Fractional diffusion model generalised by the distributed-order operator involving variable diffusion coefficients

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Abstract

The diffusion process plays a crucial role in various fields, such as fluid dynamics, microorganisms, heat conduction and food processing. Since molecular diffusion usually takes place in complex materials and disordered media, there still exist many challenges in describing the diffusion process in the real world. Fractional calculus is a powerful tool for modelling complex physical processes due to its non-local property. This research generalises a fractional diffusion model by using the distributed-order operator in time and the Riesz fractional derivative in space. Moreover, variable diffusion coefficients are introduced to better capture the diffusion complexity. The fractional diffusion model is discretised by the finite element method in space. The approximation

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of the distributed-order operator is implemented by Simpson's rule and the L2-1_σ formula. A numerical example is provided to verify the effectiveness of the proposed numerical methods. This generalised fractional diffusion model may offer more insights into characterising diffusion behaviours in complex and disordered media.

Contents

1 Introduction

Fractional calculus, developed by generalising the integrals and derivatives from the integer order to an arbitrary order, plays an important role in modelling complex phenomena in various dynamic systems [7]. Due to the non-local property of fractional derivatives, a variety of fractional partial differential equations (FPDEs) have been explored to simulate real-world physical phenomena such as the heat transfer, signal processing, viscoelastic dissipation and anomalous diffusion [12].

Diffusion behaviours are very dependent on the environment of the diffusing particles. There are limitations in capturing the diffusion processes in complex and disordered media using only constant-order fractional derivatives [9]. Barriers in disordered media lead to the hindrance or restriction of molecular movements, which results in varying non-local properties over multiple scales [6]. The distributed-order (DO) fractional operator, constructed by integrating the fractional order over a given range, has the potential to describe behaviours associated with multi-scale memory effects [3]. However, it is difficult to model the diffusion processes in complex environments by regarding the diffusion coefficient as a constant. The diffusing media exhibit various properties such as temperature, permeability and porosity [5]. Variable diffusion coefficients are more suitable for capturing diffusion complexities in disordered media. However, the research on DO fractional diffusion models with variable coefficients is still limited.

The limitations of current diffusion models motivates the study of the two-dimensional DO fractional diffusion equation (DO-FDE) combined with variable coefficients $0 < P(x, y) \leqslant P_{\max}$ on the irregular domain $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} \mathcal{D}_t^{\omega(\alpha)} u(x,y,t) = P(x,y)\mu^{2(\beta-1)}\nabla^{2\beta}u(x,y,t) + f(x,y,t)\,, \\ u(x,y,0) = u_0(x,y)\,, \quad (x,y)\in\Omega\,, \\ u(x,y,t)|_{\partial\Omega} = 0\,, \quad t\in[0,T]\,, \end{cases} \tag{1}$$

where $\partial\Omega$ is the boundary of the two-dimensional domain, and μ, with $0 < \mu < \mu_{\max}$, is introduced to preserve the balance of units. The DO operator is $\mathcal{D}_t^{\omega(\alpha)}$ and is defined by

$$\mathcal{D}_t^{\omega(\alpha)} u(x,y,t) = \int_0^1 \tau^{\alpha-1}\omega(\alpha)\, {}_0D_t^{\alpha} u(x,y,t)\, d\alpha\,, \tag{2}$$

with weight function $\omega(\alpha)$ which satisfies $0 < \int_0^1 \omega(\alpha)\, d\alpha < \infty$ and $\omega(\alpha) > 0$, and where τ, with $0 < \tau < \tau_{\max}$, preserves the balance of units. The derivative operator

$$\nabla^{2\beta} := \frac{\partial^{2\beta}}{\partial|x|^{2\beta}} + \frac{\partial^{2\beta}}{\partial|y|^{2\beta}}\,, \tag{3}$$

for $1/2 < \beta < 1$ is the Riesz fractional operator. For example,

$$\frac{\partial^{2\beta}u(x,y,t)}{\partial|x|^{2\beta}} = -\frac{1}{2\cos(\pi\beta)}\left[{}_{a(y)}^{RL}D_x^{2\beta}u(x,y,t) + {}_{x}^{RL}D_{b(y)}^{2\beta}u(x,y,t) \right]$$

