Asymmetrical suction and injection in laminar channels with porous walls: a fixed point approach

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Abstract

The problem of laminar flow in a rectangular channel with a pair of porous walls is considered. The porous walls allow fluid to be injected into or sucked out of the channel at constant velocities normal to the walls; the velocities at each wall are not necessarily of equal magnitude nor symmetrical in direction. In this article, a unique solution to this problem is shown to exist for sufficiently low Reynolds numbers through the application of Banach's fixed point theorem. This serves to further the discussion about the uniqueness of solutions for this problem, whilst also demonstrating the suitability of a fixed point approach to this family of fluid dynamics problems.

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Contents

1 Introduction

The flow of fluid through rectangular channels with porous walls has attracted much interest since its original study by Abraham Berman in 1953 [1, 2, 3, 5, 6]. A large portion of this effort has gone into extending Berman's analysis and results for larger magnitudes of Reynolds number at the porous walls. The original investigation involved the idealistic assumption of symmetric wall fluid velocities in which the fluid extracted through the walls of the channel had equal speeds [2]. Terrill and Shrestha [5] extended the investigation to find solutions in the case where flow velocities at the walls were not necessarily equal, in either magnitude or direction. This extension on Berman's original assumptions is desirable to study as it is arguably closer to modelling real-world phenomena given that symmetric flows are somewhat idealised.

In this article, a unique solution is found to exist for low Reynolds numbers. The approach taken herein follows that undertaken by Almuthaybiri and Tisdell [1] whereby Banach's Fixed Point Theorem (Banach's FPT herein) is used to approximate the solution to the original Berman problem. This approach is, as far as the author is aware, novel for solving this family of fluid problems. Thus, it is hoped that this article encourages its further adoption for study of other types of flow. The study of such flows is also well-motivated in practice; with this type of flow being found in applications such as transpiration cooling, gaseous diffusion, boundary layer separation control and systems of fluid filtration [3, 6].

2 Mathematical formulation

Consider the steady laminar flow of a viscous, incompressible fluid through a rectangular cross-section channel, as in Figure 1. This channel has porous top and bottom walls such that fluid may be injected in or extracted out of the channel perpendicular to the porous walls. We set up the coordinate system such that the origin is midway between the porous walls. The x axis lies parallel to the porous walls, whilst the y axis is perpendicular to the walls. A constant pressure gradient drives a flow along the channel in the x direction. A dimensionless coordinate $\eta = y/h$ is introduced, where h is half the height of the channel. The z axis lies perpendicular to the x and y axes, out of the page.

We assume that the width (z direction) of the channel is much larger than the height (y direction). This assumption yields fluid flow independent of the z direction, thus reducing it to a two-dimensional problem. As is customary, $u(x, y)$ and $v(x, y)$ represent the velocity components in the x and y directions, respectively. We consider the case whereby fluid is injected/extracted into the channel with velocity v_1 through the bottom wall ($v_1 > 0$ for injection, $v_1 < 0$ for suction), and with velocity v_2 through the top wall ($v_2 > 0$ for suction, $v_2 < 0$ for injection). All four combinations of signs of these velocities are possible. However, as found by Terrill and Shrestha [5], the mixed case, where fluid is injected at one wall and extracted at the other (as in Figure 1), leads to different results depending on the relative magnitudes of the wall velocities [5]. For the current investigation we derive results only for the case where $|v_2| \geqslant |v_1|$ with Reynolds number defined as $\mathcal{R} = v_2 h/\nu$, where ν is the kinematic viscosity of the fluid.

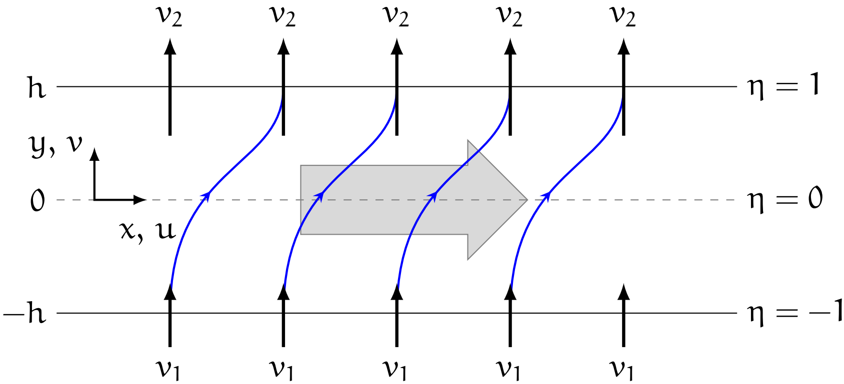

Figure 1: Coordinate system with labelled wall velocities; top wall: suction, bottom wall: injection. The grey arrow shows the main horizontal flow of fluid through a channel driven by a constant pressure gradient. Illustrative streamlines are given in blue and are not to scale.

