An interior penalty method for a two dimensional curl-curl and grad-div problem

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Abstract

We study an interior penalty method for a two dimensional curl-curl and grad-div problem that appears in electromagnetics and in fluid-structure interactions. The method uses discontinuous P_1 vector fields on graded meshes and satisfies optimal convergence rates (up to an arbitrarily small parameter) in both the energy norm and the L_2 norm. These theoretical results are corroborated by results of numerical experiments.

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1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. We consider an interior penalty method for the following curl-curl and grad-div problem.

Problem 1 *Find* $\mathbf{u} \in H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$ *such that*

$$(\nabla \times \mathbf{u}, \nabla \times \boldsymbol{v}) + \gamma(\nabla \cdot \mathbf{u}, \nabla \cdot \boldsymbol{v}) + \alpha(\mathbf{u}, \boldsymbol{v}) = (\mathbf{f}, \boldsymbol{v}) \tag{1}$$

for all $\boldsymbol{v} \in H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$, *where* $(\cdot,\cdot)$ *denotes the inner product of* $[L_2(\Omega)]^2$, $\alpha \in \mathbb{R}$ *and* $\gamma > 0$ *are constants, and* $\mathbf{f} \in [L_2(\Omega)]^2$.

The variational Problem 1 appears in electromagnetics [29, 30] and fluid-structure interactions [27, 9, 8, 10] (after interchanging the roles of curl and div). The main difficulty in the numerical solution of (1) is that standard H^1-conforming finite element methods fail when Ω is not convex [22]. Such methods produce numerical solutions that converge to a vector field that is not the solution of (1). Special treatments are therefore necessary for capturing the correct solution, either by augmenting standard H^1 finite element vector fields by singular vector fields [11, 5, 28, 3, 4], or by solving a regularized version of (1) [24, 25, 21].

Brenner et al. [15] introduced a nonconforming finite element method for (1). It uses the Crouzeix–Raviart weakly continuous P_1 vector fields [26] on graded meshes and has optimal convergence rates in both the energy norm and the L_2 norm. In this article we study an interior penalty version of the method by Brenner et al. [15]. By removing the weak continuity condition of the vector fields, the interior penalty method applies to meshes with hanging nodes. This method belongs to a growing family of finite element methods for problems posed on $H(\text{curl};\Omega)\cap H(\text{div};\Omega)$ [16, 15, 17, 20, 18].

Section 2 recalls definitions of function spaces and properties of Problem 1, and Section 3 defines the numerical scheme whose analysis is then carried out in Section 4. Section 5 discusses the extension of the method to non-conforming meshes and Section 6 presents numerical results that corroborate the theoretical results. We end with some concluding remarks in Section 7.

2 Preliminaries

The function spaces $H_0(\text{curl};\Omega)$ and $H(\text{div};\Omega)$ are defined as

$$H(\text{curl};\Omega) = \left\{ \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in [L_2(\Omega)]^2 : \nabla \times \boldsymbol{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \in L_2(\Omega) \right\},$$
$$H_0(\text{curl};\Omega) = \{ \boldsymbol{v} \in H(\text{curl};\Omega) : \boldsymbol{n}\times\boldsymbol{v} = 0 \text{ on } \partial\Omega \},$$

with $\boldsymbol{n}$ being the unit outer normal, and

$$H(\text{div};\Omega) = \left\{ \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in [L_2(\Omega)]^2 : \nabla \cdot \boldsymbol{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \in L_2(\Omega) \right\}.$$

The unique solvability of problem (1), in the case where $\alpha > 0$, is guaranteed by the Riesz representation theorem for the Hilbert space $H_0(\text{curl};\Omega)\cap H(\text{div};\Omega)$ with the inner product $(\!(\cdot,\cdot)\!)$ defined by

$$(\!(\boldsymbol{v},\boldsymbol{w})\!) = (\nabla\times\boldsymbol{v}, \nabla\times\boldsymbol{w}) + (\nabla\cdot\boldsymbol{v}, \nabla\cdot\boldsymbol{w}) + (\boldsymbol{v},\boldsymbol{w}).$$

For $\alpha \leqslant 0$, problem (1) is uniquely solvable if α is different from a sequence of exceptional values [30]. Indeed there exists a sequence of nonnegative numbers $0 \leqslant \lambda_{\gamma,1} \leqslant \lambda_{\gamma,2} \leqslant \cdots \to \infty$ such that there exists a nontrivial solution $\boldsymbol{w} \in H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$ for the eigenproblem

$$(\nabla \times \boldsymbol{w}, \nabla \times \boldsymbol{v}) + \gamma(\nabla \cdot \boldsymbol{w}, \nabla \cdot \boldsymbol{v}) = \lambda_{\gamma,j}(\boldsymbol{w}, \boldsymbol{v}) \tag{2}$$

for all $\boldsymbol{v} \in H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$. For $\alpha \leqslant 0$, problem (1) is well-posed as long as $\alpha \neq -\lambda_{\gamma,j}$ for $j \geqslant 1$.

The regularity of the solution $\boldsymbol{u}$ of (1) is well established [6, 23, 15, e.g.]. Below we summarize the results as stated by Brenner et al. [15].

First of all, $\nabla \times \boldsymbol{u}$ and $\nabla \cdot \boldsymbol{u}$ belong to $H^1(\Omega)$, and

$$\|\nabla \times \boldsymbol{u}\|_{H^1(\Omega)} + \|\nabla \cdot \boldsymbol{u}\|_{H^1(\Omega)} \leqslant C\|\boldsymbol{f}\|_{L_2(\Omega)} . \tag{3}$$

Secondly, we have $\boldsymbol{u} \in [H^2(\Omega_\delta)]^2$ and the following estimate is valid:

$$\|\boldsymbol{u}\|_{H^2(\Omega_\delta)} \leqslant C\|\boldsymbol{f}\|_{L_2(\Omega)} , \tag{4}$$

where the domain Ω_δ is obtained from Ω by excising δ-neighborhoods from the corners $c_1, \ldots, c_L$ of Ω, that is,

$$\Omega_\delta = \{x \in \Omega : |x - c_\ell| > \delta \text{ for } 1 \leqslant \ell \leqslant L\}.$$

Thirdly, in the neighborhood $\mathcal{N}_{\ell,3\delta/2} = \{x \in \Omega : |x - c_\ell| < 3\delta/2\}$ of the corner c_ℓ, we have

$$\boldsymbol{u} = \boldsymbol{u}_R + \boldsymbol{u}_S , \tag{5}$$

where $\boldsymbol{u}_R \in [H^{2-\epsilon}(\mathcal{N}_{\ell,3\delta/2})]^2$ for any $\epsilon > 0$,

$$\boldsymbol{u}_S = \sum_{\substack{j \in \mathbb{N} \\ j(\pi/\omega_\ell) \in (0,2) \setminus \{1\}}} \nu_{\ell,j} r_\ell^{j(\pi/\omega_\ell)-1} \begin{bmatrix} \sin\big(j(\pi/\omega_\ell) - 1\big)\theta_\ell \\ \cos\big(j(\pi/\omega_\ell) - 1\big)\theta_\ell \end{bmatrix} , \tag{6}$$

and $\nu_{\ell,j}$ are constants. Moreover, we have the following corner regularity estimates:

$$\sum_{\ell=1}^{L} \|\mathbf{u}_R\|_{H^{2-\epsilon}(\mathcal{N}_{\ell,3\delta/2})} \leqslant C_\epsilon \|\mathbf{f}\|_{L_2(\Omega)}\,; \tag{7a}$$

$$\sum_{\ell=1}^{L} \sum_{\substack{j\in\mathbb{N} \\ j(\pi/\omega_\ell)\in(0,2)\setminus\{1\}}} |\nu_{\ell,j}| \leqslant C\|\mathbf{f}\|_{L_2(\Omega)}\,. \tag{7b}$$

