \leqslant \alpha(G) - \kappa(G)$. Moreover, the measure $\mu_f(G)$ can take only values in $\{m/2\colon m \in$

Table 1: We summarize classes of graphs for which $\mathrm{isc}(G)$ can be computed in polynomial time.

Class	Reference	Time
Bipartite graphs	Wang et al. [8] (2011)	$O(\sqrt{\lvert V\rvert}\cdot\lvert E\rvert)$
Split graphs	Li et al. [6] (2017)	polynomial
Interval graphs	Li et al. [5] (2017)	$O(\lvert V\rvert^4)$
Kőnig–Egerváry graphs	this article (Theorem 2)	$O(\lvert V\rvert\cdot\lvert E\rvert)$

$\{0, 1, \dots, |V(G)|\}\}$ [7] and $\mu(G) \geqslant 0$ (from (1)). Thus, if $\mu(G) < \mu_f(G)$, then

$$|V(G)| - 2\mu_f(G) \leqslant |V(G)| - 2(\mu(G) + 1/2) < |V(G)| - 2\mu(G) \leqslant \alpha(G) - \kappa(G).$$

Now let $\mu_f(G) = |V(G)|/2$ and $\kappa(G) \leqslant \alpha(G)$ (respectively $\kappa(G) < \alpha(G)$). Then $|V(G)| - 2\mu_f(G) = 0 \leqslant \alpha(G) - \kappa(G)$ (respectively $|V(G)| - 2\mu_f(G) = 0 < \alpha(G) - \kappa(G)$). Now let $\mu_f(G) = (|V(G)| - 1)/2$ and $\kappa(G) + 1 \leqslant \alpha(G)$ (respectively $\kappa(G) + 1 < \alpha(G)$). Then $|V(G)| - 2\mu_f(G) = 1 \leqslant \alpha(G) - \kappa(G)$ (respectively $|V(G)| - 2\mu_f(G) = 1 < \alpha(G) - \kappa(G)$).

We now prove that (4) implies (i), (ii) or (iii). Suppose that $|V(G)| - 2\mu_f(G) \leqslant \alpha(G) - \kappa(G)$ and (i), (ii) or (iii) does not hold. Let $\mu_f(G) > |V(G)|/2 - 1$. If $\mu_f(G) \neq |V(G)|/2$ and $\kappa(G) + 1 > \alpha(G)$, then $|V(G)| - 2\mu_f(G) = 1$. Summing the last two formulas, we obtain $|V(G)| - 2\mu_f(G) > \alpha(G) - \kappa(G)$, and we obtain a contradiction. The analysis of the remaining cases are similar and left to the reader. ♠

In Table 1 we summarize classes of graphs for which $\mathrm{isc}(G)$ can be computed in polynomial time. It is worth noting that the class of bipartite graphs is contained in the class of Kőnig–Egerváry graphs [4]. In Table 2, we experimentally compare the upper bounds from (2) and (3) for small connected graphs.

At the end of this section, we determine the isolated scattering number of the so-called corona of a graph, which can be interpreted as an expanding network.

Table 2: We experimentally compare $\mathcal{B}_{\text{Old}}(G) = \alpha(G) - \kappa(G)$ and $\mathcal{B}_{\text{New}}(G) = |V(G)| - 2\mu_f(G)$ for graphs in $\mathcal{G}_n^c$, that is, for all connected graphs on $|V(G)| = n$ vertices.

n	$\lvert\mathcal{G}_n^c\rvert$	$\mathcal{B}_{\text{Old}}(G) \geqslant \mathcal{B}_{\text{New}}(G)$	$\mathcal{B}_{\text{New}}(G) > \mathcal{B}_{\text{Old}}(G)$
1	1	1 (100.0%)	0 (0.0%)
2	1	1 (100.0%)	0 (0.0%)
3	2	1 (50.0%)	1 (50.0%)
4	6	5 (83.33%)	1 (16.67%)
5	21	18 (85.71%)	3 (14.29%)
6	112	99 (88.39%)	13 (11.61%)
7	853	787 (92.26%)	66 (7.74%)
8	11 117	10 585 (95.21%)	532 (4.79%)
9	261 080	247 071 (94.63%)	14 009 (5.37%)
1 − 9	273 193	258 568 (94.65%)	14 625 (5.35%)

