Biological self-heating in industrial compost piles: an informal discussion of students applying prior mathematical skills within an industrial case study

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Abstract

We consider a simple 'toy model' for the spontaneous combustion of industrial compost stockpiles. The model is a scalar non-linear differential equation which can be analysed using techniques taught in an introductory subject on non-linear ordinary differential equations. This model was used as a case study in a third year subject. We discuss how students approached some of the questions. Could they transfer their prior knowledge about differential equations to an industrial case study? How would they cope with a problem which required both pen-and-paper and numerical calculations? Students used a variety of approaches but the worked solutions only showed one. It would be

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beneficial for students to see that there is not one correct method to solve such problems.

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1 Introduction

In New South Wales, organic waste material, both 'garden organics' and 'food organics', from households is placed in green bins. Through stockpiling Food Organics and Garden Organics (FOGO) is made into a quality compost, an example of environmental recycling. Given the chemical processes underlying FOGO composting and the size of the stockpiles held at many council and commercial facilities, it is perhaps unsurprising that spontaneous combustion of FOGO piles has been reported in NSW councils. More generally, the spontaneous combustion of compost piles has been reported throughout Australia and beyond.

In Spring session 2021, one of the authors taught the subject MATH313 (Case Studies in Applied Mathematics). This contains a module on spontaneous combustion which investigates the Semenov model

$$\frac{d\theta}{dt^*} = \psi \exp(\theta) - \theta\,, \tag{1}$$

where θ is a scaled temperature, proportional to the difference between the temperature of the stockpile and ambient temperature, t^* is a scaled time, and ψ is a control parameter. Students are taught how to analyse equation (1) using standard techniques. The model assumes that there are two mechanisms for heat transfer. Firstly, heat is generated by an exothermic chemical reaction. Secondly, heat is lost from the stockpile to the surrounding atmosphere through convection. These terms are modelled as a first-order Arrhenius reaction and by Newtonian cooling, respectively. Barnes and Fulford [1] provide further details of the model.

The recent incidences of compost ignition in NSW motivated an assignment using a toy model for the spontaneous combustion of industrial compost piles [2]. This provided a topical case study for students, who were aware of FOGO recycling in New South Wales.

Students made heavy weather of this case study and their marks were lower than our expectations. Students in this subject usually submit hardcopy solutions. However, in 2021, as a consequence of COVID, they submitted their assignments via moodle. This meant that at the end of session it was possible to examine the solutions. We discuss some of the assignment questions/solutions provided by 19 students.

One motivation for our article is to bring the underlying model to a wider audience. We believe this to be of interest because it can be analysed using techniques commonly taught to undergraduates. Our second motivation is to bring to the attention of prospective users some of the problems that our students experienced in the hope that to be forewarned is forearmed.

2 The model

Extending the basic Semenov model to include the rate of heat generation due to the microbial reactions in compost piles, whilst excluding reactant consumption, results in the equation [2]

$$\frac{d\theta}{dt^*} = \frac{\psi_b \exp(\theta)}{1 + \beta \exp(\alpha_d \theta)} + \psi_o \exp(\alpha_o \theta) - \theta \,. \tag{2}$$

The left-hand side of (2) is the rate of change of the dimensionless temperature difference between the stockpile and its environment. The first two-terms on the right-hand of (2) are, respectively, the rate of heat generation due to microbial reactions and the oxidation of organic materials. The final term models convective heat transfer between the stockpile and its surroundings. (Note that the chemical oxidation term in (2) contains a parameter α_o which does not appear in (1). This is due to the underlying models being scaled in different ways.) All parameters are non-negative.

To simplify the model we only consider heat generation due to biological reactions, that is $\psi_o = 0$. The main experimental control parameter is the biomass Semenov number ψ_b, which is a function of the size of the compost pile, that is it is a quantity that the pile operator can control. The parameters β and α_d are related to the kinetics of biomass deactivation. The former is a deactivation rate whilst the latter is a scaled acivation energy.

3 Heat release rate

Sometimes the behaviour that we are modelling imposes constraints on parameter values. The following question explores this concept.