Note that the regularity of $\nabla \times \mathbf{u}$ and $\nabla \cdot \mathbf{u}$ imply that the boundary value problem corresponding to (1) is

$$\nabla \times (\nabla \times \mathbf{u}) - \gamma \nabla(\nabla \cdot \mathbf{u}) + \alpha \mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \tag{8a}$$

$$\mathbf{n} \times \mathbf{u} = 0 \quad \text{on } \partial\Omega, \tag{8b}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{on } \partial\Omega. \tag{8c}$$

3 The interior penalty method

We need graded meshes to recover optimal convergence rates for a general polygonal domain Ω. We assume therefore that the triangulation $\mathcal{T}_h$ of Ω satisfies

$$C_1 h_T \leqslant h\Phi_\mu(T) \leqslant C_2 h_T \quad \text{for all } T \in \mathcal{T}_h\,, \tag{9}$$

where h_T is the diameter of the triangle T, $h = \max_{T\in\mathcal{T}_h} h_T$ is the mesh parameter, and the positive constants C_1 and C_2 are independent of h. The weight $\Phi_\mu(T)$ in (9) is defined by

$$\Phi_\mu(T) = \prod_{\ell=1}^{L} |c_\ell - c_T|^{1-\mu_\ell}\,, \tag{10}$$

where c_T is the center of T, and the grading parameters $\mu_1, \ldots, \mu_L$ are chosen according to the following rule:

$$\begin{cases} \mu_\ell = 1 & \text{if } \omega_\ell \leqslant \frac{\pi}{2}\,, \\ \mu_\ell < \frac{\pi}{2\omega_\ell} & \text{if } \omega_\ell > \frac{\pi}{2}\,. \end{cases} \tag{11}$$

The construction of $\mathcal{T}_h$ satisfying the mesh condition (9) is described elsewhere [1, 2, 14, 7, e.g.]. Note that $\mathcal{T}_h$ satisfies the minimum angle condition for any given grading parameters.

Remark 2 The choice of grading parameter in (11), which is dictated by the regularity of the solution $\boldsymbol{u}$ of (1), indicates that grading is needed around any corner whose angle is larger than a right angle. This is different from the grading strategy for the Laplace operator, where grading is needed only around re-entrant corners, and it is due to the singularity of the differential operator in (8a) being one order more severe than the singularity of the Laplace operator.

We take V_h to be the space of (discontinuous) P_1 vector fields, that is,

$$V_h = \left\{ \boldsymbol{v} \in [L_2(\Omega)]^2 : \boldsymbol{v}_T = \boldsymbol{v}\big|_T \in [P_1(T)]^2 \quad \text{for all } T \in \mathcal{T}_h \right\}.$$

Since the vector fields in V_h are (in general) discontinuous, their jumps across the edges of $\mathcal{T}_h$ play an important role in interior penalty methods. Below are the definitions of the tangential and normal jumps of the vector fields.

We denote by $\mathcal{E}_h$ (respectively $\mathcal{E}_h^i$) the set of the edges (respectively interior edges) of $\mathcal{T}_h$. Let $e \in \mathcal{E}_h^i$ be shared by the two triangles $T_\pm \in \mathcal{T}_h$ (compare with Figure 1) and $\boldsymbol{n}_+$ (respectively $\boldsymbol{n}_-$) be the unit normal of e pointing towards the outside of T_+ (respectively T_-). We define, on e,

$$[\![\boldsymbol{n} \times \boldsymbol{v}]\!] = \boldsymbol{n}_+ \times \boldsymbol{v}_{T_+}\big|_e + \boldsymbol{n}_- \times \boldsymbol{v}_{T_-}\big|_e\,, \tag{12a}$$

$$[\![\boldsymbol{n} \cdot \boldsymbol{v}]\!] = \boldsymbol{n}_+ \cdot \boldsymbol{v}_{T_+}\big|_e + \boldsymbol{n}_- \cdot \boldsymbol{v}_{T_-}\big|_e\,. \tag{12b}$$

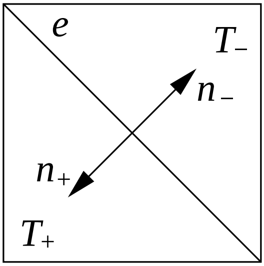

FIGURE 1: Triangles and normals in the definitions of $[\![\mathbf{n} \times \boldsymbol{v}]\!]$ and $[\![\mathbf{n} \cdot \boldsymbol{v}]\!]$.

For an edge $e \in \mathcal{E}_h^b$, we take $\mathbf{n}_e$ to be the unit normal of e pointing towards the outside of Ω and define

$$[\![\mathbf{n} \times \boldsymbol{v}]\!] = \mathbf{n}_e \times \boldsymbol{v}\big|_e . \tag{13}$$

We now define the discrete problem for the interior penalty method.

Problem 3 *Find* $\mathbf{u}_h \in V_h$ *such that*

$$a_h(\mathbf{u}_h, \boldsymbol{v}) = (\mathbf{f}, \boldsymbol{v}) \quad \textit{for all } \boldsymbol{v} \in V_h , \tag{14}$$

where

$$\begin{aligned} a_h(\boldsymbol{w}, \boldsymbol{v}) = (\nabla_h \times \boldsymbol{w}, \nabla_h \times \boldsymbol{v}) + \gamma(\nabla_h \cdot \boldsymbol{w}, \nabla_h \cdot \boldsymbol{v}) + \alpha(\boldsymbol{w}, \boldsymbol{v}) \\ + \sum_{e \in \mathcal{E}_h} \frac{[\Phi_\mu(e)]^2}{|e|} \int_e [\![\mathbf{n} \times \boldsymbol{w}]\!]\, [\![\mathbf{n} \times \boldsymbol{v}]\!]\, ds \\ + \sum_{e \in \mathcal{E}_h^i} \frac{[\Phi_\mu(e)]^2}{|e|} \int_e [\![\mathbf{n} \cdot \boldsymbol{w}]\!] [\![\mathbf{n} \cdot \boldsymbol{v}]\!]\, ds \\ + h^{-2} \sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \int_e (\Pi_e^0 [\![\mathbf{n} \times \boldsymbol{w}]\!])\, (\Pi_e^0 [\![\mathbf{n} \times \boldsymbol{v}]\!])\, ds \\ + h^{-2} \sum_{e \in \mathcal{E}_h^i} \frac{1}{|e|} \int_e (\Pi_e^0 [\![\mathbf{n} \cdot \boldsymbol{w}]\!])(\Pi_e^0 [\![\mathbf{n} \cdot \boldsymbol{v}]\!])\, ds , \end{aligned} \tag{15}$$

and where $|e|$ denotes the length of the edge e, and Π_e^0 is the orthogonal projection from $L_2(e)$ to $P_0(e)$ (the space of constant functions on e). The edge weight $\Phi_\mu(e)$ in (15) is defined by

$$\Phi_\mu(e) = \prod_{\ell=1}^{L} |c_\ell - m_e|^{1-\mu_\ell}, \tag{16}$$

where $c_1, \dots, c_L$ are the corners of Ω and m_e is the midpoint of the edge e.

