

On some overdetermined free boundary problems

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Abstract

This article deals with some free boundary problems for the Laplacian operator. We first give sufficient conditions of existence of free boundaries. Then combining the maximum principle to the monotonicity of the mean curvature, we prove a symmetry result in the case where the source term is constant. The method presented in this work is new and all the results obtained here can be extended to more general divergence operators. The technique used to prove the symmetry result can be applied for other symmetric overdetermined value problems.

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1 Introduction

Let D be an open and bounded subset of \mathbb{R}^N ($N \geq 2$) which contains all the domains we use in this article. Let $c > 0$ be a parameter and denote by f a positive function. We look for an open set $\Omega \subset D$ such that the following overdetermined free boundary problem has a solution.

$$F(c, f) \begin{cases} -\Delta u_\Omega = f & \text{in } \Omega, & u_\Omega = 0 & \text{on } \partial\Omega & \} P(\Omega, f), \\ -\Delta v_\Omega = u_\Omega & \text{in } \Omega, & v_\Omega = 0 & \text{on } \partial\Omega & \} P(\Omega, u_\Omega), \\ \frac{\partial u_\Omega}{\partial \nu} \times \frac{\partial v_\Omega}{\partial \nu} = c & \text{on } \partial\Omega & \text{(overdetermined condition).} \end{cases}$$

Here ν is the outward normal vector to $\partial\Omega$. Notice that since $u_\Omega = 0 = v_\Omega$ on $\partial\Omega$ then

$$|\nabla u_\Omega| = -\frac{\partial u_\Omega}{\partial \nu} \quad \text{and} \quad |\nabla v_\Omega| = -\frac{\partial v_\Omega}{\partial \nu}.$$

Therefore the overdetermined condition becomes

$$|\nabla u_\Omega| |\nabla v_\Omega| = c \quad \text{on } \partial\Omega.$$

In general without any assumptions on data, the problem $F(\mathbf{c}, f)$ has no solution.

Section 2 contains some preliminary results. In Section 3, we study the problem $F(\mathbf{c}, f)$ when f is a positive function belonging to $L^2(D)$ and having a compact support with nonempty interior: the solution we look for must strictly contain the convex hull of the support of f , say C . Our approach here consists of solving the shape optimization problem associated to $F(\mathbf{c}, f)$. The boundary of the minimum Ω we obtain can touch ∂C . So, the shape derivative allows us to get the overdetermined condition only on $\partial\Omega \setminus C$. Our goal in this section is to give sufficient conditions in order to have C strictly contained in Ω and $|\nabla \mathbf{u}_\Omega| |\nabla \mathbf{v}_\Omega| = \mathbf{c}$ on $\partial\Omega$, see Theorems 18 and 20. For Theorem 18 we apply the maximum principle, whereas for Theorem 20 we use the monotonicity of the mean curvature for the domains which are of class C^2 . Section 4 concerns the case where $f \equiv 1$. This section proves that any solution of $F(\mathbf{c}, 1)$ must be a ball. To get this symmetry result, we combine the maximum principle to the monotonicity of the mean curvature. Recall that when $f \equiv 2$, the problem $F(\mathbf{c}, f)$ arises from a variational problem in Probability [11]. Kinader and McDonald [11] used the first exit time for Brownian motion from a smoothly bounded domains in Euclidian space and defined two natural functionals on the space of embedded, compact, oriented, unparametrized hypersurfaces in Euclidian space. They developed explicit formulas for the first variation of the functionals and characterized the critical points as balls. Huang and Miller [10] established the variational formulas for maximizing the functionals (they considered) over C^k domains with a volume constraint and obtained the same symmetry result for their maximizers. We present a new method to derive this symmetry result. We can also extend it to the p -Laplacian case (see Remark 26 at the end of Section 4). The last section concludes.

2 Preliminaries

Definition 1 Let K_1 and K_2 be two compact subsets of D . We call a Hausdorff distance of K_1 and K_2 (or briefly $d_H(K_1, K_2)$) the following positive number:

$$d_H(K_1, K_2) = \max[\rho(K_1, K_2), \rho(K_2, K_1)],$$

where $\rho(K_i, K_j) = \max_{x \in K_i} d(x, K_j)$, $i, j = 1, 2$, and $d(x, K_j) = \min_{y \in K_j} |x - y|$.