Let G be a graph. An edge of G incident to a leaf is called a *pendant* edge. Let $C \subseteq V(G)$. The *generalized corona* of a graph G, denoted by $\text{cor}(G, C)$, is the graph obtained from G by adding a pendant edge to each vertex v of G such that $v \in C$. The *corona* of G, denoted by $\text{cor}(G)$, is the graph $\text{cor}(G, V(G))$. Let n be a positive integer. We write $\text{cor}^n(G, C)$ to denote the *generalized corona* n*th power* of G, that is, $\text{cor}^n(G, C) = \text{cor}(\text{cor}^{n-1}(G, C), C)$ if $n > 1$ and $\text{cor}^1(G, C) = \text{cor}(G, C)$. We assume that $\text{cor}^0(G, C) = G$.

Let S be a cut set of G. Let S^* be a cut set such that $\text{isc}(G) = i(G - S^*) - |S^*|$ and I (respectively I^*) be a set of all components which are trivial in $G - S$ (respectively $G - S^*$). Using Theorem 2, we establish the isolated scattering number of some coronas.

Lemma 4. *Let G be a noncomplete connected graph. Then*

$$\text{isc}(\text{cor}(G, C)) \geqslant \text{isc}(G) + |C \cap S^*| - |C \cap I^*|. \tag{6}$$

Furthermore, if $C = V(G)$, then the equality holds in (6).

Proof: The set $S_C = S \cup C$ is a cut set of $\mathrm{cor}(G, C)$. Moreover, $S_C = S \cup (C \cap I) \cup (C \setminus (I \cup S))$ since S and I are disjoint and therefore $|S_C| = |S| + |C \cap I| + |C \setminus (I \cup S)|$. On the other hand,

$$i(\mathrm{cor}(G, C) - S_C) \geqslant |I \setminus C| + |C| = |I| + |C \cap S| + |C \setminus (I \cup S)|.$$

Thus

$$\mathrm{isc}(\mathrm{cor}(G, C)) \geqslant i(\mathrm{cor}(G, C) - S_C) - |S_C| = |I| - |S| + |C \cap S| - |C \cap I|$$

and finally

$$\mathrm{isc}(\mathrm{cor}(G, C)) \geqslant |I^*| - |S^*| + |C \cap S^*| - |C \cap I^*| = \mathrm{isc}(G) + |C \cap S^*| - |C \cap I^*|.$$

Let $C = V(G)$. Then, from (6),

$$\begin{aligned}\mathrm{isc}(\mathrm{cor}(G, V(G))) = \mathrm{isc}(\mathrm{cor}(G)) &\geqslant \mathrm{isc}(G) + |V(G) \cap S^*| - |V(G) \cap I^*| \\ &= |I^*| - |S^*| + |S^*| - |I^*| = 0.\end{aligned}$$

Furhermore, $\mathrm{cor}(G)$ has a perfect matching (i.e., all added pendant edges) and $\mu_f(\mathrm{cor}(G)) = \mu(\mathrm{cor}(G)) = |V(\mathrm{cor}(G))|/2$ (from (1)). Hence, from Theorem 2, we get $0 = |V(\mathrm{cor}(G))| - 2\mu_f(\mathrm{cor}(G)) \geqslant \mathrm{isc}(\mathrm{cor}(G)) \geqslant 0$. ♠

Corollary 5. *Let G be a connected graph and n be a positive integer. Then*

$$\mathrm{isc}(\mathrm{cor}^n(G)) = 0.$$

Proof: From the proof of Lemma 4, we have $\mathrm{isc}(\mathrm{cor}(G)) = 0$ for a noncomplete connected graph G and $0 \geqslant \mathrm{isc}(\mathrm{cor}(G))$ for a complete graph G. Let G be a trivial graph. Then $\mathrm{isc}(\mathrm{cor}(G)) = 2 - |V(\mathrm{cor}(G))| = 0$. Now let G be a nontrivial complete graph. Take $S = V(G)$, then $i(\mathrm{cor}(G, V(G)) - S) - |S| = |V(G)| - |V(G)| = 0$ and hence $\mathrm{isc}(\mathrm{cor}(G)) = \mathrm{isc}(\mathrm{cor}(G, V(G))) \geqslant 0$.