The equations of momentum for this problem (in terms of η) are the two-dimensional incompressible Navier–Stokes equations

$$\begin{aligned} u\frac{\partial u}{\partial x} + \frac{v}{h}\frac{\partial u}{\partial \eta} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{h^2}\frac{\partial^2 v}{\partial \eta^2}\right), \\ u\frac{\partial v}{\partial x} + \frac{v}{h}\frac{\partial v}{\partial \eta} &= -\frac{1}{h\rho}\frac{\partial p}{\partial \eta} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{1}{h^2}\frac{\partial^2 v}{\partial \eta^2}\right). \end{aligned}$$

Additionally, the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{1}{h}\frac{\partial v}{\partial \eta} = 0\,.$$

Finally, the boundary conditions for the problem are

$$\begin{aligned} u(x,h) &= 0\,, \quad u(x,-h) = 0\,, && \text{(No-slip condition)} \\ v(x,h) &= v_2\,, \quad v(x,-h) = v_1\,. && \text{(Suction/Injection at walls)} \end{aligned}$$

Using a stream function

$$\Psi(x,\eta) = \left[\frac{hU(0)}{\alpha_2} - \nu_2 x\right] f(\eta)\,,$$

where $\alpha_2 = 1 - \nu_2/\nu_1$ and $U(0)$ is an arbitrary average horizontal velocity at $x = 0$, Terrill and Shrestha [5], following Berman's original work, reduced the Navier–Stokes equations for this problem to

$$f'''(\eta) + \mathcal{R}[f'^2(\eta) - f(\eta)f''] = K\,,$$

where K is a constant. And so, by differentiation with respect to η, we have

$$f^{(iv)}(\eta) + \mathcal{R}[f'(\eta)f''(\eta) - f(\eta)f'''(\eta)] = 0\,. \tag{1}$$

The boundary conditions in terms of this new unknown function f are

$$f'(1) = 0\,, \quad f'(-1) = 0\,, \quad f(1) = 1\,, \quad f(-1) = 1 - \alpha_2\,. \tag{2}$$

Equation (1) with boundary conditions (2) define the boundary value problem (BVP) to be solved using Banach's FPT [1, Theorem 4.3].

3 Green's function

To solve the BVP using Banach's FPT, we first reformulate it as an integral equation with a Green's function kernel.

Lemma 1. *The BVP, defined by (1) and (2), is equivalent to the integral equation*

$$f(\eta) = \int_{-1}^{1} G(\eta,s)\mathcal{R}[f'(s)f''(s) - f(s)f'''(s)]\,ds + \phi(\eta)\,, \quad \eta \in [-1,1]\,,$$

where the Green's function is

$$G(\eta,s) = \frac{1}{24}\begin{cases}(s+1)^2(\eta-1)^2[\eta(s-2)+2s-1] & \text{for } -1 \leqslant s \leqslant \eta \leqslant 1\,,\\ (\eta+1)^2(s-1)^2[s(\eta-2)+2\eta-1] & \text{for } -1 \leqslant \eta \leqslant s \leqslant 1\,.\end{cases}$$

And $\phi(\eta) = [\alpha_2(3\eta - \eta^3) + 2(2-\alpha_2)]/4\,,$ *with* $\alpha_2 := 1 - \nu_1/\nu_2\,.$

Proof: A sketch of the proof is given as the method to convert a BVP into an equivalent integral equation is readily available [e.g., 4, 1]. It is sufficient to find f of the form

$$f(\eta) = \phi(\eta) + \phi_1(\eta)\,, \tag{3}$$

where $\phi(\eta)$ is the solution to the homogeneous ODE $\phi^{(iv)} = 0$ with non-homogeneous boundary conditions. And $\phi_1(\eta)$ is found using Cauchy's formula for repeated integration on the non-homogeneous ODE:

$$\phi_1^{(iv)}(\eta) = -\mathcal{R}[\phi_1'(\eta)\phi_1''(\eta) - \phi_1(\eta)\phi_1'''(\eta)]\,,$$

with boundary conditions determining the constants of integration. ♠

The next lemma gives bounds for the Green's function and its partial derivatives which are used in the application of Banach's FPT in Section 4.