Remark 4 Comparing (10) and (16) we have

$$\Phi_\mu(e) \approx \Phi_\mu(T) \quad \text{if } e \subset \partial T\,, \tag{17}$$

where the positive constants in the equivalence are independent of h. This relation is important for the derivation of optimal a priori error estimates.

We use the Crouzeix–Raviart interpolation operator in the analysis of the interior penalty method. For $s > 1/2$ and $T \in \mathcal{T}_h$, we define $\Pi_T : [H^s(T)]^2 \to [P_1(T)]^2$ by

$$\int_{e_j} (\Pi_T \zeta)\, ds = \int_{e_j} \zeta\, ds \quad \text{for } 1 \leqslant j \leqslant 3\,, \tag{18}$$

where e_1, e_2 and e_3 are the edges of T. The operator Π_T satisfies the following standard error estimate [26]:

$$\|\zeta - \Pi_T \zeta\|_{L_2(T)} + h_T^{\min(s,1)} |\zeta - \Pi_T \zeta|_{H^{\min(s,1)}(T)} \leqslant C_T h_T^s |\zeta|_{H^s(T)} \tag{19}$$

for all $\zeta \in [H^s(T)]^2$ and $s \in (1/2, 2]$, where the positive constant C_T depends on the minimum angle of T (and also on s when s approaches $1/2$).

Since $H_0(\text{curl};\Omega) \cap H(\text{div};\Omega) \subset [H^s(\Omega)]^2$ for some $s > 1/2$ [30, 15, cf. e.g.], we can define a global interpolation operator

$$\Pi_h : H_0(\text{curl};\Omega) \cap H(\text{div};\Omega) \to V_h$$

by piecing together the local interpolation operators, that is,

$$(\Pi_h \boldsymbol{v})_T = \Pi_T \boldsymbol{v}_T \quad \text{for all } T \in \mathcal{T}_h\,. \tag{20}$$

We also denote the piecewise defined curl and div operator by $\nabla_h\times$ and $\nabla_h\cdot$, that is,

$$(\nabla_h \times \boldsymbol{v})_T = \nabla \times (\boldsymbol{v}_T) \quad \text{for all } T \in \mathcal{T}_h\,, \tag{21}$$
$$(\nabla_h \cdot \boldsymbol{v})_T = \nabla \cdot (\boldsymbol{v}_T) \quad \text{for all } T \in \mathcal{T}_h\,. \tag{22}$$

It follows from (18), (20)–(22) and Green's theorem that

$$\nabla_h \times (\Pi_h \boldsymbol{v}) = \Pi_0^h(\nabla \times \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in H_0(\text{curl};\Omega) \cap H(\text{div};\Omega), \tag{23}$$
$$\nabla_h \cdot (\Pi_h \boldsymbol{v}) = \Pi_0^h(\nabla \cdot \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in H_0(\text{curl};\Omega) \cap H(\text{div};\Omega), \tag{24}$$

where Π_0^h is the orthogonal projection from $L_2(\Omega)$ onto the space of piecewise constant functions associated with $\mathcal{T}_h$.

4 Error analysis

The discretization error is measured in both the L_2 norm and the mesh dependent energy norm $\|\cdot\|_h$ defined by

$$\begin{aligned}\|\boldsymbol{v}\|_h^2 = {}& \|\nabla_h \times \boldsymbol{v}\|_{L_2(\Omega)}^2 + \gamma\|\nabla_h \cdot \boldsymbol{v}\|_{L_2(\Omega)}^2 + \|\boldsymbol{v}\|_{L_2(\Omega)}^2 \\ & + \sum_{e\in\mathcal{E}_h} \frac{[\Phi_\mu(e)]^2}{|e|}\|[\![\boldsymbol{n}\times\boldsymbol{v}]\!]\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}_h^i} \frac{[\Phi_\mu(e)]^2}{|e|}\|[\![\boldsymbol{n}\cdot\boldsymbol{v}]\!]\|_{L_2(e)}^2 \\ & + h^{-2}\left(\sum_{e\in\mathcal{E}_h} \frac{1}{|e|}\|\Pi_e^0[\![\boldsymbol{n}\times\boldsymbol{v}]\!]\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}_h^i} \frac{1}{|e|}\|\Pi_e^0[\![\boldsymbol{n}\cdot\boldsymbol{v}]\!]\|_{L_2(e)}^2\right).\end{aligned} \tag{25}$$

Note that $a_h(\cdot,\cdot)$ is bounded by this energy norm, that is,

$$|a_h(\boldsymbol{w},\boldsymbol{v})| \leqslant (|\alpha|+1)\|\boldsymbol{w}\|_h\|\boldsymbol{v}\|_h \tag{26}$$

for all $\boldsymbol{v},\boldsymbol{w} \in H_0(\text{curl};\Omega)\cap H(\text{div};\Omega) + V_h\,.$

For $\alpha > 0$, $a_h(\cdot,\cdot)$ is also coercive with respect to $\|\cdot\|_h$, that is,

$$a_h(\mathbf{v},\mathbf{v}) \geqslant \min(1,\alpha)\|\mathbf{v}\|_h^2 \tag{27}$$

for all $\mathbf{v} \in H_0(\text{curl};\Omega) \cap H(\text{div};\Omega) + V_h$. In this case the discrete problem is well-posed and we have the following abstract error estimate, whose proof is identical with our earlier proof [18, Lemma 3.5].

Lemma 5 *Let α be positive, $\beta = \min(1,\alpha)$, $\mathbf{u} \in H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$ be the solution of (1), and $\mathbf{u}_h$ satisfy the discrete problem (14). Then*

$$\|\mathbf{u}-\mathbf{u}_h\|_h \leqslant \left(\frac{1+\alpha+\beta}{\beta}\right) \inf_{\mathbf{v}\in V_h} \|\mathbf{u}-\mathbf{v}\|_h + \frac{1}{\beta} \sup_{\mathbf{w}\in V_h\setminus\{0\}} \frac{a_h(\mathbf{u}-\mathbf{u}_h,\mathbf{w})}{\|\mathbf{w}\|_h}. \tag{28}$$

For $\alpha \leqslant 0$, the following Gårding (in)equality holds

$$a_h(\mathbf{v},\mathbf{v}) + (|\alpha|+1)(\mathbf{v},\mathbf{v}) = \|\mathbf{v}\|_h^2 \tag{29}$$

for all $\mathbf{v} \in H_0(\text{curl};\Omega) \cap H(\text{div};\Omega) + V_h$. In this case the discrete problem is indefinite and the following lemma provides an abstract error estimate for the scheme (14) under the assumption that it has a solution. Its proof, which is based on (26) and (29), is also identical to our earlier proof [18, Lemma 3.6].