Algorithm 4.1 Greedy Algorithm Min

function Min(G)
    $I \leftarrow \emptyset$
    while $V(G) \neq \emptyset$ **do**
        choose $v \in V(G)$ with $d_G(v) = \delta(G)$
        $G \leftarrow G - N_G[v]$
        $I \leftarrow I \cup \{v\}$
    return I

Since a corona of a connected graph is connected, we have $\mathrm{isc}(\mathrm{cor}^n(G)) = 0$ for a positive integer n. ♠

From the previous consideration, we conjecture that if n is a positive integer, $C \subseteq V(G)$, and $C \neq \emptyset$, then

$$\mathrm{isc}(\mathrm{cor}^n(G, C)) = |V(\mathrm{cor}^n(G, C))| - 2\mu_f(\mathrm{cor}^n(G, C)) .$$

In addition, we pose the following question: Is $\mathrm{isc}(\mathrm{cor}^n(G, C))$ a monotonic function with respect to n?

4 Greedy algorithm for isolated scattering number

In this section, we present a greedy algorithm that determines a lower bound of the isolated scattering number. We achieve this by a modification of Algorithm 4.1, the so-called greedy algorithm Min [3]. The algorithm Min recursively chooses a vertex with the smallest neighborhood (i.e., a vertex with minimum degree) in a graph and then, it removes the closed neighborhood of the vertex from that graph. A set produced by Min is an independent set. The algorithm Min has complexity $O(n^2)$.

We perform a little modification of the Min Algorithm 4.1 and we obtain

Algorithm 4.2 Greedy Algorithm MIN-ISC

```
function MIN-ISC(G)
    |I| ← 0, |S| ← 0, max ← −∞
    while V(G) ≠ ∅ do
        choose v ∈ V(G) with d_G(v) = δ(G)
        G ← G − N_G[v]
        |S| ← |S| + |N_G[v]| − 1
        |I| ← |I| + 1
        if |I| − |S| > max then
            max = |I| − |S|
    return max
```

MIN-ISC Algorithm 4.2. The MIN-ISC works properly since the union of open neighborhoods of the v is a cut set. This algorithm has the same complexity as MIN, that is $O(n^2)$, since we only add several constant time operations.

We perform some preliminary experiments using MIN-ISC Algorithm 4.2 and we report them in Table 3. This table compares lower bounds computed from MIN-ISC with bounds computed from (2).

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Table 3: For graphs in $\mathcal{G}_n^c$, that is, for all connected graphs on $|V(G)| = n$ vertices, we experimentally compare the old lower bound $\mathcal{B}_{\text{Old}}(G) = 2\alpha(G) - |V(G)|$ and $\mathcal{B}_{\text{New}}(G)$, which here means the value produced by Min-Isc Algorithm 4.2 for G.

n	$\lvert\mathcal{G}_n^c\rvert$	$\mathcal{B}_{\text{New}}(G) \geqslant \mathcal{B}_{\text{Old}}(G)$	$\mathcal{B}_{\text{Old}}(G) > \mathcal{B}_{\text{New}}(G)$
1	1	1 (100.0%)	0 (0.0%)
2	1	1 (100.0%)	0 (0.0%)
3	2	2 (100.0%)	0 (0.0%)
4	6	6 (100.0%)	0 (0.0%)
5	21	21 (100.0%)	0 (0.0%)
6	112	112 (100.0%)	0 (0.0%)
7	853	850 (99.65%)	3 (0.35%)
8	11 117	11 060 (99.49%)	57 (0.51%)
9	261 080	260 016 (99.59%)	1 064 (0.41%)
10	11 716 571	11 685 617 (99.74%)	30 954 (0.26%)
1 – 10	11 989 764	11 957 686 (99.73%)	32 078 (0.27%)

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