Lemma 2. *The Green's function* G *and its partial derivatives for all* $\eta \in [-1, 1]$ *satisfy*

$$\begin{aligned}
\int_{-1}^{1} |G(\eta, s)|\, ds &\leqslant \frac{1}{24} =: \beta_0\,,\\
\int_{-1}^{1} \left|\frac{\partial G(\eta, s)}{\partial \eta}\right| ds &\leqslant \frac{2}{27}(26\sqrt{13} - 92) =: \beta_1\,,\\
\int_{-1}^{1} \left|\frac{\partial^2 G(\eta, s)}{\partial \eta^2}\right| ds &\leqslant \frac{1}{2} =: \beta_2\,,\\
\int_{-1}^{1} \left|\frac{\partial^3 G(\eta, s)}{\partial \eta^3}\right| ds &\leqslant 1 =: \beta_3\,.
\end{aligned}$$

Proof: A sketch proof is given as the details are cumbersome and uninformative.

The first bound β_0 uses the fact that $G \leqslant 0$ on $[-1, 1] \times [-1, 1]$. The remaining bounds are found by maximising the respective partial derivatives by means of the max-min theorem, using this maximum value in the integrand, and finally simplifying the expression. ♠

4 Fixed point theorem approach

The perturbation series solution to the BVP (1) and (2) was found previously by Terrill and Shrestha [5]. But, a discussion on the uniqueness of such a solution is yet to be undertaken. The following use of Banach's FPT enables a better understanding of the uniqueness of the solution of this laminar flow problem.

To apply Banach's FPT to the BVP (1) and (2) a choice of non-empty set X, operator T and metric d needs to be made. They are to be chosen such that the operator T is a self-map (i.e., $T : X \to X$) and also a contraction (i.e., $d(Tf, Tg) \leqslant k\, d(f, g)$ with $k < 1$). These two properties ultimately lead to a pair of inequalities relating the Reynolds number $\mathcal{R}$ and R. Constant R is a half the width (as measured using the metric d) of the solution strip around the homogeneous solution $\phi(\eta)$, and is unrelated to the half width of the channel h.

In the following section a choice of metric and operator is made so that the necessary conditions for the application of Banach's FPT can be constructed.

4.1 Set, metric and operator choice

Let $C^3([-1, 1])$ be the set of thrice differentiable continuous functions in the closed interval $[-1, 1]$. For all $f, g \in C^3([-1, 1])$ define the metric d, such that for $i = \{0, 1, 2, 3\}$,

$$d(f, g) = \max_{i=0,1,2,3} \left\{ W_i \max_{\eta\in[-1,1]} \left| f^{(i)}(\eta) - g^{(i)}(\eta) \right| \right\},$$

where $f^{(i)}(\eta)$ denotes the ith derivative of f with respect to η and constants $W_i = \beta_0/\beta_i$. Recall that β_i are the upper bounds found in Lemma 2.

For completeness, these constants are explicitly given by $W_0 = 1$, $W_1 = 9/[16(13\sqrt{13} - 46)]$, $W_2 = 1/6$ and $W_3 = 1/24$. It can be shown that the pair $(C^3([-1, 1]), d)$ form a complete metric space [1].

The domain of the operator T is chosen to be

$$X = \mathcal{B}_R = \{f \in C^3([-1, 1]) : d(f, \phi) \leqslant R\}, \tag{4}$$

where ϕ is given in Lemma 1. Now, $\mathcal{B}_{\mathcal{R}}$ is a closed subspace of $C^3([-1, 1])$ and thus the chosen metric is still complete in this space.