Lemma 6 *Let $\mathbf{u} \in H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$ satisfy (1) and $\mathbf{u}_h$ be a solution of (14). Then*

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_h \leqslant (2|\alpha|+3) \inf_{\mathbf{v}\in V_h} \|\mathbf{u}-\mathbf{v}\|_h + \sup_{\mathbf{w}\in V_h\setminus\{0\}} \frac{a_h(\mathbf{u}-\mathbf{u}_h,\mathbf{w})}{\|\mathbf{w}\|_h} \\ + (|\alpha|+1)\|\mathbf{u}-\mathbf{u}_h\|_{L_2(\Omega)}.\end{aligned} \tag{30}$$

From here on we consider α and γ to be fixed and drop the dependence on these constants in our estimates.

Remark 7 The first term on the right-hand side of (28) and (30) measures the approximation property of V_h with respect to the energy norm. The second term measures the consistency error. The third term on the right-hand side of (30) addresses the indefiniteness of the problem when $\alpha \leqslant 0$.

Since the interpolation operator Π_h defined in Section 3 is also the one employed in earlier work [16], we use in our analysis the following results from that article [16, Lemma 5.1 and Lemma 5.2] which were obtained by using (11), (17), (19) and the regularity estimates (3)–(7).

Lemma 8 *Let* $\mathbf{u} \in H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$ *be the solution of* (1). *We have the following interpolation error estimates*:

$$\|\mathbf{u} - \Pi_h \mathbf{u}\|_{L_2(\Omega)} \leqslant h^{2-\epsilon} \|\mathbf{f}\|_{L_2(\Omega)}\,, \tag{31}$$

$$\sum_{e \in \mathcal{E}_h} \frac{[\Phi_\mu(e)]^2}{|e|} \|[\![\mathbf{u} - \Pi_h \mathbf{u}]\!]\|^2_{L_2(e)} \leqslant C_\epsilon h^{2-\epsilon} \|\mathbf{f}\|^2_{L_2(\Omega)}\,, \tag{32}$$

for any $\epsilon > 0$, *where* $[\![\mathbf{u} - \Pi_h \mathbf{u}]\!]$ *is the jump of* $\mathbf{u} - \Pi_h \mathbf{u}$ *across the interior edges of* $\mathcal{T}_h$ *and* $[\![\mathbf{u} - \Pi_h \mathbf{u}]\!]$ *is* $\mathbf{u} - \Pi_h \mathbf{u}$ *on the boundary edges of* $\mathcal{T}_h$.

The approximation property of V_h is established by the following lemma.

Lemma 9 *Let* $\mathbf{u} \in H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$ *be the solution of* (1). *Then*

$$\inf_{\mathbf{v} \in V_h} \|\mathbf{u} - \mathbf{v}\|_h \leqslant \|\mathbf{u} - \Pi_h \mathbf{u}\|_h < C_\epsilon h^{1-\epsilon} \|\mathbf{f}\|_{L_2(\Omega)} \tag{33}$$

for any $\epsilon > 0$.

Proof: It follows from (18) that $\Pi_e^0 [\![\mathbf{n} \times (\mathbf{u} - \Pi_h \mathbf{u})]\!] = 0$ for all $e \in \mathcal{E}_h$ and $\Pi_e^0 [\![\mathbf{n} \cdot (\mathbf{u} - \Pi_h \mathbf{u})]\!] = 0$ for all $e \in \mathcal{E}_h^i$. Therefore we have

$$\begin{aligned}\|\mathbf{u} - \Pi_h \mathbf{u}\|_h^2 = \|\nabla_h \times (\mathbf{u} - \Pi_h \mathbf{u})\|^2_{L_2(\Omega)} \\ + \gamma \|\nabla_h \cdot (\mathbf{u} - \Pi_h \mathbf{u})\|^2_{L_2(\Omega)} + \|\mathbf{u} - \Pi_h \mathbf{u}\|^2_{L_2(\Omega)}\end{aligned}$$

$$+\sum_{e\in\mathcal{E}_h}\frac{[\Phi_\mu(e)]^2}{|e|}\|[\![\mathbf{n}\times(\mathbf{u}-\Pi_h\mathbf{u})]\!]\|^2_{L_2(e)} \qquad (34)$$
$$+\sum_{e\in\mathcal{E}_h^i}\frac{[\Phi_\mu(e)]^2}{|e|}\|[\![\mathbf{n}\cdot(\mathbf{u}-\Pi_h\mathbf{u})]\!]\|^2_{L_2(e)}\,.$$

Lemma 8 estimates the last three terms on the right-hand side of (34), and we bound the first two terms on the right-hand side of (34) by (3), (19), (23) and (24) as

$$\begin{aligned}&\|\nabla_h\times(\mathbf{u}-\Pi_h\mathbf{u})\|^2_{L_2(\Omega)}+\gamma\|\nabla_h\cdot(\mathbf{u}-\Pi_h\mathbf{u})\|^2_{L_2(\Omega)}\\&=\|\nabla\times\mathbf{u}-\Pi_0^h(\nabla\times\mathbf{u})\|^2_{L_2(\Omega)}+\gamma\|\nabla\cdot\mathbf{u}-\Pi_0^h(\nabla\cdot\mathbf{u})\|^2_{L_2(\Omega)}\\&\leqslant Ch^2\|\mathbf{f}\|^2_{L_2(\Omega)}\,.\end{aligned}$$

♠

Next we turn to the consistency error, where we need the following result proved in earlier work [16, Lemma 5.3].

Lemma 10

$$\sum_{e\in\mathcal{E}_h}|e|\,[\Phi_\mu(e)]^{-2}\|\eta-\hat{\eta}_{T_e}\|^2_{L_2(e)}\leqslant Ch^2|\eta|^2_{H^1(\Omega)}\quad \textit{for all } \eta\in H^1(\Omega),$$

where $\hat{\eta}_{T_e}=|T_e|^{-1}\int_{T_e}\eta\,dx$ *is the mean of* η *over* T_e*, one of the triangles in* $\mathcal{T}_h$ *that has* e *as an edge.*

The following lemma provides an optimal bound for the consistency error.

Lemma 11 *Let* $\mathbf{u}\in H_0(\text{curl};\Omega)\cap H(\text{div};\Omega)$ *be the solution of* (1) *and* $\mathbf{u}_h\in V_h$ *satisfy* (14)*. Then*

$$\sup_{\boldsymbol{w}\in V_h\setminus\{0\}}\frac{a_h(\mathbf{u}-\mathbf{u}_h,\boldsymbol{w})}{\|\boldsymbol{w}\|_h}\leqslant Ch\|\mathbf{f}\|_{L_2(\Omega)}\,. \qquad (35)$$

Proof: Let $w \in V_h$ be arbitrary. Since the strong form of (1) is given by (8), we find, by (12), (13), (15) and integration by parts,

$$\begin{aligned} a_h(u, w) &= \sum_{T\in\mathcal{T}_h} \int_T (\nabla \times u)(\nabla \times w)\,dx \\ &\quad + \sum_{T\in\mathcal{T}_h} \gamma \int_T (\nabla \cdot u)(\nabla \cdot w)\,dx + \alpha(u, w) \\ &= (f, w) + \sum_{e\in\mathcal{E}_h} \int_e (\nabla \times u)[\![n \times w]\!]\,ds \\ &\quad + \sum_{e\in\mathcal{E}_h^i} \gamma \int_e (\nabla \cdot u)[\![n \cdot w]\!]\,ds\,. \end{aligned} \tag{36}$$

Subtracting (14) from (36) gives

$$a_h(u - u_h, w) = \sum_{e\in\mathcal{E}_h} \int_e (\nabla \times u)[\![n \times w]\!]\,ds + \sum_{e\in\mathcal{E}_h^i} \gamma \int_e (\nabla \cdot u)[\![n \cdot w]\!]\,ds\,. \tag{37}$$

We rewrite the first term on the right-hand side of (37) as

$$\begin{aligned} \sum_{e\in\mathcal{E}_h} \int_e (\nabla \times u)[\![n \times w]\!]\,ds &= \sum_{e\in\mathcal{E}_h} \int_e (\nabla \times u - (\widehat{\nabla \times u})_{T_e})[\![n \times w]\!]\,ds \\ &\quad + \sum_{e\in\mathcal{E}_h} \int_e (\widehat{\nabla \times u})_{T_e} (\Pi_e^0[\![n \times w]\!])\,ds\,, \end{aligned} \tag{38}$$

where $(\widehat{\nabla \times u})_{T_e}$ is the mean of $\nabla \times u$ on T_e, one of the triangles in $\mathcal{T}_h$ that has e as an edge.