Definition 2 Let ω_n be a sequence of open subsets of D and ω be an open subset of D . Let K_n and K be their complements in D . We say that the sequence ω_n converges in the Hausdorff sense, to ω (or briefly $\omega_n \xrightarrow{H} \omega$) if

$$\lim_{n \rightarrow +\infty} d_H(K_n, K) = 0.$$

Definition 3 Let ω_n be a sequence of open subsets of D and ω be an open subset of D . We say that the sequence ω_n converges in the compact sense to ω (or briefly $\omega_n \xrightarrow{K} \omega$) if

- every compact subset of ω is included in ω_n , for n large enough, and
- every compact subset of $\bar{\omega}^c$ is included in $\bar{\omega}_n^c$, for n large enough.

Definition 4 Let ω_n be a sequence of open subsets of D and ω be an open subset of D . We say that the sequence ω_n converges in the sense of characteristic functions, to ω (or briefly $\omega_n \xrightarrow{L} \omega$) if χ_{ω_n} converges to χ_ω in $L_{loc}^p(\mathbb{R}^N)$, $p \neq \infty$, (χ_ω is the characteristic function of ω).

Lemma 5 [5, 15] If ω_n is a sequence of open subsets of D then

1. there exists a subsequence (still denoted by ω_n) which converges, in the Hausdorff sense, to some open subset of D ,
2. $|\partial\Omega| \leq \liminf_{n \rightarrow +\infty} |\partial\Omega_n|$.

Definition 6 [3] Let C be a compact convex set and let $\text{int}(C)$ be its interior. The bounded domain ω satisfies C-GNP if

1. $\omega \supset \text{int}(C)$,
2. $\partial\omega \setminus C$ is locally Lipschitz,
3. for any $c \in \partial C$ there is an outward normal ray Δ_c such that $\Delta_c \cap \omega$ is connected, and
4. for every $x \in \partial\omega \setminus C$ the inward normal ray to ω (if exists) meets C .

Put

$$\mathcal{O}_C = \{\omega \subset D : \omega \text{ satisfies C-GNP}\}.$$

Theorem 7 If $\omega_n \in \mathcal{O}_C$, then there exists an open subset $\omega \subset D$ and a subsequence (again denoted by ω_n) such that (i) $\omega_n \xrightarrow{H} \omega$, (ii) $\omega_n \xrightarrow{K} \omega$, (iii) χ_{ω_n} converges to χ_ω in $L^1(D)$ and (iv) $\omega \in \mathcal{O}_C$. Furthermore, the assertions (i), (ii) and (iii) are equivalent.

Barkatou proved this theorem [3, Theorem 3.1] and the equivalence between (i), (ii) and (iii) [3, Propositions 3.4, 3.5, 3.6, 3.7 and 3.8]. Notice that, in general, we do not have the equivalence between (i), (ii) and (iii) [9].

Proposition 8 Let $\{\omega_n, \omega\} \subset \mathcal{O}_C$ such that $\omega_n \xrightarrow{H} \omega$. Let u_n and u_ω be respectively the solutions of $P(\omega_n, f)$ and $P(\omega, f)$. Then u_n converges strongly in $H_0^1(D)$ to u_ω (u_n and u_ω are extended by zero in D).

This proposition was proven for $N = 2$ or 3 [3, Theorem 4.3].

Proposition 9 [4, Theorem 3.5] *Let v_n and v_ω be respectively the solutions of $P(\omega_n, g_n)$ and $P(\omega, g)$. If g_n converges strongly in $H^{-1}(D)$ to g then v_n converges strongly in $H_0^1(D)$ to v_ω (v_n and v_ω are extended by zero in D).*

Definition 10 *Let C be a convex set. We say that an open subset ω has the C-SP, if*

1. $\omega \supset \text{int}(C)$,
2. $\partial\omega \setminus C$ is locally Lipschitz,
3. for any $c \in \partial C$ there is an outward normal ray Δ_c such that $\Delta_c \cap \omega$ is connected, and
4. for all $x \in \partial\omega \setminus C$ $K_x \cap \omega = \emptyset$, where K_x is the closed cone defined by

$$\{y \in \mathbb{R}^N : (y - x) \cdot (z - x) \leq 0, \text{ for all } z \in C\}.$$

Remark 11 K_x is the normal cone to the convex hull of C and $\{x\}$.