Define $T : \mathcal{B}_R \to C^3([-1, 1])$ such that

$$(Tf)(\eta) = \int_{-1}^{1} G(\eta, s)\mathcal{R}[f'(s)f''(s) - f(s)f'''(s)]\, ds + \phi(\eta), \tag{5}$$

where $\phi(\eta)$ and $G(\eta, s)$ were found in Lemma (1). Finding the fixed point of operator T equates to finding an expression for f, the solution to the BVP (1) and (2). Thus the choice of operator is reasonable. Finding this fixed point requires some intermediary lemmas, which aid in showing that the operator T, as chosen here, is a self-map and a contraction on $\mathcal{B}_R$. The next section discusses these lemmas.

4.2 Results and lemmas for applying Banach's FPT

Recall that $\phi(\eta) = [\alpha_2(3\eta - \eta^2) + 2(2 - \alpha_2)]/4$ from Lemma 1. We state the bounds on the absolute value of ϕ and its derivatives up to the third order. The bounds are found by use of absolute value properties and the triangle inequality. We have

$$|\phi(\eta)| \leqslant \tfrac{3}{2}, \quad |\phi'(\eta)| \leqslant \tfrac{3}{2}, \quad |\phi''(\eta)| \leqslant 3, \quad |\phi'''(\eta)| \leqslant 3. \tag{6}$$

In the following lemma we use these bounds and the set

$$\begin{aligned} B = \big\{(\eta, u, v, w, z) \in \mathbb{R}^5 : \eta \in [-1, 1], |u - \phi(\eta)| \leqslant R, \\ |v - \phi'(\eta)| \leqslant 32(26\sqrt{13} - 46)R/9, \\ |w - \phi''(\eta)| \leqslant 12R, |z - \phi'''(\eta)| \leqslant 24R\big\}. \end{aligned} \tag{7}$$

Lemma 3. *Define a function* $H : [-1,1]\times\mathbb{R}^4 \to \mathbb{R}$ *such that* $H(\eta,u,v,w,z) = \mathcal{R}(vw-uz)$. *Then* H *is bounded above on* B *by*

$$N := |\mathcal{R}|\left[\tfrac{8}{3}(208\sqrt{13}-727)R^2+\tfrac{1}{3}(416\sqrt{13}-1265)R+\tfrac{21}{2}\right].$$

Proof: For (η,u,v,w,z) on B consider

$$\begin{aligned}
|H(\eta,u,v,w,z)| &= |\mathcal{R}(vw-uz)| \\
&\leqslant |\mathcal{R}|(|v||w|+|u||z|) \\
&\leqslant |\mathcal{R}|[(|v-\phi'(\eta)|+|\phi'(\eta)|)(|w-\phi''(\eta)|+|\phi''(\eta)|) \\
&\quad +(|u-\phi(\eta)|+|\phi(\eta)|)(|z-\phi'''(\eta)|+|\phi'''(\eta)|)] \\
&= |\mathcal{R}|\left[\tfrac{8}{3}(208\sqrt{13}-727)R^2+\tfrac{1}{3}(416\sqrt{13}-1265)R+\tfrac{21}{2}\right] \\
&=: N\,.
\end{aligned}$$

Above we have repeatedly used the triangle inequality, and also utilised the bounds (6) on $\phi^{(i)}(\eta)$ along with the definition (7) of the set B. ♠

Another useful lemma which defines the local Lipschitz constants for the function H follows.

Lemma 4. *As in Lemma 3, define* $H(\eta,u,v,w,z) = \mathcal{R}(vw-uz)$ *for* $(\eta,u,v,w,z)\in B$. *For all* $R>0$ *the function* H *is Lipschitz on* B *so that there exist constants* L_i *such that, for all* $(\eta,u_0,u_1,u_2,u_3),(\eta,v_0,v_1,v_2,v_3)\in B$,

$$|H(\eta,u_0,u_1,u_2,u_3)-H(\eta,v_0,v_1,v_2,v_3)| \leqslant L_i\sum_{i=0}^{3}|u_i-v_i|\,.$$