It follows from (3), (25), Lemma 10 and the Cauchy–Schwarz inequality that the first term on the right-hand side of (38) satisfies

$$\sum_{e\in\mathcal{E}_h} \int_e \left(\nabla \times u - (\widehat{\nabla \times u})_{T_e}\right) [\![n \times w]\!]ds$$

$$\leqslant \left(\sum_{e \in \mathcal{E}_h} |e| \, [\Phi_\mu(e)]^{-2} \left\| \nabla \times \mathbf{u} - (\widehat{\nabla \times \mathbf{u}}) \right\|^2_{L_2(e)} \right)^{1/2} \tag{39}$$

$$\times \left(\sum_{e \in \mathcal{E}_h} |e|^{-1} \, [\Phi_\mu(e)]^2 \, \|\mathbf{n} \times \boldsymbol{w}\|^2_{L_2(e)} \right)^{1/2}$$

$$\leqslant Ch \|\mathbf{f}\|_{L_2(\Omega)} \|\boldsymbol{w}\|_h .$$

For the second term on the right-hand side of (38), we find, by using the Cauchy–Schwarz inequality, (3) and (25),

$$\sum_{e \in \mathcal{E}_h} \int_e (\widehat{\nabla \times \mathbf{u}})_{T_e} \left(\Pi^0_h [\![\mathbf{n} \times \boldsymbol{w}]\!] \right) ds$$

$$\leqslant \sum_{e \in \mathcal{E}_h} \left(|e|^{1/2} \|(\widehat{\nabla \times \mathbf{u}})_{T_e}\|_{L_2(e)} \right) \left(|e|^{-1/2} \|\Pi^0_e [\![\mathbf{n} \times \boldsymbol{w}]\!]\|_{L_2(e)} \right)$$

$$\leqslant Ch \left(\sum_{e \in \mathcal{E}_h} \|(\widehat{\nabla \times \mathbf{u}})_{T_e}\|^2_{L_2(T_e)} \right)^{1/2} \tag{40}$$

$$\times \left(h^{-2} \sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \|\Pi^0_e [\![\mathbf{n} \times \boldsymbol{w}]\!]\|^2_{L_2(e)} \right)^{1/2}$$

$$\leqslant Ch \|\nabla \times \mathbf{u}\|_{L_2(\Omega)} \|\boldsymbol{w}\|_h$$

$$\leqslant Ch \|\mathbf{f}\|_{L_2(\Omega)} \|\boldsymbol{w}\|_h .$$

Here we have also used the fact that, if e is an edge of a triangle T, then

$$|e| \|q\|^2_{L_2(e)} \leqslant C_T \|q\|^2_{L_2(T)} \quad \text{for any constant function } q, \tag{41}$$

where the positive constant C_T depends only on the shape of T.

Combining (38)–(40), we have

$$\sum_{e \in \mathcal{E}_h} \int_e (\nabla \times \mathbf{u}) [\![\mathbf{n} \cdot \boldsymbol{w}]\!] \, ds \leqslant Ch \|\mathbf{f}\|_{L_2(\Omega)} \|\boldsymbol{w}\|_h , \tag{42}$$

and similarly,

$$\sum_{e\in\mathcal{E}_h^i}\int_e(\nabla\cdot\mathbf{u})[\![\mathbf{n}\cdot\boldsymbol{w}]\!]\,ds \leqslant Ch\|\mathbf{f}\|_{L_2(\Omega)}\|\boldsymbol{w}\|_h\,. \tag{43}$$

The estimate (35) follows from (37), (42) and (43). ♠

The following lemma gives an L_2 error estimate under the assumption that the discrete problem (14) has a solution.

Lemma 12 *Let* $\mathbf{u}\in H_0(\text{curl};\Omega)\cap H(\text{div};\Omega)$ *be the solution of* (1) *and* $\mathbf{u}_h\in V_h$ *satisfy* (14). *Then*

$$\|\mathbf{u}-\mathbf{u}_h\|_{L_2(\Omega)} \leqslant C_\epsilon\left(h^{2-\epsilon}\|\mathbf{f}\|_{L_2(\Omega)} + h^{1-\epsilon}\|\mathbf{u}-\mathbf{u}_h\|_h\right) \tag{44}$$

for any $\epsilon>0$.

Proof: The proof is based on a duality argument. Let $\boldsymbol{z}\in H_0(\text{curl};\Omega)\cap H(\text{div};\Omega)$ satisfy

$$(\nabla\times\boldsymbol{v},\nabla\times\boldsymbol{z})+\gamma(\nabla\cdot\boldsymbol{v},\nabla\cdot\boldsymbol{z})+\alpha(\boldsymbol{v},\boldsymbol{z}) = (\boldsymbol{v},(\mathbf{u}-\mathbf{u}_h)) \tag{45}$$

for all $\boldsymbol{v}\in H_0(\text{curl};\Omega)\cap H(\text{div};\Omega)$. The strong form of (45) is

$$\nabla\times(\nabla\times\boldsymbol{z})-\gamma\nabla(\nabla\cdot\boldsymbol{z})+\alpha\boldsymbol{z} = \mathbf{u}-\mathbf{u}_h \quad \text{in } \Omega\,, \tag{46a}$$

$$\mathbf{n}\times\boldsymbol{z}=0 \quad \text{on } \partial\Omega\,, \tag{46b}$$

$$\nabla\cdot\boldsymbol{z}=0 \quad \text{on } \partial\Omega\,, \tag{46c}$$

and we have the following analog of (3):

$$\|\nabla\times\boldsymbol{z}\|_{H^1(\Omega)}+\|\nabla\cdot\boldsymbol{z}\|_{H^1(\Omega)} \leqslant C\|\mathbf{u}-\mathbf{u}_h\|_{L_2(\Omega)}\,. \tag{47}$$

Furthermore we can write (45) as

$$a_h(\boldsymbol{v},\boldsymbol{z}) = (\boldsymbol{v},(\mathbf{u}-\mathbf{u}_h)) \quad \text{for all } \boldsymbol{v}\in H_0(\text{curl};\Omega)\cap H(\text{div};\Omega). \tag{48}$$

