# A gradient recovery approach for nonconforming finite element methods with boundary modification

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**Abstract**

We use orthogonal and biorthogonal projections to post-process the gradient of the finite element solution produced by a nonconforming finite element approach. This leads to a better approximation property of the recovered gradient. We use an $L^2$-projection, where the trial and test spaces are different but form a biorthogonal system. This leads to an efficient numerical approach. We also modify our projection by applying the boundary modification method to obtain a higher order approximation on the boundary patch. Numerical examples are presented to demonstrate the efficiency and optimality of the approach.

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# Contents



# 1 Introduction

Gradient recovery is a technique of post-processing a finite element solution to achieve a better accuracy than the standard gradient. There are many methods proposed and implemented to post-process the gradient of a finite element solution. The majority of the existing methods, such as those based on local or global least square fittings [13, 6, 10, 12], global or local projections [1, 5], or averaging methods [8, 2], assume that the elements used are conforming. There are only a few publications dealing with the gradient recovery for nonconforming finite elements [4]. We introduce a technique of recovering the gradient of a nonconforming finite element solution using a biorthogonal system proposed earlier for the conforming finite element method [7]. In this article, we use two nonconforming finite element methods. One of them is the standard nonconforming triangular element method introduced by Crouzeix and Raviart [3] and the other one is the nonconforming quadrilateral element introduced by Rannacher and Turek [11].

Ilyas, Lamichhane and Meylan [7] introduced a new type of gradient recovery method to the standard conforming linear triangular element, where the gradient of the finite element solution is projected by using a biorthogonal system. Due to the use of the biorthogonal system, the projection matrix is diagonal and hence the computation is very efficient. Moreover, they proposed an efficient boundary modification technique to improve the accuracy of the recovered gradient on the boundary patch. The boundary modification was inspired by a technique used in the mortar finite element [9].

In this article, we extend the idea of Ilyas, Lamichhane and Meylan [7] to the triangular and quadrilateral nonconforming elements. Since the triangular Crouzeix–Raviart element basis functions are orthogonal in the $L^2$-norm, we do not need to construct a biorthogonal system in this case. However, the orthogonality does not hold for the quadrilateral nonconforming finite element basis function. In this case, we construct the biorthogonal basis functions to achieve the efficiency of the $L^2$-projection.

Let our domain $\Omega \subset \mathbb{R}^2$ be bounded and have polygonal boundary $\partial\Omega$. Let $\mathcal{T}_h$ be a quasi-uniform partition of $\Omega$ into triangles or quadrilaterals. We define $\mathcal{N}_v$ to be set of vertices of the partition $\mathcal{T}_h$, and $\mathcal{N}_m$ to be the set of midpoints of edges on the partition $\mathcal{T}_h$. We use standard Sobolev spaces:

$$
\begin{aligned}
L^2(\Omega) &= \left\{ f : \int_\Omega |f(x)|^2 \, dx < \infty \right\}, \\
H^1(\Omega) &= \left\{ f : f \in L^2(\Omega), \nabla f \in \left[L^2(\Omega)\right]^2 \right\}.
\end{aligned}
$$

With the triangular elements we denote the conforming standard (ST) element space and the nonconforming (NC) Crouzeix–Raviart element spaces as, respectively,

$$
\begin{aligned}
V_h^{\mathrm{ST}} &= \left\{ v \in C^0(\Omega) : v|_T \in \mathcal{P}_1(T), T \in \mathcal{T}_h \right\}, \\
V_h^{\mathrm{NC}} &= \left\{ v \in L^2(\Omega) : v|_T \in \mathcal{P}_1(T), T \in \mathcal{T}_h, v \text{ is continuous at all } p \in \mathcal{N}_m \right\}.
\end{aligned}
$$