is the Riesz fractional derivative with respect to x which is composed of the left and right Riemann–Liouville fractional derivatives [8], respectively,

$$
\begin{aligned}
{}^{RL}_{a(y)}D^{2\beta}_{x}u(x,y,t) &= \frac{1}{\Gamma(2-2\beta)}\frac{\partial^2}{\partial x^2}\int_{a(y)}^{x}\frac{u(\xi,y,t)}{(x-\xi)^{2\beta-1}}\,d\xi\,,\\
{}^{RL}_{x}D^{2\beta}_{b(y)}u(x,y,t) &= \frac{(-1)^2}{\Gamma(2-2\beta)}\frac{\partial^2}{\partial x^2}\int_{x}^{b(y)}\frac{u(\xi,y,t)}{(\xi-x)^{2\beta-1}}\,d\xi\,,
\end{aligned}
$$

where $a(y)$ and $b(y)$ are the lower and upper boundaries of an irregular domain. The Caputo fractional derivative [8] is defined as

$$
{}_0D^{\alpha}_{t}u(x,y,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\xi)^{-\alpha}\frac{\partial}{\partial\xi}u(x,y,\xi)\,d\xi\,, \tag{4}
$$

for $0<\alpha<1$.

It is challenging to derive analytical solutions of fractional partial differential equations (1) (FPDEs) with DO operators [3]. Efficient and robust numerical methods needs to be explored to solve them. In this research, the integral in the DO operator (2) is first transformed into a multi-term form using Simpson's rule [2]. Then we employ the L2-1_σ formula [1] as an approximation of the Caputo fractional derivative (4). The L2-1_σ formula has been widely used in solving FPDEs and has the advantage of high order accuracy [4]. The space Riesz fractional operator (3) is discretised by the finite element method (FEM) [14] which is efficient in dealing with fractional systems in irregularly shaped domains [13]. However, there is little research on solving DO fractional models using the L2-1_σ formula combined with the FEM.

In this study, a generalised fractional framework (DO-FDE) is developed by introducing the DO operator and variable diffusion coefficients to simulate anomalous diffusion in disordered media. In Section 2, the generalised DO-FDE is solved numerically by employing Simpson's rule, the L2-1_σ formula and the FEM. Then in Section 3, a numerical example is performed to verify the effectiveness of the proposed numerical schemes. Finally, some conclusions are drawn in Section 4.

2 Numerical techniques

This section develops numerical schemes for solving the DO-FDE efficiently.

2.1 Time discretisation

The time discretisation is implemented by utilising Simpson's rule and the L2-1_σ formula. Firstly, Simpson's rule [2] is employed to discretise the integral in the DO operator (2).

Let $h(\alpha) \in C^4[0,1]$, $\Delta\alpha = 1/2L$, and $\alpha_j = j\Delta\alpha$ for $j = 0, 1, \ldots, 2L$. There exists a $\xi \in (0,1)$, such that

$$\int_0^1 h(\alpha)\, d\alpha = J^S_{\Delta\alpha} h(\alpha) - \frac{(\Delta\alpha)^4}{180} h^{(4)}(\xi)\,, \tag{5}$$

where

$$J^S_{\Delta\alpha} h(\alpha) = \Delta\alpha \sum_{j=0}^{2L} \eta_j h(\alpha_j)\,,$$

with

$$\eta_j = \begin{cases} 1/3\,, & j = 0, 2L\,, \\ 2/3\,, & j = 2, 4, \ldots, 2L-4, 2L-2\,, \\ 4/3\,, & j = 1, 3, \ldots, 2L-3, 2L-1\,. \end{cases}$$