Proposition 12 [3, Proposition 2.3] *ω has the C-GNP if and only if ω satisfies the C-SP.*

Theorem 13 [2, Theorem 1.4] *Let L be a compact subset of \mathbb{R}^N . Let f_n be a sequence a functions defined on L . We assume that the f_n are of class C^3 and*

$$\left| \frac{\partial f_n}{\partial x_i} \right| \leq M, \quad \left| \frac{\partial^2 f_n}{\partial x_i \partial x_j} \right| \leq M, \quad \left| \frac{\partial^3 f_n}{\partial x_i \partial x_j \partial x_k} \right| \leq M,$$

where M is a strictly positive constant and is independent of n .

Define a sequence Ω_n , by $\Omega_n = \{x \in L : f_n(x) > 0\}$ and suppose there exists $\alpha > 0$ such that $|f_n(x)| + |\nabla f_n(x)| \geq \alpha$ for all x in L . If the Ω_n have the C-GNP, then there exists Ω of class C^2 and a subsequence (still denoted by Ω_n) such that Ω_n converges in the compact sense to Ω .

Remark 14 The aim of Theorem 13 is to give the C^2 regularity of the minimum Ω (respectively Ω^*) of J (respectively j) defined below. This is in order to use the shape derivative and so to resolve Problem $F(c, f)$. The proof of this theorem uses the following lemma

Lemma 15 Let L be a compact subset of \mathbb{R}^N . Let f_n be a sequence of functions defined as Theorem 13. Suppose that Ω is an open subset of L such that

$$\Omega = \{x \in L : h(x) > 0\} \quad \text{and} \quad \partial\Omega = \{x \in L : h(x) = 0\},$$

where h is a continuous function defined in L . If f_n converges uniformly to h in L , then the Ω_n converge in the compact sense, to Ω .

3 Problem $F(c, f)$

In this section, $N \in \{2, 3\}$. If ω is an open subset of D , let $|\partial\omega|$ (respectively $|\omega|$) be the perimeter (respectively the volume) of ω and $H_{\partial\omega}$ be the mean curvature of $\partial\omega$.

3.1 Shape optimization problems

Consider the following shape optimization problem (OP).

$$\text{Find } \Omega \in \mathcal{O}_C \quad \text{such that} \quad J(\Omega) = \min_{\omega \in \mathcal{O}_C} J(\omega),$$

where

$$J(\omega) = c|\omega| - \frac{1}{2} \int_{\omega} u_{\omega}^2 dx,$$

with $|\omega|$ the volume of ω and u_{ω} (respectively v_{ω}) the solution of $P(\omega, f)$ (respectively $P(\omega, u_{\omega})$). Notice that by Green's formula,

$$J(\omega) = c|\omega| - \frac{1}{2} \int_{\omega} f v_{\omega} dx.$$

3.1.1 Existence of the minima

Theorem 16 *There exists $\Omega \in \mathcal{O}_C$ which is of class C^2 and minimizes the functional J on \mathcal{O}_C . Furthermore, u_{Ω} (respectively v_{Ω}) is the solution of $P(\Omega, f)$ (respectively $P(\Omega, u_{\Omega})$).*

Proof: Let u_D be the solution of the Dirichlet problem $P(D, f)$, by the Maximum Principle, $0 \leq u_{\omega} \leq u_D$ so

$$J(\omega) = c|\omega| - \frac{1}{2} \int_{\omega} u_{\omega}^2 dx \geq -\frac{1}{2} \int_D u_D^2,$$