Question 1 *At sufficiently high temperatures micro-organisms die quicker than they can reproduce. Thus the rate at which heat is released by biological reactions must eventually be a decreasing function of temperature. Does this impose any restrictions on the values of any of the model parameters?*

The rate of heat release is the first term on the right-hand side of equation (2)

$$\mathcal{R} = \frac{\psi_b \exp(\theta)}{1 + \beta \exp(\alpha_d \theta)}, \tag{3}$$

$$\Rightarrow \frac{d\mathcal{R}}{d\theta} = \frac{\psi_b \exp(\theta)}{[1 + \beta \exp(\alpha_d \theta)]^2} \cdot [1 + \beta(1 - \alpha_d) \exp(\alpha_d \theta)]. \tag{4}$$

From this we find that

$$\frac{d\mathcal{R}}{d\theta} = 0 \iff 1 = \beta\,(\alpha_d - 1) \exp(\alpha_d \theta). \tag{5}$$

Consequently, for the rate of heat release to eventually be a decreasing function we require $\alpha_d > 1$.

- Seven students went wrong at the start by incorrectly identifying the expression for the rate of heat release, writing

  $$\mathcal{R} = \frac{\psi_b \exp(\theta)}{1 + \beta \exp(\alpha_d \theta)} - \theta.$$

  This indicates that students did not see a distinction between the rate of change of heat content of a stockpile and the rate at which heat is released by biological reactions. This uncertainty could have been clarified by recourse to the discussion of the Semenov model in the lecture notes.

- Three students correctly obtained the condition $\alpha_d > 1$ through the use of a calculus argument. An additional five students obtained equation (4). However, four did not see how to proceed from this point.

- One student commented "*not sure where to go from here or if I'm even on the right track*". Finally, one student obtained the inequality

  $$\frac{1}{\beta(\alpha_d - 1)} > 0,$$

  but incorrectly deduced from this that $\alpha_d > 0$.

- Four students scored full marks via a different argument that considered the limiting value of the heat release rate as $\theta \to \infty$:

$$\begin{aligned}\lim_{\theta\to\infty} \mathcal{R} &= \lim_{\theta\to\infty} \frac{\psi_b \exp(\theta)}{1+\beta\exp(\alpha_d\theta)} = \lim_{\theta\to\infty} \frac{\psi_b}{\exp(-\theta)+\beta\exp\left[(\alpha_d-1)\,\theta\right]} \\ &= \lim_{\theta\to\infty} \frac{\psi_b}{\beta\exp\left[(\alpha_d-1)\,\theta\right]} \\ &= \begin{cases} \text{DNE} & 0<\alpha_d<1\,, \\ \frac{\psi_b}{\beta} & \alpha_d = 1\,, \\ 0 & \alpha_d > 1\,. \end{cases}\end{aligned} \tag{6}$$

4 Steady-state diagram: S-shaped behaviour

Consider the non-linear differential equation model

$$\frac{dx}{dt} = f(x,\mu)\,,$$

where μ is a parameter. A standard approach to study such models is to construct the steady-state diagram. This shows how the steady-state solutions and their stability change as the control parameter is varied. The steady-state solutions are found by solving the equation

$$f(x,\mu) = 0\,. \tag{7}$$

There are two common ways to determine the stability of a steady-state solution x^*. We may either evaluate the eigenvalue $\lambda = f'(x^*,\mu)$ or we can determine the stability by sketching the function $y = f(x,\mu)$.

Question 2 In this question we take $\beta = 0.01$ and $\alpha_d = 1.1$.

1. *Obtain the steady-state diagram showing the dimensionless temperature as a function of the biomass Semenov number over the region* $0 \leqslant \theta \leqslant 12$.

A more challenging question would remove the restriction $0 \leqslant \theta \leqslant 12$. Students would have to determine the appropriate range by finding the location of the limit points.