Then denote $\Delta t = T/N_0$ and $t_n = n\Delta t$ for $n = 0, 1, \ldots, N_0$. The L2-1_σ formula [1] is developed by combining the linear interpolation on $[t_k, t_{k+1}]$, with $0 \leqslant k < n$, and quadratic interpolation on $[t_{k-1}, t_{k+1}]$, with $1 \leqslant k \leqslant n-1$, and is used to approximate the time fractional operator. Denote

$$H(\sigma) = \sum_{j=0}^{2L} \frac{\Delta\alpha \eta_j \tau^{\alpha_j - 1} \omega(\alpha_j)}{\Gamma(3-\alpha_j)} \sigma^{1-\alpha_j} \left[\sigma - \left(1 - \tfrac{\alpha_j}{2}\right)\right] (\Delta t)^{2-\alpha_j}\,, \quad \sigma \geqslant 0\,.$$

Lemma 1. *[4] There exists a unique positive root*

$$\sigma^* \in \left[\min_{0\leqslant j\leqslant L} \left\{1 - \tfrac{\alpha_j}{2}\right\}, \max_{0\leqslant j\leqslant L} \left\{1 - \tfrac{\alpha_j}{2}\right\}\right]$$

that satisfies $H(\sigma) = 0$.

For simplicity, denote $\sigma = \sigma^*$ and suppose $u(t) \in C^3[0, T]$, then the Caputo fractional derivative (4) at $t = t_{n+\sigma}$ for $n < N_0$) is discretised to obtain the L2-1_σ formula [1]

$$\begin{aligned} {}_0D_t^\alpha u(t_{n+\sigma}) &= \frac{1}{\Gamma(1-\alpha)} \left[\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{u'(\eta)\, d\eta}{(t_{n+\sigma}-\eta)^\alpha} + \int_{t_n}^{t_{n+\sigma}} \frac{u'(\eta)\, d\eta}{(t_{n+\sigma}-\eta)^\alpha} \right] \\ &\approx \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n} s_{n-k}^{(\alpha,\sigma)} [u(t_{k+1}) - u(t_k)] := \mathbb{D}_t^\alpha u^{n+\sigma} , \end{aligned} \tag{6}$$

where $s_0^{(\alpha,\sigma)} = \sigma^{1-\alpha}$ when $n = 0$, and when $n \geqslant 1$ the coefficients satisfy

$$s_k^{(\alpha,\sigma)} = \begin{cases} a_0^{(\alpha,\sigma)} + b_1^{(\alpha,\sigma)}, & k = 0 , \\ a_k^{(\alpha,\sigma)} + b_{k+1}^{(\alpha,\sigma)} - b_k^{(\alpha,\sigma)}, & 1 \leqslant k \leqslant n-1 , \\ a_n^{(\alpha,\sigma)} - b_n^{(\alpha,\sigma)}, & k = n , \end{cases} \tag{7}$$

where

$$b_k^{(\alpha,\sigma)} = \tfrac{1}{2-\alpha}[(k+\sigma)^{2-\alpha} - (k+\sigma-1)^{2-\alpha}] - \tfrac{1}{2}[(k+\sigma)^{1-\alpha} + (k+\sigma-1)^{1-\alpha}] ,$$

and for $k \geqslant 1$,

$$a_k^{(\alpha,\sigma)} = (k+\sigma)^{1-\alpha} - (k+\sigma-1)^{1-\alpha} .$$

Based on the composite Simpson's rule (5), from (2) we obtain

$$\mathcal{D}_t^{\omega(\alpha)} u(x, y, t) = J_{\Delta\alpha}^S \left[\tau^{\alpha-1} \omega(\alpha)\, {}_0D_t^\alpha u(x, y, t) \right] + R_{\Delta\alpha}^S , \tag{8}$$

where

$$R_{\Delta\alpha}^S = \widetilde{C}(\Delta\alpha)^4 \max_{\xi \in (0,1)} \left| \partial_\xi^4 \left[\tau^{\xi-1} \omega(\xi)\, {}_0D_t^\xi u(x, y, t) \right] \right| ,$$

with $\widetilde{C}$ a constant.