and $\inf J$ exists. Let Ω_n be a minimizing sequence in \mathcal{O}_C . We choose Ω_n as in Theorem 13 above and get the existence of a subsequence Ω_{n_k} and of Ω which is of class C^2 such that $\Omega_{n_k} \xrightarrow{K} \Omega$. Then, from Theorem 7, (i) implies $\Omega_{n_k} \xrightarrow{H} \Omega$, (iv) gives $\Omega \in \mathcal{O}_C$ and by (iii) $|\Omega_{n_k}|$ converges to $|\Omega|$. The convergence of $\int_D u_{n_k}^2 \chi_{\Omega_{n_k}}$ to $\int_D u_{\Omega}^2 \chi_{\Omega}$ is given by (iii) of Theorem 7 together with Proposition 8. We then conclude that $J(\Omega) = \min_{\omega \in \mathcal{O}_C} J(\omega)$. Now, on one hand Proposition 8 implies that u_{Ω} is solution of $P(\Omega, f)$. On the other hand, Proposition 8 together with Proposition 9 implies that v_{Ω} is the solution of $P(\Omega, u_{\Omega})$. ♠

Put

$$\mathcal{O}_{\Omega} = \{\omega \subset \Omega : \omega \text{ satisfies C-GNP}\}.$$

Consider

$$j(\omega) = c|\partial\omega| + \int_{\partial\omega} \frac{\partial u_\omega}{\partial \nu} dx,$$


where u_ω the solution of the Dirichlet problem $P(\omega, f)$. By Green's formula,

$$j(\omega) = c|\partial\omega| - \int_\omega f(x) dx.$$

Theorem 17 *There exists $\Omega^* \in \mathcal{O}_\Omega$ which minimizes the functional j on \mathcal{O}_Ω . Ω^* is of class C^2 .*

Proof: Since f is positive, then for all $\omega \subset D$,

$$j(\omega) \geq - \int_D f(x) dx.$$

So $\inf j$ exists. As in the proof of Theorem 16 above, there exists a subsequence $\Omega_{n_k}^*$ and Ω^* which is of class C^2 such that $\Omega_{n_k}^* \xrightarrow{H} \Omega$. Now, from Theorem 7, (iv) gives $\Omega^* \in \mathcal{O}_C$ and by (iii) $\int_{\Omega_{n_k}^*} f(x) dx$ converges to $\int_{\Omega^*} f(x) dx$. Then item 2 of Lemma 5 permits to get $j(\Omega^*) = \min_{\omega \in \mathcal{O}_C} j(\omega)$. 

3.1.2 The optimality conditions

In this subsection, we use the standard tool of the domain derivative to write down the optimality condition. Before doing this, recall the definition of the domain derivative [17, 14]. Suppose that Ω is of class C^2 . Consider a deformation field $V \in C^2(\mathbb{R}^N; \mathbb{R}^N)$ and set $\Omega_t = \{x + tV(x) : x \in \Omega\}$, $t > 0$. The application $\text{Id} + tV$ (a perturbation of the identity) is a Lipschitz

diffeomorphism for t small enough and by definition, the derivative of J at Ω in the direction V is

$$dJ(\Omega, V) = \lim_{t \rightarrow 0} \frac{J(\Omega_t) - J(\Omega)}{t}.$$

As the functional J depends on the domain Ω through the solution of the Dirichlet problem $P(\Omega, f)$, we need to define also the domain derivative u'_Ω of u_Ω :

$$u'_\Omega = \lim_{t \rightarrow 0} \frac{u_{\Omega_t} - u_\Omega}{t}.$$

Furthermore, u'_Ω is the solution of the following problem [17, 14]:

$$\begin{cases} -\Delta u'_\Omega = 0 & \text{in } \Omega, \\ u'_\Omega = -\frac{\partial u_\Omega}{\partial \nu} V \cdot \nu & \text{on } \partial\Omega, \end{cases} \quad (1)$$

Now to compute the the shape derivatives of J and j , recall the following [9].