With the quadrilateral elements we denote the conforming standard element space and the nonconforming Rannacher–Turek element space as, respectively,

$$
V_h^{\mathrm{ST}} = \left\{ v_h \in C^0(\Omega) : v_h|_T = \hat{v}_h \circ F_T^{-1}, \hat{v}_h|_{\hat{T}} \in \operatorname{span}\{1, x, y, xy\}, T \in \mathcal{T}_h \right\},
$$

$$V_h^{\text{NC}} = \big\{v \in L^2(\Omega) : v_h|_T = \hat{v}_h \circ F_T^{-1}\,,\ \hat{v}_h|_{\hat{T}} \in \operatorname{span}\{1, x, y, x^2 - y^2\}\,,\\ T \in \mathcal{T}_h\,, v_h \text{ continuous at } \mathcal{N}_m\big\}\,,$$

where $F_T : (0,1)^2 \to T$ is an element map, and $\hat{T} = (0,1)^2$.

# 2 New gradient recovery method

## 2.1 Biorthogonal projection

Let $\{\phi_1, \ldots, \phi_N\}$ and $\{\mu_1, \ldots, \mu_N\}$ be two sets of functions. We say that these two sets are biorthogonal if they satisfy the biorthogonality condition

$$\int_\Omega \phi_i \mu_j \, dx = c_i \delta_{i,j}\,, \quad c_i \neq 0\,, \quad \text{for all } i, j \in \{1, \ldots, N\}\,. \tag{1}$$

For the gradient recovery methods, we require a projection operator. In this article we use two projection operators, $\mathcal{P}^{\text{ST}}$ and $\mathcal{P}^{\text{NC}}$, which are orthogonal or biorthogonal projections to project $\tilde{\nabla} u_h$ onto $[V_h^{\text{ST}}]^2$ and $[V_h^{\text{NC}}]^2$, respectively. Here $\tilde{\nabla} = \left(\tilde{\partial}/\tilde{\partial} x_1, \tilde{\partial}/\tilde{\partial} x_2\right)$ is the piecewise gradient operator where partial derivatives are applied element-wise. If we have a triangular mesh and Crouzeix–Raviart finite elements, then the projection $\mathcal{P}^{\text{NC}}$ is orthogonal, as Crouzeix–Raviart finite element basis functions are orthogonal with respect to the $L^2$-inner product [3]. Otherwise, both $\mathcal{P}^{\text{ST}}$ and $\mathcal{P}^{\text{NC}}$ are biorthogonal projections.

Now we briefly describe a biorthogonal projection. Let $\mathcal{P}$ be a biorthogonal projection of $u \in L^2(\Omega)$ onto $V_h$. Let $\{\phi_1, \ldots, \phi_N\}$ be a set of basis functions of $V_h$ and $\{\mu_1, \ldots, \mu_N\}$ be another set of basis functions satisfying the biorthogonality relationship (1). Then $\mathcal{P}u \in V_h$ is defined by

$$\int_\Omega (\mathcal{P}u)\mu_j \, dx = \int_\Omega u \mu_j \, dx\,, \quad 1 \leqslant j \leqslant N\,. \tag{2}$$

Since $\mathcal{P}u \in V_h$ we write

$$\mathcal{P}u = \sum_{i=1}^{N} u_i \phi_i \,,$$

and thus (2) can be written as

$$\sum_{i=1}^{N} u_i \int_\Omega \phi_i \mu_j \, dx = \int_\Omega u \mu_j \, dx \,, \quad 1 \leqslant j \leqslant N \,. \tag{3}$$

Since the Gram matrix corresponding to the inner product $\int_\Omega \phi_i \mu_j \, dx$ is diagonal, the solution of the above system of equations is very efficient. Here, $\mathcal{P}$ projects $\tilde{\nabla} u_h$ onto $[V_h]^2$ as $\mathcal{P}\left(\tilde{\nabla} u\right) = \left(\mathcal{P}\left(\tilde{\partial} u / \tilde{\partial} x\right), \mathcal{P}\left(\tilde{\partial} u / \tilde{\partial} y\right)\right)$.