Denote $u^{n+\sigma}$ as the numerical solution of the function $u(x,y,t)$ at $t = t_{n+\sigma}$, for $n = 0, 1, \ldots, N_0 - 1$. With the combination of equation (5) and equation (6), the DO fractional operator (2) is approximated as

$$\begin{aligned} J^S_{\Delta\alpha}\left[\tau^{\alpha-1}\omega(\alpha)\, {}_0D^\alpha_t u(t_{n+\sigma})\right] &\approx J^S_{\Delta\alpha}\left(\tau^{\alpha-1}\omega(\alpha)\mathbb{D}^\alpha_t u^{n+\sigma}\right) \\ &= \sum_{k=0}^{n} \hat{s}^{(n+1)}_k (u^{k+1} - u^k) := \mathcal{D}^\alpha_t u^{n+\sigma}, \end{aligned} \tag{9}$$

where

$$\hat{s}^{(n+1)}_k = \sum_{j=0}^{2L} W^{(\Delta t,\Delta\alpha)}_{\alpha_j} s^{(\alpha_j,\sigma)}_{n-k}, \quad k = 0, 1, \ldots, n,$$

with the coefficient

$$W^{(\Delta t,\Delta\alpha)}_{\alpha_j} = \Delta\alpha\eta_j\tau^{\alpha_j-1}\omega(\alpha_j)\frac{(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}.$$

In equation (9), $\mathcal{D}^\alpha_t u^{n+\sigma}$ is the approximation of the multi-term formulation $J^S_{\Delta\alpha}\left[\tau^{\alpha-1}\omega(\alpha)\, {}_0D^\alpha_t u(t_{n+\sigma})\right]$ obtained from applying the L2-1_σ formula (6).

Lemma 2. *[4] Suppose $u(t) \in C^3(0,T)$, then we have*

$$\begin{aligned} &\left|J^S_{\Delta\alpha}\left[\tau^{\alpha-1}\omega(\alpha)\, {}_0D^\alpha_t u(t_{n+\sigma})\right] - \mathcal{D}^\alpha_t u^{n+\sigma}\right| \\ &\leqslant \max_{0\leqslant t\leqslant T} |\partial^3_t u(t)| \sum_{j=0}^{2L} \frac{\lambda_j}{\Gamma(2-\alpha_j)}\left(\frac{1-\alpha_j}{12} + \frac{\sigma}{6}\right)\sigma^{-\alpha_j}(\Delta t)^{3-\alpha_j}, \end{aligned}$$

where $\lambda_j = \Delta\alpha\eta_j\tau^{\alpha_j-1}\omega(\alpha_j)$.

2.2 Spatial discretisation

The variable diffusion coefficient in equation (1) give rise to some challenges in implementing the discretisation of the diffusion term. Following Xu et al. [13], we divide both sides of equation (1) by the variable coefficient $P(x,y)$

to tackle this issue. This technique provides the possibility for employing the important Lemma 3, which decreases the order of the space fractional derivative. Hence, the coefficient $P(x,y)$ is incorporated in the time fractional derivative term and the source term in the fully discrete scheme. Then, both sides of the fully discrete form are multiplied by the coefficient $P(x,y)$ in the numerical computation process (returning to the original form of the model). Thus, in the matrix system, the coefficient $P(x,y)$ is involved with the diffusion term and associated with the assembly of the stiffness matrix.

Firstly, we divide both sides of equation (1) by $P(x,y)$ to obtain

$$\mathcal{D}_t^{\omega(\alpha)}u(x,y,t)/P(x,y) = \mu^{2(\beta-1)}\nabla^{2\beta}u(x,y,t) + g(x,y,t)\,, \tag{10}$$

where $g(x,y,t) = f(x,y,t)/P(x,y)$. Then define the finite element space

$$V_h := \{v_h \mid v_h \in C(\Omega) \cap H_0^\beta(\Omega)\,,\ v_h|_E \text{ is linear } \forall E \in \mathcal{T}_h \text{ and } v_h|_{\partial\Omega} = 0\}\,,$$

where $\{\mathcal{T}_h\}$ is a mesh partition constructed from a number of triangular elements with h being the maximum length of these triangles.