1. The shape derivatives of the volume and the perimeter are respectively

$$\int_{\partial\Omega} V \cdot \nu \, d\sigma \quad \text{and} \quad \int_{\partial\Omega} \text{NH}_{\partial\Omega} V \cdot \nu \, d\sigma.$$

2. Since $u_\Omega \in H_0^1(D)$ and Ω is of class C^2 , if we put

$$F(\Omega) = \int_{\Omega} u_\Omega^2 \, dx,$$

then the Hadamard formula gives

$$dF(\Omega, V) = 2 \int_{\Omega} u_\Omega u'_\Omega \, dx.$$

But v_Ω is solution to Dirichlet problem $P(\Omega, u_\Omega)$, so by Green's formula

$$dF(\Omega, V) = 2 \int_{\partial\Omega} |\nabla u_\Omega| |\nabla v_\Omega| V \cdot \nu \, d\sigma.$$

Now since $J(\omega) = c|\omega| - \frac{1}{2}F(\omega)$ then

$$dJ(\Omega, V) = \int_{\partial\Omega} (c - |\nabla u_\Omega| |\nabla v_\Omega|) V \cdot \nu \, d\sigma. \quad (2)$$

Ω being the minimum of the functional J , $dJ(\Omega, V) \geq 0$ for every admissible direction V . Therefore

$$\int_{\partial\Omega} (c - |\nabla u_\Omega| |\nabla v_\Omega|) V \cdot \nu \, d\sigma \geq 0 \quad \text{for every admissible direction } V.$$

We mean by admissible direction the one which allows us to keep the C-GNP or the C-SP (according to Proposition 12). Since Ω has the C-GNP, it satisfies the C-SP. Then

$$\text{for all } x \in \partial\Omega \setminus C, \quad K_x \cap \Omega = \emptyset.$$

For t sufficiently small, let $\Omega_t = \Omega + tV(\Omega)$ be the deformation of Ω in the direction V . Let $x_t \in \partial\Omega_t$. There exists $x \in \partial\Omega$ such that $x_t = x + tV(x)$. Using the definition of K_{x_t} and the equality above, we get (for t small enough and for every displacement V)

$$\text{for all } x_t \in \partial\Omega_t \setminus C, \quad K_{x_t} \cap \Omega_t = \emptyset,$$

which means that Ω_t satisfies the C-SP (and so the C-GNP) for every direction V when t is sufficiently small. Then, using V and $-V$, and that the set of the functions $V \cdot \nu$ is dense in $L^2(\partial\Omega)$, we deduce

$$|\nabla u_\Omega| |\nabla v_\Omega| = c \quad \text{on } \partial\Omega \setminus \partial C. \quad (3)$$

On the other hand, the admissible directions V on $\partial\Omega \cap \partial C$ must satisfy

$$V(x) \cdot \nu(x) \geq 0,$$

and one gets

$$|\nabla u_\Omega| |\nabla v_\Omega| \leq c \quad \text{on } \partial\Omega \cap \partial C. \quad (4)$$

Now, thanks to Hadamard formula, the shape derivative of j at Ω^* is

$$dj(\Omega^*, V) = \int_{\partial\Omega^*} (NcH_{\partial\Omega^*} - f)V \cdot \nu \, d\sigma \geq 0,$$

for every admissible direction V .

Arguing as above and observing that the deformation domain Ω_t^* is in Ω if the directions V on $\partial\Omega^* \cap \partial\Omega$ satisfy $V(x) \cdot \nu(x) \leq 0$, we get

$$\begin{cases} H_{\partial\Omega^*} = 0 & \text{on } \partial\Omega^* \setminus (\partial C \cup \partial\Omega), \\ H_{\partial\Omega^*} \leq 0 & \text{on } \partial\Omega^* \cap \partial\Omega, \quad \text{and} \\ H_{\partial\Omega^*} \geq \frac{f}{Nc} & \text{on } \partial\Omega^* \cap \partial C. \end{cases} \quad (5)$$

3.1.3 Main theorems

Theorem 18 *Let Ω be as in Theorems 16 and C be of class C^2 . Let u_C (respectively v_C) be the solution of $P(\text{int}(C), f)$ (respectively $P(\text{int}(C), u_C)$). Suppose that both u_C and v_C are in $C^2(C)$ and*

$$|\nabla u_C| |\nabla v_C| > c \quad \text{on } \partial C. \quad (6)$$

Then Ω is a solution of $F(c, f)$ which strictly contains C .

Proof: $\partial\Omega \neq \partial C$, otherwise $\Omega = \text{int}(C)$, $u_\Omega = u_C$ and $v_\Omega = v_C$. But the optimality conditions (3) and (4) give

$$|\nabla u_C| |\nabla v_C| = |\nabla u_\Omega| |\nabla v_\Omega| \leq c \quad \text{on } \partial C,$$

which contradicts (6).