It follows from (46), (48) and integration by parts that the following analog of (36) holds:

$$\begin{aligned} a_h(\mathbf{u}_h, \boldsymbol{z}) &= \sum_{T\in\mathcal{T}_h} \int_T (\nabla\times\mathbf{u}_h)(\nabla\times\boldsymbol{z})\,dx \\ &\quad + \sum_{T\in\mathcal{T}_h} \gamma \int_T (\nabla\cdot\mathbf{u}_h)(\nabla\cdot\boldsymbol{z})\,dx + \alpha(\mathbf{u}_h, \boldsymbol{z}) \\ &= (\mathbf{u}_h, (\mathbf{u}-\mathbf{u}_h)) + \sum_{e\in\mathcal{E}_h} \int_e [\![\mathbf{n}\times\mathbf{u}_h]\!](\nabla\times\boldsymbol{z})\,ds \\ &\quad + \sum_{e\in\mathcal{E}_h^i} \gamma \int_e [\![\mathbf{n}\cdot\mathbf{u}_h]\!](\nabla\cdot\boldsymbol{z})\,ds\,. \end{aligned} \tag{49}$$

Combining (48) and (49) gives

$$\begin{aligned} \|\mathbf{u}-\mathbf{u}_h\|_{L_2(\Omega)}^2 &= (\mathbf{u}, \mathbf{u}-\mathbf{u}_h) - (\mathbf{u}_h, \mathbf{u}-\mathbf{u}_h) \\ &= a_h(\mathbf{u}-\mathbf{u}_h, \boldsymbol{z}) + \sum_{e\in\mathcal{E}_h} \int_e [\![\mathbf{n}\times\mathbf{u}_h]\!](\nabla\times\boldsymbol{z})\,ds \\ &\quad + \sum_{e\in\mathcal{E}_h^i} \gamma \int_e [\![\mathbf{n}\cdot\mathbf{u}_h]\!](\nabla\cdot\boldsymbol{z})\,ds\,, \end{aligned} \tag{50}$$

and we estimate the three terms on the right-hand side of (50) separately.

Using (37) and the fact that $\Pi_e^0[\![\mathbf{n}\times(\Pi_h\boldsymbol{z})]\!]$ (respectively $\Pi_e^0[\![\mathbf{n}\cdot(\Pi_h\boldsymbol{z})]\!]$) vanishes for all $e\in\mathcal{E}_h$ (respectively $e\in\mathcal{E}_h^i$), we rewrite the first term (following the notation in (38)) as

$$\begin{aligned} a_h(\mathbf{u}-\mathbf{u}_h, \boldsymbol{z}) &= a_h(\mathbf{u}-\mathbf{u}_h, \boldsymbol{z}-\Pi_h\boldsymbol{z}) + a_h(\mathbf{u}-\mathbf{u}_h, \Pi_h\boldsymbol{z}) \\ &= a_h(\mathbf{u}-\mathbf{u}_h, \boldsymbol{z}-\Pi_h\boldsymbol{z}) \\ &\quad + \sum_{e\in\mathcal{E}_h} \int_e \Big(\nabla\times\mathbf{u} - (\widehat{\nabla\times\mathbf{u}})_{T_e}\Big) [\![\mathbf{n}\times(\Pi_h\boldsymbol{z})]\!]\,ds \end{aligned}$$

$$+ \sum_{e\in\mathcal{E}_h^i} \int_e \gamma \left(\nabla\cdot\mathbf{u} - (\widehat{\nabla\cdot\mathbf{u}})_{T_e} \right) [\![\mathbf{n}\cdot(\Pi_h \boldsymbol{z})]\!] \, ds \,,$$

from which we obtain the following estimate using (3), and Lemmas 8 and 10:

$$a_h(\mathbf{u}-\mathbf{u}_h, \boldsymbol{z}) \leqslant C_\epsilon \left(h^{2-\epsilon}\|f\|_{L_2(\Omega)} + h^{1-\epsilon}\|\mathbf{u}-\mathbf{u}_h\|_h \right) \|\mathbf{u}-\mathbf{u}_h\|_{L_2(\Omega)} . \quad (51)$$

Brenner et al. [15, pp. 526–527] gave details where an identical estimate is derived.

We now consider the second term on the right-hand side of (50). First

$$\begin{aligned} \sum_{e\in\mathcal{E}_h} \int_e [\![\mathbf{n}\times\mathbf{u}_h]\!] (\nabla\times\boldsymbol{z}) ds &= \sum_{e\in\mathcal{E}_h} \int_e [\![\mathbf{n}\times\mathbf{u}_h]\!] \left(\nabla\times\boldsymbol{z} - (\widehat{\nabla\times\boldsymbol{z}})_{T_e} \right) ds \\ &\quad + \sum_{e\in\mathcal{E}_h} \int_e \left(\Pi_e^0 [\![\mathbf{n}\times\mathbf{u}_h]\!] \right) (\widehat{\nabla\times\boldsymbol{z}})_{T_e} \, ds \,, \end{aligned} \quad (52)$$

where $(\widehat{\nabla\times\boldsymbol{z}})_{T_e}$ is the mean of $\nabla\times\boldsymbol{z}$ on one of the triangles $\mathcal{T}_e \in \mathcal{T}_h$ that has e as an edge.

The estimate below follows from (25), Lemma 10, (47) and the Cauchy–Schwarz inequality:

$$\begin{aligned} &\sum_{e\in\mathcal{E}_h} \int_e [\![\mathbf{n}\times\mathbf{u}_h]\!] \left(\nabla\times\boldsymbol{z} - (\widehat{\nabla\times\boldsymbol{z}})_{T_e} \right) ds \\ &\leqslant Ch\|\mathbf{u}-\mathbf{u}_h\|_{L_2(\Omega)} \|\mathbf{u}-\mathbf{u}_h\|_h \,. \end{aligned} \quad (53)$$

On the other hand, as in the derivation of (40), we obtain by the Cauchy–Schwarz inequality, (25), (41) and (47),

$$\begin{aligned} &\sum_{e\in\mathcal{E}_h} \int_e (\Pi_e^0 [\![\mathbf{n}\times\mathbf{u}_h]\!]) (\widehat{\nabla\times\boldsymbol{z}})_{T_e} \, ds \\ &= \sum_{e\in\mathcal{E}_h} \int_e \left(\Pi_e^0 [\![\mathbf{n}\times(\mathbf{u}_h-\mathbf{u})]\!] \right) (\widehat{\nabla\times\boldsymbol{z}})_{T_e} \, ds \end{aligned} \quad (54)$$

$$\leqslant Ch\|\mathbf{u}-\mathbf{u}_h\|_h\|\nabla\times\mathbf{z}\|_{L_2(\Omega)}$$
$$\leqslant Ch\|\mathbf{u}-\mathbf{u}_h\|_h\|\mathbf{u}-\mathbf{u}_h\|_{L_2(\Omega)}\,.$$

Combining (52)–(54), we obtain

$$\sum_{e\in\mathcal{E}_h}\int_e [\![\mathbf{n}\times\mathbf{u}_h]\!](\nabla\times\mathbf{z})\,ds \leqslant Ch\|\mathbf{u}-\mathbf{u}_h\|_{L_2(\Omega)}\|\mathbf{u}-\mathbf{u}_h\|_h\,. \tag{55}$$

Similarly, we have the following bound on the third term on the right-hand side of (50):

$$\sum_{e\in\mathcal{E}_h^i}\int_e \gamma[\![\mathbf{n}\cdot\mathbf{u}_h]\!](\nabla\cdot\mathbf{z})\,ds \leqslant Ch\|\mathbf{u}-\mathbf{u}_h\|_{L_2(\Omega)}\|\mathbf{u}-\mathbf{u}_h\|_h\,. \tag{56}$$

The estimate (44) follows from (50), (51), (55) and (56). ♠

In the case where $\alpha > 0$, the following theorem is an immediate consequence of Lemmas 5, 9, 11 and 12.