The irregular domain is defined as

$$\begin{aligned}
\Omega &= \{(x,y) \mid a(y) < x < b(y)\,, c_1 < y < d_1\} \quad \text{or} \\
\Omega &= \{(x,y) \mid c(x) < y < d(x)\,, a_1 < x < b_1\}\,,
\end{aligned}$$

where $c_1 = \min_{(x,y)\in\Omega} c(x)$, $d_1 = \min_{(x,y)\in\Omega} d(x)$, $a_1 = \min_{(x,y)\in\Omega} a(y)$ and $b_1 = \min_{(x,y)\in\Omega} b(y)$. Then we define the inner product

$$(u\,,v) := \int_{c_1}^{d_1}\int_{a(y)}^{b(y)} u(x,y)v(x,y)\,dxdy = \int_{a_1}^{b_1}\int_{c(x)}^{d(x)} u(x,y)v(x,y)\,dydx\,.$$

Lemma 3. *[11] If* $\mu \in (1/2, 1)$, $u, v \in J^{2\mu}_{L,0}(\Omega) \cap J^{2\mu}_{R,0}(\Omega)$, *then*

$$\begin{aligned}
\left({}^{RL}_{a(y)}D^{2\mu}_{x} u, v\right) &= \left({}^{RL}_{a(y)}D^{\mu}_{x} u, {}^{RL}_{x}D^{\mu}_{b(y)} v\right),\\
\left({}^{RL}_{x}D^{2\mu}_{b(y)} u, v\right) &= \left({}^{RL}_{x}D^{\mu}_{b(y)} u, {}^{RL}_{a(y)}D^{\mu}_{x} v\right),\\
\left({}^{RL}_{c(x)}D^{2\mu}_{y} u, v\right) &= \left({}^{RL}_{c(x)}D^{\mu}_{y} u, {}^{RL}_{y}D^{\mu}_{d(x)} v\right),\\
\left({}^{RL}_{y}D^{2\mu}_{d(x)} u, v\right) &= \left({}^{RL}_{y}D^{\mu}_{d(x)} u, {}^{RL}_{c(x)}D^{\mu}_{y} v\right).
\end{aligned}$$

Let $u_h^{n+\sigma} \in V_h$ be the numerical solution of $u(x, y, t)$ at $t = t_{n+\sigma}$. From the time discretisation (9), for all $v_h \in V_h$ the fully discrete variational formulation is

$$\begin{cases} \left(\frac{1}{P}\mathcal{D}^{\alpha}_{t} u_h^{n+\sigma}, v_h\right) + \mathcal{B}\left(u_h^{n+\sigma}, v_h\right) = \left(g^{n+\sigma}, v_h\right), \\ u_h(0) = \varphi_{0h}, \end{cases} \tag{11}$$

where $g^{n+\sigma} = g(x, y, t^{n+\sigma})$ and $\varphi_{0h} \in V_h$ is an approximation of $u_0(x, y)$. Using Lemma 3, the bilinear form is

$$\begin{aligned}
\mathcal{B}(u_h^{n+\sigma}, v_h) = c_\beta \mu^{2(\beta-1)} \Big[&\left({}^{RL}_{a(y)}D^{\beta}_{x} u_h^{n+\sigma}, {}^{RL}_{x}D^{\beta}_{b(y)} v_h\right) \\
&+ \left({}^{RL}_{x}D^{\beta}_{b(y)} u_h^{n+\sigma}, {}^{RL}_{a(y)}D^{\beta}_{x} v_h\right) + \left({}^{RL}_{c(x)}D^{\beta}_{y} u_h^{n+\sigma}, {}^{RL}_{y}D^{\beta}_{d(x)} v_h\right) \\
&+ \left({}^{RL}_{y}D^{\beta}_{d(x)} u_h^{n+\sigma}, {}^{RL}_{c(x)}D^{\beta}_{y} v_h\right) \Big],
\end{aligned}$$

where $c_\beta = 1/2\cos(\pi\beta)$. According to the interpolated shape function of the triangular element [10], define a basis function $\{\phi_i(x, y)\}_{i=1}^{N_p}$ so that

$$u_h(x, y, t_{n+\sigma}) \approx \sum_{i=1}^{N_p} u_i^{n+\sigma} \phi_i(x, y), \tag{12}$$

where N_p is the total number of the grid nodes.