Now, suppose that $\partial\Omega \cap \partial C \neq \emptyset$. Since u_Ω and u_C are in $C^2(C)$,

$$\Delta u_\Omega = -f = \Delta u_C \quad \text{in } \text{int}(C) \quad \text{and} \quad u_\Omega \geq 0 = u_C \quad \text{on } \partial C,$$

the maximum principle implies that

$$u_\Omega \geq u_C \quad \text{in } \text{int}(C).$$

But $u_\Omega \neq u_C$ in $\text{int}(C)$, then

$$u_\Omega > u_C \quad \text{in } \text{int}(C).$$

Now, since $u_\Omega = u_C$ on $\partial\Omega \cap \partial C$, then by the maximum principle we get

$$\frac{\partial u_\Omega}{\partial \nu} < \frac{\partial u_C}{\partial \nu} \quad \text{on } \partial\Omega \cap \partial C,$$

or again, since $|\nabla u_\Omega| = -\frac{\partial u_\Omega}{\partial \nu}$,

$$|\nabla u_C| < |\nabla u_\Omega| \quad \text{on } \partial\Omega \cap \partial C.$$

Now by the maximum principle, $u_\Omega \geq 0$ on $\bar{\Omega}$. Then replacing f by u_Ω , u_Ω by v_Ω and u_C by v_C , and arguing as above, we obtain

$$|\nabla v_C| < |\nabla v_\Omega| \quad \text{on } \partial\Omega \cap \partial C.$$

But (6) together with (4) implies

$$c < |\nabla u_C| |\nabla v_C| < |\nabla u_\Omega| |\nabla v_\Omega| \leq c \quad \text{on } \partial C \cap \partial\Omega.$$

which is absurd. It then follows that C is strictly contained in Ω and so

$$|\nabla u_\Omega| |\nabla v_\Omega| = c \quad \text{on } \partial\Omega.$$



Theorem 19 *Let Ω and Ω^* be as in Theorems 16 and 17. Suppose*

$$|\nabla u_{\Omega^*}| |\nabla v_{\Omega^*}| > c \quad \text{on } \partial\Omega^*. \tag{7}$$

Then Ω is a solution of $F(c, f)$ which strictly contains Ω^ .*

Replacing Ω by Ω^* , the proof is the same as above.

Theorem 20 *Let Ω and Ω^* be as in Theorems 16 and 17. Suppose that C is of class C^2 and*

$$H_{\partial C} < \frac{f}{N_c} \quad \text{on } \partial C. \quad (8)$$

Then

1. C is strictly contained in Ω^* ,
2. Ω is a solution of $F(c, f)$ which contains Ω^* .

Proof:

1. Suppose by contradiction that $\partial\Omega^* \cap \partial C \neq \emptyset$. Since $\text{int}(C) \subset \Omega^*$ and C and Ω^* are of class C^2 , then by using the monotonicity of the mean curvature together with (5) and (8) we have on $\partial\Omega^* \cap \partial C$

$$\frac{f}{N_c} \leq H_{\partial\Omega^*} \leq H_{\partial C} < \frac{f}{N_c},$$

which gives a contradiction.

2. C is strictly contained in Ω^* which is contained in Ω (by definition). So C is strictly contained in Ω and (3) gives the overdetermined condition on $\partial\Omega$.



4 A symmetry result

In this section, we suppose that $f \equiv 1$ and $N \geq 2$.

Definition 21 [6] *We say that a domain ω satisfies the ε -cone property if for all $x \in \partial\omega$ there exist a direction vector $\xi \in \mathbb{R}^N$ such that the cone $C(y, \xi, \varepsilon) \subset \omega$ for all $y \in B(x, \varepsilon) \cap \bar{\omega}$. ε denotes both angle and height of the cone.*

Denoting by \mathcal{O}_ε the class of domains which have the ε -cone property, we have this lemma.