The steady-state diagram 1 is the classic S-shaped response curve that is obtained in many combustion problems. It contains three branches: a 'low' temperature branch, an 'intermediate' branch, and a 'high' temperature branch. We show in the answer to the next question that these branches are respectively stable, unstable, and stable. From the perspective of composting, the lowest branch is undesirable, representing stockpiles with a negligible degree of composting. The highest branch is desirable, it represents stockpiles where a significant amount of composting occurs. The control parameter ψ_b is a function of several parameters, but most importantly it is a function of the geometry of the compost pile: $\psi_b \propto V/S$, where V is the volume and S the surface area of the stockpile.

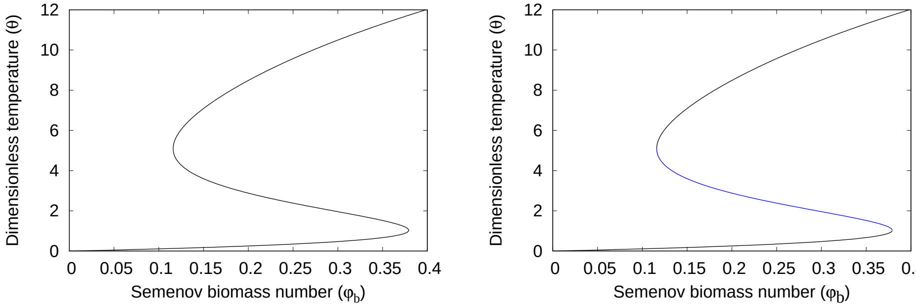
2. *Your diagram should indicate the stability of the solution branches: you must show how you have determined this.*

The first step in the solution is to construct the steady-state diagram without determining the stability of the solutions. This is shown in Figure 1(a). At the steady-state the right-hand side of equation (2) is rearranged to give

$$\psi_b = \frac{1 + \beta \exp(\alpha_d \theta)}{\exp(\theta)} \cdot \theta. \tag{8}$$

- Eight students generated values for θ and used equation (8) to calculate the corresponding values for ψ_b. They used a variety of packages to plot θ as a function of ψ_b: Excel (one student), MATLAB (three), Python (one), and RStudio (three). Five of the students provided details on the spacing of the θ points: two took $\Delta\theta = 1$, and one each used $\Delta\theta = 0.119$, $\Delta\theta = 0.01$, and $\Delta\theta = 0.001$.

- Equation (8) gives ψ_b as a function of θ. A number of packages can use this definition to plot θ as a function of ψ_b. Six students took this approach: five used DESMOS and one used "an online grapher".



(a) Steady-state diagram without stability.

(b) Steady-state diagram with stability.

Figure 1: Steady state diagram for the compost model without chemical heating (S-shaped behaviour). In (b) the black and blue lines represent stable and unstable steady-state solutions, respectively. Parameter values: $\alpha = 1.1$, $\beta = 0.01$, $\psi_0 = 0$.

- Five students provided no insights into how they obtained Figure 1(a).

The next task is to deduce the stability of the steady-state solutions shown in Figure 1(a). This figure has two limit-points. These divide the steady-state diagram into three regions. Denote the location of the limit-points by $(\psi_{b,e}, \theta_e)$ and $(\psi_{b,i}, \theta_i)$ with $\psi_{b,i} > \psi_{b,e}$. The first region is given by $0 \leqslant \psi_b < \psi_{b,e}$, the second region is given by $\psi_{b,e} < \psi_b < \psi_{b,i}$, and the third region is given by $\psi_b > \psi_{b,i}$. In these regions, for a fixed value of ψ_b, there are one, three, and one steady-state solutions, respectively. (For readers familiar with combustion theory, the subscripts e and i denote extinction and ignition limit points respectively.)

In each region the stability can be determined by sketching the derivative (2) as a function of the scaled temperature. The derivative is positive when $\theta = 0$ and is negative for sufficiently large θ. Thus, when there is a single steady-state solution it is stable. When there are three steady-state solutions the lowest and highest are stable whilst the middle is unstable. No student took this approach—perhaps indicating a lack of expertise in graphing functions.