Since the Crouzeix–Raviart basis functions are $L^2$-orthogonal, we do not need to construct a biorthogonal system in this case. However, the Rannacher–Turek basis functions are not orthogonal and we need to construct a set of basis functions which form a biorthogonal system. We construct the basis functions in the reference element $\hat{T} = (0,1)^2$ as follows. Let $\hat{\phi}_1$, $\hat{\phi}_2$, $\hat{\phi}_3$ and $\hat{\phi}_4$ be the four basis functions associated with the four mid-points of the edges $(1/2, 0)$, $(1, 1/2)$, $(1/2, 1)$ and $(0, 1/2)$, respectively. Then,

$$\hat{\phi}_1 = -x^2 + x + y^2 - 2y + \tfrac{3}{4} \,, \qquad \hat{\phi}_2 = x^2 - y^2 + y - \tfrac{1}{4} \,,$$
$$\hat{\phi}_3 = -x^2 + x + y^2 - \tfrac{1}{4} \,, \qquad \hat{\phi}_4 = x^2 - 2x - y^2 + y + \tfrac{3}{4} \,.$$

We construct the following local basis functions which form a biorthogonal system with the standard local Rannacher–Turek basis functions:

$$\hat{\mu}_1 = 4 - \tfrac{57}{2} y + \tfrac{45}{2}\left(x - x^2 + y^2\right), \qquad \hat{\mu}_2 = -2 - \tfrac{33}{2} x + \tfrac{45}{2}\left(y + x^2 - y^2\right),$$
$$\hat{\mu}_3 = -2 - \tfrac{33}{2} y + \tfrac{45}{2}\left(x - x^2 + y^2\right), \qquad \hat{\mu}_4 = 4 - \tfrac{57}{2} x + \tfrac{45}{2}\left(y + x^2 - y^2\right).$$

## 2.2 Boundary modification

The boundary modification follows the same procedure as Ilyas, Lamichhane and Meylan [7], with only minor changes made to accommodate the change in element type and shape. First we define the set of nodes on the boundary $\mathcal{N}^{\text{out}}$ and interior nodes $\mathcal{N}^{\text{in}}$ as

$$\mathcal{N}^{\text{out}} = \{p : p \in \mathcal{N}, p \in \partial\Omega\}, \quad \mathcal{N}^{\text{in}} = \mathcal{N} \setminus \mathcal{N}^{\text{out}},$$

where $\mathcal{N}$ is defined as $\mathcal{N}_v$ or $\mathcal{N}_m$ for the standard or nonconforming element space, respectively. We then expand this notion to the elements, such that

$$\mathcal{T}^{\text{in}} = \left\{T : T \in \mathcal{T}_h, p \notin \overline{T}, \forall p \in \mathcal{N}^{\text{out}}\right\}, \quad \mathcal{T}^{\text{out}} = \mathcal{T}_h \setminus \mathcal{T}^{\text{in}},$$

where $\overline{T}$ is the closure of $T$. For each $x_i \in \mathcal{N}^{\text{out}}$ we find the closest element $T_i \in \mathcal{T}^{\text{in}}$, where we judge the distance from the element as the distance between $x_i$ and the centre of the triangle. Then, we remove the basis function corresponding to $x_i$, which we denote as $\phi_i$, and modify the basis functions corresponding to the nodes in $T_i$. We denote the nodes in $T_i$ as $x_{i_j}$ and their corresponding functions as $\phi_{i_j}$, where $1 \leqslant j \leqslant d$, where $d$ is three for triangular elements and four for quadrilateral elements.