Theorem 13 *Let α be positive. The following two discretization error estimates hold for the solution $\mathbf{u}_h$ of (14):*

$$\|\mathbf{u}-\mathbf{u}_h\|_h \leqslant C_\epsilon h^{1-\epsilon}\|\mathbf{f}\|_{L_2(\Omega)} \quad \textit{for any } \epsilon>0\,;$$
$$\|\mathbf{u}-\mathbf{u}_h\|_{L_2(\Omega)} \leqslant C_\epsilon h^{2-\epsilon}\|\mathbf{f}\|_{L_2(\Omega)} \quad \textit{for any } \epsilon>0\,.$$

In the case where $\alpha \leqslant 0$, we have the following convergence theorem for the scheme (14), whose proof is based on Lemmas 6, 9, 11 and 12, and the approach of Schatz for indefinite problems [32]. The arguments are identical to those of an earlier proof [16, Theorem 4.5].

Theorem 14 *Assume $-\alpha \geqslant 0$ is not one of the eigenvalues $\lambda_{\gamma,j}$ defined by (2). There exists a positive number h_* such that the discrete problem (14) is uniquely solvable for all $h \leqslant h_*$, in which case the following two discretization error estimates are valid:*

$$\|\mathbf{u}-\mathbf{u}_h\|_h \leqslant C_\epsilon h^{1-\epsilon}\|\mathbf{f}\|_{L_2(\Omega)} \quad \textit{for any } \epsilon>0\,;$$

$$\|\mathbf{u}-\mathbf{u}_h\|_{L_2(\Omega)} \leqslant C_\epsilon h^{2-\epsilon}\|\mathbf{f}\|_{L_2(\Omega)} \quad \textit{for any } \epsilon > 0\,.$$

5 Nonconforming meshes

For simplicity we developed and analyzed the interior penalty method for triangulations (conforming meshes) of Ω. But of course one of the main reasons for using an interior penalty method is that it can be applied to partitions with hanging nodes (nonconforming meshes). Here we indicate briefly how the scheme and results extend with minor modifications to such meshes.

Let $\mathcal{P}_h$ be a partition of Ω with hanging nodes satisfying the following condition: whenever a closed edge of a triangle in $\mathcal{P}_h$ contains a hanging node, then it is the union of closed edges of triangles in $\mathcal{T}_h$. An example of such a partition is depicted in Figure 2. For such a partition, we modify the definition of $\mathcal{E}_h$ as follows. Let e be an (open) edge of a triangle in $\mathcal{P}_h$. Then $e \in \mathcal{E}_h$ if and only if (i) it contains at least one hanging node, (ii) it is the common edge of two triangles in $\mathcal{P}_h$, that is, its endpoints are the common vertices of these triangles, or (iii) it is a subset of $\partial\Omega$. For example, the edge of the largest triangle (the diagonal of the square) in Figure 2 belongs to $\mathcal{E}_h$ while the three edges on the diagonal from the three triangles on the other side do not. All together there are 12 edges in $\mathcal{E}_h$ for the partition in Figure 2.

All the definitions in Section 3 can be extended to $\mathcal{P}_h$ in a straightforward fashion. For example, if $e \in \mathcal{E}_h^i$ has at least one hanging node, then e is the edge of a triangle $T_- \in \mathcal{P}_h$ and also the union of edges $e_1, \ldots, e_m$ of the triangles $T_{+,1}, \ldots, T_{+,m}$ in $\mathcal{P}_h$ that are on the other side of e (compare with Figure 3 where $m = 4$). We define, on e,

$$[\![\mathbf{n}\times\mathbf{v}]\!] = (\mathbf{n}_-\times\mathbf{v}_{T_-})\big|_e + \sum_{j=1}^{m}(\mathbf{n}_+\times\mathbf{v}_{T_{+,j}})\big|_{e_j}\,,$$

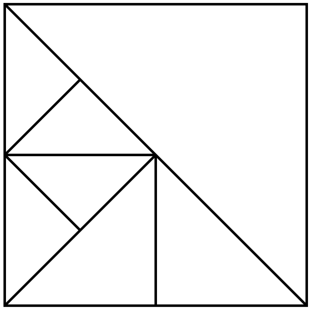

FIGURE 2: A triangular mesh with hanging nodes.

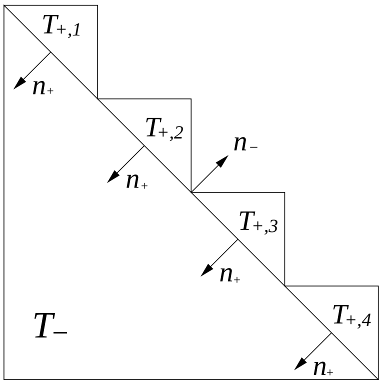

FIGURE 3: A triangular mesh with hanging nodes.

$$[\![\mathbf{n}\cdot\boldsymbol{v}]\!] = (\mathbf{n}_-\cdot\boldsymbol{v}_{T_-})\big|_e + \sum_{j=1}^{m}(\mathbf{n}_+\cdot\boldsymbol{v}_{T_{+,j}})\big|_{e_j}\,.$$

The analysis in Section 4 remains valid for the type of nonconforming meshes under consideration because the crucial relations $\Pi_e^0[\![\mathbf{n}\times(\mathbf{u}-\Pi_h\mathbf{u})]\!]=0$ for all $\boldsymbol{e}\in\mathcal{E}_h$ and $\Pi_e^0[\![\mathbf{n}\cdot(\mathbf{u}-\Pi_h\mathbf{u})]\!]=0$ for all $\boldsymbol{e}\in\mathcal{E}_h^i$ hold for the modified definition of $\mathcal{E}_h$. Furthermore, all the estimates for $\mathbf{u}-\Pi_h\mathbf{u}$ can be carried out triangle by triangle and hence also hold for partitions with hanging nodes.

Of course, the constants in the estimates now depend on the shape regularity of the partition, which roughly speaking involves the shape regularity of the triangles in $\mathcal{P}_h$ and the distribution of the hanging nodes on the edges in $\mathcal{E}_h$. Brenner [12, 13] detailed the concept of shape regularity of partitions.

6 Numerical experiments

In this section we report the results of a series of numerical experiments that corroborate our theoretical results. Both the L_2 error $\|\mathbf{u}-\mathbf{u}_h\|_{L_2(\Omega)}$ and the energy error $\|\mathbf{u}-\mathbf{u}_h\|_h$ are computed for $\gamma = 1$ in all the experiments.

In the first experiment we examine the convergence behavior of our numerical scheme on the square domain $(0,1)^2$ with conforming uniform meshes (Figure 4, left), where the exact solution is

$$\mathbf{u} = \begin{bmatrix} y(1-y) \\ x(1-x) \end{bmatrix}. \tag{57}$$

Table 1 gives the results for $\alpha = 1$, 0 and -1. They show that the scheme (14) is second order accurate in the L_2 norm and first order accurate in the energy norm, which agrees with the error estimates in Theorems 13 and 14.