To assemble the matrix system and simplify the calculation of the variable coefficients, both sides of the fully discrete form are multiplied by $P(x, y)$.

Taking $v_h = \phi_r(x,y)$, with $r = 1,2,\dots,N$, and combining equation (12) and the fully discrete scheme (11), we have

$$\sum_{i=1}^{N_p} \mathcal{D}_t^{\alpha} u_i^{n+\sigma}(\phi_i,\phi_r) + \sum_{i=1}^{N_p} u_i^{n+\sigma}\mathcal{B}(P\phi_i,\phi_r) = (f^{n+\sigma},\phi_r)\,, \quad r = 1,2,\dots,N\,,$$

where

$$\begin{aligned}\mathcal{B}(P\phi_i,\phi_r) = c_\beta \mu^{2(\beta-1)} \Big[& \Big(P\, {}^{RL}_{a(y)}D_x^\beta \phi_i\,,\, {}^{RL}_{x}D_{b(y)}^\beta \phi_r\Big) + \Big(P\, {}^{RL}_{x}D_{b(y)}^\beta \phi_i\,,\, {}^{RL}_{a(y)}D_x^\beta \phi_r\Big)\\ & + \Big(P\, {}^{RL}_{c(x)}D_y^\beta \phi_i\,,\, {}^{RL}_{y}D_{d(x)}^\beta \phi_r\Big) + \Big(P\, {}^{RL}_{y}D_{d(x)}^\beta \phi_i\,,\, {}^{RL}_{c(x)}D_y^\beta \phi_r\Big)\Big]\,,\end{aligned}$$

One of these inner products is approximated by [13]

$$\begin{aligned}\Big(P\, {}^{RL}_{a(y)}D_x^\beta \phi_i\,,\, {}^{RL}_{x}D_{b(y)}^\beta \phi_r\Big) &= \sum_{p=1}^{N_e} \int_{e_p} P\, {}^{RL}_{a(y)}D_x^\beta \phi_i\, {}^{RL}_{x}D_{b(y)}^\beta \phi_r \, dx dy\\ &\approx \sum_{p=1}^{N_e} \sum_{q=1}^{N_G^p} w_q \Big({}^{RL}_{a(y)}D_x^\beta \phi_i|_{(x_q,y_q)}\Big)\Big({}^{RL}_{x}D_{b(y)}^\beta \phi_r|_{(x_q,y_q)}\Big) P(x_q,y_q)\,,\end{aligned}$$

where N_G^p refers to the total number of Gauss points in a triangular element e_p, and w_q are their corresponding weights.

Let $u_i^{n+\sigma} = \sigma u_i^{n+1} - (1-\sigma)u_i^n$ in the bilinear form, then the fully discrete scheme is transformed into the matrix system

$$\sum_{k=0}^{n} \hat{s}_k^{(n+1)} \mathbf{A}(\mathbf{U}^{k+1} - \mathbf{U}^k) + \mathbf{K}[\sigma\mathbf{U}^{n+1} + (1-\sigma)\mathbf{U}^n] = \mathbf{F}^{n+\sigma}, \qquad (13)$$

where $\mathbf{A}_{ri} = (l_i\,,l_r)$ is the mass matrix, $\mathbf{K}_{ri} = \mathcal{B}(Pl_i\,,l_r)$ is the stiffness matrix, $\mathbf{U} = [u_1^n, u_2^n, \dots, u_{N_p}^n]^T$ and $\mathbf{F}^{n+\sigma} = [(f^{n+\sigma},l_1),(f^{n+\sigma},l_2),\dots,(f^{n+\sigma},l_N)]^T$.

3 Numerical example and discussions

This section gives a numerical example to verify the effectiveness and reliability of the developed numerical methods.