The modification to the basis functions is as follows:

1. remove the basis function $\phi_i$;
2. replace the basis functions $\phi_{i_j}$ with the functions

$$\tilde{\phi}_{i_j} = \phi_{i_j} + \alpha_j \phi_i, \quad 1 \leqslant j \leqslant d,$$

where $\alpha_j$ are scalars satisfying

$$\sum_{j=1}^{d} \alpha_j p\left(x_{i_j}\right) = p\left(x_i\right), \quad \begin{cases} p \in \mathcal{P}_1(\Omega) & \text{triangular elements,} \\ p \in \operatorname{span}\{1, x, y, xy\} & \text{quadrilateral elements.} \end{cases}$$

Here $\{\alpha_j\}_{j=1}^{d}$ are the barycentric coordinates of $x_i$ in relation to $T_i$. We denote this boundary modified projection as $\mathcal{P}^{\text{ST}*}$ for the standard projection and $\mathcal{P}^{\text{NC}*}$ for the nonconforming projection.

# 3 Results

In this section we present three numerical examples to illustrate the numerical effectiveness of our proposed method. We investigate the $L^2$-error and rate of convergence for the standard projection and modified boundary projection for both conforming and nonconforming elements, while using the original gradient of the solution as a baseline. This investigation is performed on triangular and quadrilateral elements. The different errors of the gradient of the solution are:

- $E(u_h) = \left\|\nabla u - \tilde{\nabla} u_h\right\|_{L^2(\Omega)}$;
- $E_{\text{ST}}(u_h) = \left\|\nabla u - \mathcal{P}^{\text{ST}}\left(\tilde{\nabla} u_h\right)\right\|_{L^2(\Omega)}$ for projection to the standard element space;
- $E_{\text{NC}}(u_h) = \left\|\nabla u - \mathcal{P}^{\text{NC}}\left(\tilde{\nabla} u_h\right)\right\|_{L^2(\Omega)}$ for projection to the nonconforming element space.
- $E_{\text{ST}^*}(u_h) = \left\|\nabla u - \mathcal{P}^{\text{ST}^*}\left(\tilde{\nabla} u_h\right)\right\|_{L^2(\Omega)}$ for boundary modified projection to the standard element space;
- $E_{\text{NC}^*}(u_h) = \left\|\nabla u - \mathcal{P}^{\text{NC}^*}\left(\tilde{\nabla} u_h\right)\right\|_{L^2(\Omega)}$ for boundary modified projection to the nonconforming element space.

The first two examples demonstrate the superconvergence of the method for a regular solution. The third example demonstrates that the method shows an improved convergence rate even when the solution is less regular.

## 3.1 Example 1

The first example has the exact solution of a simple polynomial in two variables

$$u = x^2y^2 + (x^2 - 1)(y^2 - 1) + xy\,. \tag{4}$$

Dirichlet boundary conditions are applied on $\partial\Omega$, where $\Omega = [-1, 1]^2$.

Table 1: Triangle Example 1 errors and convergence rates $R$, where all errors are operating on $u_h$.

| elem | $E$ | R | $E_{ST}$ | R | $E_{NC}$ | R | $E_{ST*}$ | R | $E_{NC*}$ | R |
|---|---|---|---|---|---|---|---|---|---|---|
| 32 | 1.2264 | — | 1.1734 | — | 0.9436 | — | 1.5197 | — | 1.7901 | — |
| 128 | 0.6339 | 1.0 | 0.5023 | 1.2 | 0.3647 | 1.4 | 0.4007 | 1.9 | 0.4600 | 2.0 |
| 512 | 0.3196 | 1.0 | 0.1888 | 1.4 | 0.1330 | 1.5 | 0.0865 | 2.2 | 0.0934 | 2.3 |
| 2048 | 0.1601 | 1.0 | 0.0682 | 1.5 | 0.0476 | 1.5 | 0.0187 | 2.2 | 0.0191 | 2.3 |
| 8192 | 0.0801 | 1.0 | 0.0243 | 1.5 | 0.0169 | 1.5 | 0.0042 | 2.1 | 0.0038 | 2.3 |

Table 2: Quadrilateral Example 1 errors and convergence rates $R$, where all errors are operating on $u_h$.