- Sixteen students picked a representative value for ψ_b in each region and used a graphics package to plot the derivative as a function of the temperature. They then determined the stability of the steady-state solutions. Of these students, four considered five cases, adding the two special cases when either $\psi_b = \psi_{b,e}$ or $\psi_b = \psi_{b,i}$ in which there are two steady-state solutions.

- One student used equation (8) to eliminate ψ_b from the eigenvalue expression. Consequently, the eigenvalue can be plotted as a function of θ. The values of θ at which the stability changes are the zeroes of this function.

- One student provided a correct answer without any working and one student assumed that the stability is identical to that in the Semenov model.

3. *You should determine the location of any bifurcation points on your steady-state diagram.*

The steady-state diagram contains two limit-point bifurcations. The steady-state equation (7) has the form

$$\mathcal{G}\left(x, \mu\right) = 0\,. \tag{9}$$

Informally, a limit point occurs at the point $(x, \mu) = (x_0, \mu_0)$ when

$$\mathcal{G}\left(x_0, \mu_0\right) = \mathcal{G}_x\left(x_0, \mu_0\right) = 0\,. \tag{10}$$

(Two non-degeneracy conditions must be checked.) The limit points are

$$(\psi_{\text{ig}}, \theta_{\text{ig}}) = (0.3791, 1.034)\,, \qquad (\psi_{\text{ext}}, \theta_{\text{ext}}) = (0.1160, 5.094)\,.$$

Three students were unable to identify the location of the bifurcation points. Four students did not explain how they identified the location of the limit points. A variety of approaches were used.

- One student determined the singularity equations (10). From these they obtained an equation only containing the state variable θ. They found the roots by plotting this θ. They commented "when graphing this question, I switched to Symbolab to plot as it was easier to manipulate than python. I attempted to use DESMOS and while the graphing was fast, this resulted in lack of accuracy and large errors". It is encouraging to see a student use a variety of packages and then picking the most suitable. A fruitful classroom activity would be for students to discuss their different approaches.

- Viewing the bifurcation parameter ψ_b as a function of the state variable θ the values of θ at the limit-points are those where $\frac{d\psi_b}{d\theta}$ is zero.

  Six students plotted this derivative as a function of θ and found its roots. Two student used Wolfram Alpha to find the value of θ when the derivative is zero. (One student used Wolfram Alpha to determine the derivative whilst the other calculated it by hand.) One student did not find the corresponding values of the bifurcation parameters.

- Using an appropriate software package the coordinates can be found by 'clicking' on the bifurcation points (three students).

- One student found the approximate location of the limit points by "examining the graph". They then "scrolled through the data" to find "the exact points at which the bifurcations could be located".

- One student estimated the location of the lower limit-point from a hardcopy print of their steady-state diagram.

These approaches are listed in decreasing degree of sophistication. We feel that students would benefit more from a classroom discussion of different approaches to this question rather than merely receiving a solutions sheet.

4. *In industrial composting, 'moderate', but not negligible, temperatures rises are required. Interpret your diagram in terms that are useful for the operator of a compost pile.*

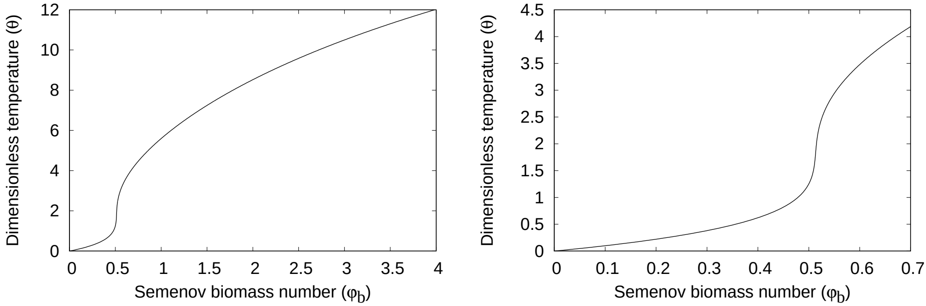
(When the oxidation reactions are included they create an additional branch of even higher temperature solutions).