We investigate the two-dimensional DO fractional dynamic system (1) on the circular domain $\Omega = \{(x, y) \mid x^2 + y^2 < 1\}$ with initial condition $u(x, y, 0) = (x^2 + y^2 - 1)^2/10$. Consider $P(x, y) = \exp[0.001(x - y)]$, $T = 1$, $\tau = \mu = 1$ and $\omega(\alpha) = \Gamma(5 - \alpha)/\Gamma(5)$. Then the source term is

$$\begin{aligned} f(x, y, t) = {} & \frac{0.1}{\ln t}(t^4 - t^3)(x^2 + y^2 - 1)^2 + 0.1(t^4 + 1)P_1(x, y)[q_1(x, a_0, b_0) \\ & + 2(y^2 - 1)q_2(x, a_0, b_0) + (y^2 - 1)^2 q_3(x, a_0, b_0) \\ & + q_1(y, c_0, d_0) + 2(x^2 - 1)q_2(y, c_0, d_0) + (x^2 - 1)^2 q_3(y, c_0, d_0)] \,, \end{aligned}$$

where

$$\begin{aligned} & a_0 = -\sqrt{1 - y^2}, \quad b_0 = \sqrt{1 - y^2}, \quad c_0 = -\sqrt{1 - x^2}, \quad d_0 = \sqrt{1 - x^2}, \\ & q_1(r, a, b) = {}_{a}^{RL}D_{r}^{2\beta}(r^4) + {}_{r}^{RL}D_{b}^{2\beta}(r^4), \quad q_2(r, a, b) = {}_{a}^{RL}D_{r}^{2\beta}(r^2) + {}_{r}^{RL}D_{b}^{2\beta}(r^2), \\ & q_3(r, a, b) = {}_{a}^{RL}D_{r}^{2\beta}(1) + {}_{r}^{RL}D_{b}^{2\beta}(1), \quad P_1(x, y) = P(x, y)\mu^{2(\beta-1)}/c_\beta \,. \end{aligned}$$

The exact solution of this example is

$$u(x, y, t) = \tfrac{1}{10}(t^4 + 1)(x^2 + y^2 - 1)^2 \,.$$

Table 1 shows the L_2 error and convergence order of various space meshes with $\beta = 0.80$, $L = 100$ and $\Delta t = 1/1000$ at $t = 1$. Results demonstrate that second-order accuracy of the L_2 error is obtained. In Table 2 the L_2 error of the time discretisation achieves second-order convergence with $\beta = 0.90$, $L = 100$ and $\Delta t \approx h$ at $t = 1$. The results suggest that the developed numerical methods are reliable for efficiently solving the fractional dynamic model.

Table 1: The L_2 error and the corresponding convergence order of the space discretisation with $\beta = 0.85$, $L = 100$ and $\Delta t = 1/1000$ at $t = 1$.

h	L_2 error	Order
4.4846×10^{-1}	1.01×10^{-2}	—
2.8917×10^{-1}	3.31×10^{-3}	2.54
1.6444×10^{-1}	1.02×10^{-3}	2.08
8.6550×10^{-2}	2.64×10^{-4}	2.11
7.1216×10^{-2}	1.74×10^{-4}	2.13

Table 2: The L_2 error and convergence order of the time discretisation with $\Delta t \approx h$, $\beta = 0.90$ and $L = 100$ at $t = 1$.

Δt	h	L_2 error	Order
$1/2$	4.4846×10^{-1}	2.30×10^{-2}	—
$1/6$	1.6444×10^{-1}	2.90×10^{-3}	1.89
$1/11$	8.6550×10^{-2}	8.62×10^{-4}	2.00
$1/14$	7.1216×10^{-2}	5.46×10^{-4}	1.89
$1/21$	4.5873×10^{-2}	2.44×10^{-4}	1.98

4 Conclusions

In summary, this research establishes a new fractional diffusion model by introducing the DO fractional operator and variable diffusion coefficients to further describe diffusion complexity. Simpson's rule is utilised to transform the integral in the DO operator into a multi-term form. We approximate the Caputo fractional derivative with the L2-1_σ formula. Then applying the FEM for spatial discretisation, the fully discrete formulation is constructed. Finally, a numerical example is provided and the validity of numerical methods is discussed. This generalised DO-FDE for a problem with variable diffusion coefficients may provide more possibilities for capturing anomalous diffusion with multi-scale properties in disordered media.

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