| elem | $E$ | R | $E_{ST}$ | R | $E_{NC}$ | R | $E_{ST*}$ | R | $E_{NC*}$ | R |
|---|---|---|---|---|---|---|---|---|---|---|
| 16 | 1.2295 | — | 1.1654 | — | 0.9065 | — | 1.1391 | — | 1.6725 | — |
| 64 | 0.6242 | 1.0 | 0.4684 | 1.3 | 0.3390 | 1.4 | 0.2974 | 1.9 | 0.3668 | 2.2 |
| 256 | 0.3133 | 1.0 | 0.1728 | 1.4 | 0.1215 | 1.5 | 0.0753 | 2.0 | 0.0626 | 2.6 |
| 1024 | 0.1568 | 1.0 | 0.0620 | 1.5 | 0.0431 | 1.5 | 0.0189 | 2.0 | 0.0114 | 2.5 |
| 4096 | 0.0784 | 1.0 | 0.0221 | 1.5 | 0.0152 | 1.5 | 0.0047 | 2.0 | 0.0024 | 2.2 |

The convergence rate of the unmodified gradient projections for the triangular and quadratic elements approaches $R = 1.5$, as shown in Tables 1 and 2. The boundary modified projection has a numerical convergence rate that appears to be at least quadratic, whereas the nonconforming boundary modified projection appears to be the best in terms of errors.

## 3.2 Example 2

The second example has a transcendental function as its solution. The exact solution for this example is

$$u = e^x(x^2 + y^2) + y^2 \cos(xy) + x^2 \sin(xy)\,. \tag{5}$$

The Dirichlet boundary conditions are constructed on $\partial\Omega$, where $\Omega = [0, 1]^2$.

Table 3: Triangle Example 2 errors and convergence rates $R$, where all errors are operating on $u_h$.

| elem | $E$ | R | $E_{ST}$ | R | $E_{NC}$ | R | $E_{ST^*}$ | R | $E_{NC^*}$ | R |
|---|---|---|---|---|---|---|---|---|---|---|
| 32 | 0.9173 | – | 0.7477 | – | 0.4355 | – | 0.4787 | – | 0.4135 | – |
| 128 | 0.4717 | 1.0 | 0.2894 | 1.4 | 0.1562 | 1.5 | 0.1223 | 2.0 | 0.1031 | 2.0 |
| 512 | 0.2382 | 1.0 | 0.1062 | 1.4 | 0.0551 | 1.5 | 0.0292 | 2.1 | 0.0237 | 2.1 |
| 2048 | 0.1195 | 1.0 | 0.0381 | 1.5 | 0.0193 | 1.5 | 0.0070 | 2.1 | 0.0054 | 2.1 |
| 8192 | 0.0598 | 1.0 | 0.0135 | 1.5 | 0.0068 | 1.5 | 0.0017 | 2.0 | 0.0013 | 2.1 |

Table 4: Quadrilateral Example 2 errors and convergence rates $R$, where all errors are operating on $u_h$.

| elem | $E$ | R | $E_{ST}$ | R | $E_{NC}$ | R | $E_{ST^*}$ | R | $E_{NC^*}$ | R |
|---|---|---|---|---|---|---|---|---|---|---|
| 16 | 0.8530 | – | 0.5914 | – | 0.4168 | – | 0.4868 | – | 0.3949 | – |
| 64 | 0.4297 | 1.0 | 0.2151 | 1.5 | 0.1495 | 1.5 | 0.1056 | 2.2 | 0.0732 | 2.4 |
| 256 | 0.2153 | 1.0 | 0.0772 | 1.5 | 0.0534 | 1.5 | 0.0221 | 2.3 | 0.0146 | 2.3 |
| 1024 | 0.1077 | 1.0 | 0.0276 | 1.5 | 0.0190 | 1.5 | 0.0046 | 2.3 | 0.0031 | 2.2 |
| 4096 | 0.0539 | 1.0 | 0.0098 | 1.5 | 0.0067 | 1.5 | 0.0010 | 2.2 | 0.0007 | 2.1 |

The convergence rates for the projected solutions approach $R = 1.5$ for both elements and projections. The boundary modified projections approach a higher rate of $R = 2$ for both elements and both projections, as shown in Tables 3 and 4.