Lemma 22 [6] *If $\omega_n \in \mathcal{O}_\varepsilon$, then there exists an open subset $\omega \subset D$ and a subsequence (again denoted by ω_n) such that (i) $\omega_n \xrightarrow{H} \omega$, (ii) $\bar{\omega}_n \xrightarrow{H} \bar{\omega}$, (iii) $\partial\omega_n \xrightarrow{H} \partial\omega$, (iv) χ_{ω_n} converges to χ_ω in $L^1(D)$ and (v) $\omega \in \mathcal{O}_\varepsilon$, (vi) v_{ω_n} converges strongly in $H_0^1(D)$ to v_ω (v_{ω_n} and v_ω are respectively the solution of $P(\omega_n, 1)$ and $P(\omega, 1)$).*

Let Ω be a solution of the problem $F(c, 1)$ and put

$$\mathcal{O}_1 = \{\omega \supset \Omega : \omega \in \mathcal{O}_\varepsilon\}.$$

Consider

$$J_1(\omega) = c^2|\partial\omega| - \frac{1}{2} \int_\omega u_\omega^2,$$

where u_ω is the solution of the Dirichlet problem $P(\omega, 1)$.

Before stating the main theorem of this section, let us state the two following propositions. The proofs are analogous to those of Theorems 16 and 17. They use Lemma 22 together with the lower semicontinuity of the perimeter and the continuity of the volume (for the Hausdorff topology). To obtain

the optimality conditions, we use the shape derivative as in Subsection 3.1.2. Using the same notations, to get Ω in $(\Omega_1)_t$ (for t small enough) we must choose a deformation directions V such that $V \cdot \nu \geq 0$ on $\partial\Omega \cap \partial\Omega_1$. In the same way $V \cdot \nu \leq 0$ on $\partial\Omega \cap \partial\Omega_2$ allows us to have $(\Omega_2)_t$ in Ω (Ω_1 and Ω_2 are given by Propositions 23 and 24).

Proposition 23 *There exists $\Omega_1 \in \mathcal{O}_1$ which minimizes J_1 on \mathcal{O}_1 . If Ω_1 is of class C^2 , then*

1. $|\nabla u_{\Omega_1}| |\nabla v_{\Omega_1}| \leq Nc^2 H_{\partial\Omega_1}$ on $\partial\Omega_1 \cap \partial\Omega$,
2. $|\nabla u_{\Omega_1}| |\nabla v_{\Omega_1}| = Nc^2 H_{\partial\Omega_1}$ on $\partial\Omega_1 \setminus \partial\Omega$.

Consider

$$\mathcal{O}_2 = \{\omega \subset \Omega : \partial\omega \cap \partial\Omega_1 \neq \emptyset \text{ and } \omega \in \mathcal{O}_\varepsilon\},$$

and

$$J_2(\omega) = c|\partial\omega| - |\omega|.$$

Proposition 24 *There exists $\Omega_2 \in \mathcal{O}_2$ which minimizes J_2 on \mathcal{O}_2 . If Ω_2 is of class C^2 , then*

1. $H_{\partial\Omega_2} \leq \frac{1}{Nc}$ on $\partial\Omega \cap \partial\Omega_2$,
2. $H_{\partial\Omega_2} = \frac{1}{Nc}$ on $\partial\Omega_2 \setminus \partial\Omega$.

In the sequel, we state and prove the main theorem of this section.

Theorem 25 *If Ω is a solution of $F(c, 1)$, then Ω is a ball of radius Nc .*

As we see in the following proof, to reach the conclusion of the theorem, we show that $\Omega = \Omega_1$.

Proof: Since $\Omega \subset \Omega_1$, it is sufficient to show that $\partial\Omega = \partial\Omega_1$. By Lemma 22 and the definition of \mathcal{O}_2 , $\partial\Omega_2 \cap \partial\Omega_1 \neq \emptyset$. So since $\Omega_2 \subset \Omega \subset \Omega_1$ then

$$\partial\Omega_2 \cap \partial\Omega \cap \partial\Omega_1 \neq \emptyset.$$

This together with the monotonicity of the mean curvature and the fact that Ω , Ω_1 and Ω_2 are of class C^2 , implies that for all $x \in \partial\Omega_2 \cap \partial\Omega \cap \partial\Omega_1$,

$$H_{\partial\Omega_1}(x) \leq H_{\partial\Omega}(x) \leq H_{\partial\Omega_2}(x) \leq \frac{1}{Nc}. \quad (9)$$