In the second experiment we check the behavior of the scheme (14) on the square $(0,1)^2$ using nonconforming meshes with hanging nodes depicted in Figure 4 (right). The results in Table 2 show that the scheme also behaves as predicted in Theorems 13 and 14.

The goal of the final experiment is to demonstrate the convergence behavior of our scheme on the L-shaped domain $(-0.5,0.5)^2 \setminus [0,0.5]^2$. The right-hand side function is chosen to be

$$\mathbf{f} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{58}$$

The meshes are graded around the re-entrant corner $(0,0)$ using the refinement procedure of Bacuta et al. [7] with the grading parameter $1/3$. The

Table 1: Errors of the scheme on the square $(0,1)^2$ with conforming uniform meshes and exact solution given by (57).

h	$\frac{\|\mathbf{u}-\mathbf{u}_h\|_{L_2(\Omega)}}{\|\mathbf{u}\|_{L_2(\Omega)}}$	order	$\frac{\|\mathbf{u}-\mathbf{u}_h\|_h}{\|\mathbf{u}\|_h}$	order
		$\alpha = 1$		
1/8	1.34E−01	1.96	3.62E−01	1.00
1/16	3.29E−02	2.03	1.79E−01	1.02
1/32	8.07E−03	2.03	8.82E−02	1.02
1/64	1.99E−03	2.02	4.38E−02	1.01
		$\alpha = 0$		
1/8	1.49E−01	2.01	3.83E−01	1.03
1/16	3.63E−02	2.04	1.88E−01	1.03
1/32	8.88E−03	2.03	9.25E−02	1.02
1/64	2.19E−03	2.02	4.59E−02	1.01
		$\alpha = -1$		
1/8	1.69E−01	2.08	4.07E−01	1.07
1/16	4.05E−02	2.06	1.98E−01	1.04
1/32	9.88E−03	2.03	9.76E−02	1.02
1/64	2.44E−03	2.02	4.84E−02	1.01

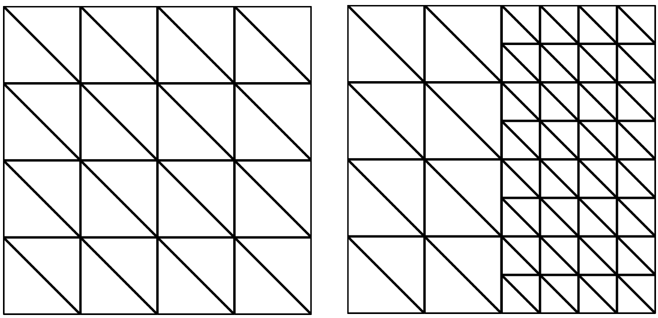

Figure 4: Conforming uniform mesh (left) and nonconforming mesh (right) on the square domain.

Table 2: Errors of the scheme on the square $(0,1)^2$ with nonconforming meshes and exact solution given by (57).

h	$\frac{\|\mathbf{u}-\mathbf{u}_h\|_{L_2(\Omega)}}{\|\mathbf{u}\|_{L_2(\Omega)}}$	order	$\frac{\|\mathbf{u}-\mathbf{u}_h\|_h}{\|\mathbf{u}\|_h}$	order
		$\alpha = 1$		
1/8	8.82E−02	1.80	2.98E−01	0.90
1/16	2.27E−02	1.96	1.51E−01	0.98
1/32	5.69E−03	2.00	7.59E−02	1.00
1/64	1.42E−03	2.00	3.81E−02	0.99
		$\alpha = 0$		
1/8	1.28E−01	1.93	3.59E−01	0.97
1/16	3.21E−02	2.00	1.80E−01	1.00
1/32	8.00E−03	2.00	8.99E−02	1.00
1/64	1.96E−03	2.03	4.03E−02	1.15
		$\alpha = -1$		
1/8	2.36E−01	2.38	4.85E−01	1.20
1/16	5.52E−02	2.10	2.35E−01	1.01
1/32	1.35E−02	2.03	1.16E−01	1.01
1/64	3.31E−03	2.03	5.80E−02	1.00

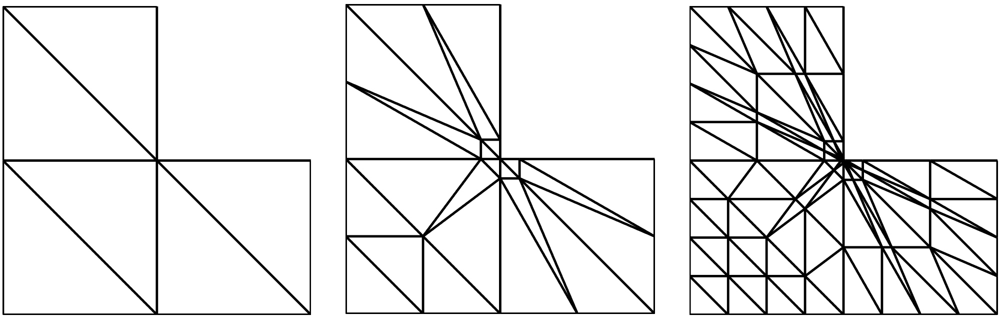

Figure 5: Graded meshes on the L-shaped domain.

first three levels of graded meshes are depicted in Figure 5. The results in Table 3 demonstrate that the scheme is second order accurate in the L_2 norm and first order accurate in the energy norm.

7 Concluding remarks

We extended the nonconforming method of our earlier work [15] to an interior penalty method that can be applied to nonconforming meshes. This interior penalty method enjoys optimal convergence rates without any tuning of penalty parameters.

The condition numbers of the discrete problems are worsened by the weak over penalization. It is therefore important to have good preconditioners in the case of fine meshes. For the Laplace operator, efficient preconditioners for interior penalty methods with weak over penalisation were developed by Owens et al. [31, 19]. The development of good preconditioners for the family of $H(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$ methods in earlier work [16, 15, 17, 20, 18] and this article is currently under investigation.

Table 3: Errors of the scheme on the L-shaped domain with graded meshes and right-hand side given by (58).

h	$\frac{\Vert\mathbf{u}-\mathbf{u}_h\Vert_{L_2(\Omega)}}{\Vert\mathbf{u}\Vert_{L_2(\Omega)}}$	order	$\frac{\Vert\mathbf{u}-\mathbf{u}_h\Vert_h}{\Vert\mathbf{u}\Vert_h}$	order
		$\alpha = 1$		
1/16	4.77E−01	1.67	1.02E+00	1.13
1/32	1.28E−01	1.89	4.65E−01	1.13
1/64	3.23E−02	1.99	2.20E−01	1.08
1/128	8.03E−03	2.01	1.07E−01	1.04
		$\alpha = 0$		
1/16	6.21E−01	2.11	1.14E+00	1.37
1/32	1.52E−01	2.03	5.01E−01	1.19
1/64	3.74E−02	2.02	2.34E−01	1.10
1/128	9.22E−03	2.02	1.13E−01	1.05
		$\alpha = -1$		
1/16	9.07E−01	3.45	1.46E+00	1.48
1/32	1.90E−01	2.26	5.47E−01	1.37
1/64	4.49E−02	2.08	2.55E−01	1.15
1/128	1.10E−02	2.04	1.22E−01	1.06

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