## 3.3 Example 3

The domain for this problem is $[-1, 1]^2$ with a slit $\{(x, y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 1\,, y = 0\}$, which is removed, giving us $\Omega = [-1, 1]^2 \setminus \{(x, y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 1\,, y = 0\}$. Therefore, this problem has a less regular solution than the previous examples because it has singularities along the slit that have been removed from the normal domain. The solution is

$$u = r^\alpha \sin(\alpha\theta)\,, \quad r = \sqrt{x^2 + y^2} \quad \theta = \arctan\left(\frac{y}{x}\right). \tag{6}$$

Table 5: Triangle Example 3 errors and convergence rates $R$, where all errors are operating on $u_h$.

| elem | $E$ | R | $E_{ST}$ | R | $E_{NC}$ | R | $E_{ST^*}$ | R | $E_{NC^*}$ | R |
|---|---|---|---|---|---|---|---|---|---|---|
| 128 | 0.2973 | — | 0.1414 | — | 0.1025 | — | 0.0673 | — | 0.0607 | — |
| 512 | 0.1506 | 1.0 | 0.0501 | 1.5 | 0.0365 | 1.5 | 0.0222 | 1.6 | 0.0175 | 1.8 |
| 2048 | 0.0757 | 1.0 | 0.0178 | 1.5 | 0.0130 | 1.5 | 0.0071 | 1.6 | 0.0056 | 1.6 |
| 8192 | 0.0379 | 1.0 | 0.0063 | 1.5 | 0.0046 | 1.5 | 0.0023 | 1.6 | 0.0018 | 1.6 |
| 32768 | 0.0190 | 1.0 | 0.0022 | 1.5 | 0.0017 | 1.5 | 0.0007 | 1.6 | 0.0006 | 1.7 |

Table 6: Quadrilateral Example 3 errors and convergence rates $R$, where all errors are operating on $u_h$.

| elem | $E$ | R | $E_{ST}$ | R | $E_{NC}$ | R | $E_{ST^*}$ | R | $E_{NC^*}$ | R |
|---|---|---|---|---|---|---|---|---|---|---|
| 64 | 0.1775 | — | 0.0981 | — | 0.0692 | — | 0.0482 | — | 0.0322 | — |
| 256 | 0.0892 | 1.0 | 0.0361 | 1.4 | 0.0255 | 1.4 | 0.0160 | 1.6 | 0.0099 | 1.7 |
| 1024 | 0.0447 | 1.0 | 0.0132 | 1.5 | 0.0093 | 1.5 | 0.0054 | 1.6 | 0.0033 | 1.6 |
| 4096 | 0.0223 | 1.0 | 0.0048 | 1.5 | 0.0034 | 1.5 | 0.0018 | 1.6 | 0.0011 | 1.6 |
| 16384 | 0.0112 | 1.0 | 0.0017 | 1.5 | 0.0012 | 1.5 | 0.0006 | 1.6 | 0.0004 | 1.6 |

We have used Dirichlet conditions on $\partial\Omega$.

The projected solutions have convergence rates that are lower than $R = 1.5$, but seem to approach a convergence rate slightly below $R = 1.5$, as seen in Tables 5 and 6. Both these convergence rates are higher than the base solution's rate of convergence.

The boundary modified solutions continue to have higher rates of convergence, even in this less regular problem. However, the convergence rates are less than the $R = 2$ found in the more regular cases of Examples 1 and 2, and a higher number of elements were required in Example 3 before the boundary modified projection could be applied since the method requires that there exists at least one element that has no nodes on $\partial\Omega$.

# 4 Conclusion

We have applied the gradient recovery method proposed by Ilyas, Lamichhane and Meylan [7] to nonconforming triangular and quadrilateral elements. We have numerically demonstrated that the recovered gradient has a better convergence rate than the standard gradient. An interesting finding is that we get best results projecting the gradient of a nonconforming space to the corresponding nonconforming space.

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