Hence, Propositions 23 and 24 imply

$$|\nabla u_{\Omega_1}(x)| |\nabla v_{\Omega_1}(x)| \leq Nc^2 H_{\partial\Omega_2}(x) \leq c.$$

Suppose that $\partial\Omega \neq \partial\Omega_1$, by the maximum principle we obtain


$$c = |\nabla u_{\Omega}(x)| |\nabla v_{\Omega}(x)| < |\nabla u_{\Omega_1}(x)| |\nabla v_{\Omega_1}(x)| \leq c,$$

which is a contradiction. It then follows that $\Omega = \Omega_1$ and so

$$c = |\nabla u_{\Omega}| |\nabla v_{\Omega}| = Nc^2 H_{\partial\Omega} \quad \text{on } \partial\Omega.$$

This implies that

$$H_{\partial\Omega} = \frac{1}{Nc},$$

which means that Ω is a ball of radius Nc thanks to Alexandrov's result [1]. In this case, (9) implies $\Omega_2 = \Omega$. 

Remark 26 Combining the Hopf's comparison principle [18] with the monotonicity of the mean curvature, one can extend the result of Theorem 25 to the case of the p -Laplace operator.

5 Concluding remarks

Remark 27 Recall that the C-GNP is satisfied by the solution of the so-called Quadrature Surface Free Boundary Problem [16, 8]. The domain which satisfies the C-GNP may be non-Lipschitz, it may contain cusps [3]. As a consequence, \mathcal{O}_C cannot be contained in \mathcal{O}_ε .

Remark 28 The hypothesis in Theorem 13 about the local regularity is not too restrictive because of, for instance, results due to DiBenditto [7], Lewis [12] and Lieberman [13].

Remark 29 For the p-Laplacian, the continuity with respect to the domain is a consequence of the γ_p -convergence [4]. So using Hopf's comparison principle and replacing the condition (6) by

$$|\nabla \mathbf{u}_C| |\nabla \mathbf{v}_C| > \mathbf{c} \quad \text{on } C.$$

one can extend Theorems 18 and 19 to the p-Laplacian case.

Remark 30 Theorem 20 can be extended to other divergence operators like $\text{div}(\mathbf{a}(x, D\mathbf{u}))$. For this kind of operators the continuity with respect to the domain is a simple consequence of Mosco convergence [4].

Remark 31 Consider the following free boundary problem. We look for a domain Ω and a couple of functions $(\mathbf{u}_\Omega; \mathbf{v}_\Omega)$ such that \mathbf{u}_Ω is solution of $P(\Omega, f)$, \mathbf{v}_Ω is solution to $P(\Omega, \mathbf{u}_\Omega)$ and

$$|\nabla \mathbf{u}_\Omega| |\nabla \mathbf{v}_\Omega| = \sigma H_{\partial\Omega} + \mathbf{c} \quad \text{on } \partial\Omega.$$

By considering the functional of domain

$$J_\sigma(\omega) = \frac{\sigma}{N} |\partial\omega| + \mathbf{c}|\omega| - \frac{1}{2} \int_\omega \mathbf{u}_\omega^2 \, dx,$$

we use the same arguments as in Theorem 16 and prove the existence of a minimum Ω for J_σ on \mathcal{O}_C which is of class C^2 . Then performing the shape derivative we obtain the following optimality conditions

$$\begin{aligned} |\nabla \mathbf{u}_\Omega| |\nabla v_\Omega| &= \sigma H_{\partial C} + c \quad \text{on } \partial\Omega \setminus \partial C, \\ |\nabla \mathbf{u}_\Omega| |\nabla v_\Omega| &\leq \sigma H_{\partial C} + c \quad \text{on } \partial\Omega \cap \partial C. \end{aligned}$$

Next, replacing in the inequality (6) of Theorem 18, c by $\sigma H_{\partial C} + c$ we obtain the same conclusion by combining the maximum principle with the monotonicity of the mean curvature. Now, according to Remark 30 we can extend this result to the p-Laplacian case.

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