The first part of the solution discusses the signifiance of the three branches, this discussion is presented following Question 2.

If the stockpile is sufficiently small, that is $\psi_b < \psi_{b,e}$, then there is only one steady-state solution. This low-temperature scenario represents 'composting failure'. If the stockpile is sufficiently large, that is $\psi_b > \psi_{b,i}$, then there is only one steady-state solution. This high-temperature scenario represents 'successful composting'. For values of the stockpiles between these extremes, that is when $\psi_{b,e} < \psi_b < \psi_{b,i}$, there are three steady-state solutions, with $0 < \theta_1 < \theta_2 < \theta_3$. The steady-state solution θ_2 represents a point of transition. If the initial temperature $\theta(0)$ is higher than θ_2, then the system evolves to the highest branch. Conversely, if $\theta(0) < \theta_2$, then the system evolves to the lowest branch. Thus, in the region in which there are three steady-state solutions, the temperature at which the stockpile is assembled determines whether the composting process is a success or a failure.

The analysis of the steady-state diagram, Figure 1, is very similar to that of the Semenov model, which is in the lecture notes. The difference is that in the composting problem the high temperature branch is a desirable outcome, whereas in the Semenov model it is undesirable. Despite this analogy this question was very poorly done.

- Only one student scored full marks whilst six students scored zero.
- Five students failed to discuss the role that the initial conditions play in determining the long-term behaviour of the stockpile, despite this being an important consideration for the Semenov model.
- Two students provided an otherwise correct description but did not link the outcomes to the underlying physical problem. This hints at an underlying problem, since relating mathematical results to the physical problem that generated the model is the raison d'être not just of this question but the subject.



(a) Steady-state diagram with stability. (b) Blow-up of figure (a).

Figure 2: Steady state diagram for the compost model without chemical heating (unique solution). Parameter values: $\alpha = 1.1$, $\beta = 0.1$, $\psi_o = 0$.

- Two students thought that the high-temperature branch was undesirable.
- Two students found other ways to lose marks.

5 Steady-state diagram: unique solution

Question 3 In this question we take $\beta = 0.1$ and $\alpha_d = 1.1$.

This question had the same format as Question 2. The steady-state diagram is shown in Figure 2: for these parameter values there is always a unique steady-state. We only consider the final part part of the question.

4. *In this application, 'moderate', but not negligible, temperatures rises are required. Interpret your diagram in terms that are useful for the operator of a compost pile.*

In answering this question, students were left to their own devices. One approach is to note that the temperature at the extinction limit-point in Figure 1(a) is $\theta_{ext} = 5.094$. Therefore, from a practical perspective, any steady-state temperature with $\theta \geqslant 5.094$ corresponds to composting. This gives a critical value for the biomass Semenov number that guarantees good composting as $\psi_b = 0.879$. A second approach is to look at the derivative $\frac{d\theta}{d\psi_b}$ along the solution branch and to argue that criticality corresponds to the value of ψ_b with the highest gradient.

6 Conclusions

Recent incidences of the self-ignition of compost stockpiles in NSW motivated the writing of an assignment about biological self-heating. We have provided an overview of how our students approached some of the questions. The questions required students to combined pen-and-pencil calculations with numerical calculations. Although students used a variety of approaches to reach their final answers only the lecturer was aware of this.

A good example of this was a question relating to parameter restriction within the context of the term modelling biological self-heating. Some students approached this through calculus whilst others approached it by taking an appropriate limit. It would be useful to have a mechanism by which students can see that there is not always one approach to solve a problem.

Our investigation would have been improved by interviewing students, to have them explain and discuss the problems they had with the case study. This would have provided their perspective on their choice of approaches. Unfortunately, as this is a third year subject the majority of students were unavailable for interview.

Smoke has sometimes been observed coming from stockpiles running the FOGO process. Such behaviour is not possible in the model considered by students; the assumption $\psi_o = 0$ removes heat generation by oxidative processes. The inclusion of this term provides insights into the safe construction of a stockpile. The resulting model is more appropriate for an honours subject.

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