Proof: The proof of this lemma closely follows that of a lemma by Almuthaybiri and Tisdell [1]. We include here only the bounds of the partial derivatives on B as these are the Lipschitz constants to be used later. We have, for all $(\eta, u, v, w, z) \in B$,

$$\begin{aligned}
\left|\frac{\partial H}{\partial u}\right| &\leqslant |\mathcal{R}|\,(24R+3) =: L_1\,,\\
\left|\frac{\partial H}{\partial v}\right| &\leqslant |\mathcal{R}|\,(12R+3) =: L_2\,,\\
\left|\frac{\partial H}{\partial w}\right| &\leqslant |\mathcal{R}|\left(\frac{32(13\sqrt{13}-46)}{9}R+\frac{3}{2}\right) =: L_3\,,\\
\left|\frac{\partial H}{\partial z}\right| &\leqslant |\mathcal{R}|\left(R+\frac{3}{2}\right) =: L_4\,.
\end{aligned}$$

♠

4.3 Application of FPT

The definitions and lemmas introduced in the previous section enable the application of Banach's FPT to our problem. The self-map and contraction mapping conditions in Banach's FPT lead to a restriction on the values of the Reynolds number $\mathcal{R}$ and R. This is summarised in the following result, whose proof comes from applying Banach's FPT to the problem.

Theorem 5. *If $R > 0$ and $\mathcal{R}$ simultaneously satisfy the two inequalities*

$$|\mathcal{R}|\left[\tfrac{1}{24}\left(\tfrac{8}{3}R^2(208\sqrt{13}-727)+\tfrac{1}{3}R(416\sqrt{13}-1301)+9\right)\right] \leqslant R\,, \tag{8}$$

$$|\mathcal{R}|\left[\tfrac{1}{72}\left(16R\left(208\sqrt{13}-727\right)+416\sqrt{13}-1301\right)\right] < 1\,, \tag{9}$$

then the operator T has a unique fixed point in $\mathcal{B}_R$. Moreover, the BVP (1) and (2) admits a unique solution f with $(\eta, f(\eta), f'(\eta), f''(\eta), f'''(\eta)) \in B$ for all $\eta \in [-1, 1]$.

Proof: We continue using the notation which has been established previously in this article. This includes: β_i for the relevant bounds concerning the Green's function, W_i the constants used in the definition of the set B, and L_i for the Lipschitz constants (recalling that H is Lipschitz on the set B). We use the operator (5) with domain $\mathcal{B}_R$.

Self-map condition We show that the operator T is a self-map, that is $T : \mathcal{B}_R \to \mathcal{B}_R$. Consider for $f \in \mathcal{B}_R$, $\eta \in [-1, 1]$:

$$\begin{aligned}
|(Tf)(\eta) - \phi(\eta)| &\leqslant \int_{-1}^{1} |G(\eta, s)||\mathcal{R}[f'(s)f''(s) - f(s)f'''(s)]| \, ds \\
&\leqslant N \int_{-1}^{1} |G(\eta, s)| \, ds && \text{(by Lemma 3)} \\
&\leqslant N\beta_0 = N/24 .
\end{aligned}$$

In the above, we utilised the Green's function bounds from Lemma 3, which assumes that $(\eta, f(\eta), f'(\eta), f''(\eta), f'''(\eta)) \in B$. Finally, since $W_0 = 1$, we have $W_0|(Tf)(\eta) - \phi(\eta)| \leqslant N\beta_0$.

Omitting repetitive details, we have in general

$$W_i|(Tf)^{(i)}(\eta) - \phi^{(i)}(\eta)| \leqslant \frac{\beta_0}{\beta_i} N\beta_i = N\beta_0 .$$

The upper bounds above hold for all $\eta \in [-1, 1]$, and thus it is also true that

$$W_i \max_{\eta \in [-1,1]} |(Tf)^{(i)}(\eta) - \phi^{(i)}(\eta)| \leqslant N\beta_0 .$$

This final observation then means that

$$\begin{aligned}
d(Tf, \phi) &= \max_{i=0,1,2,3} \left\{ W_i \max_{\eta \in [-1,1]} |(Tf)^{(i)}(\eta) - \phi^{(i)}(\eta)| \leqslant N\beta_0 \right\} \\
&\leqslant \max\{N\beta_0, N\beta_0, N\beta_0, N\beta_0\} \\
&= N\beta_0
\end{aligned}$$

$$= |\mathcal{R}| \left[\tfrac{1}{24} \left(\tfrac{8}{3} R^2 (208\sqrt{13} - 727) + \tfrac{1}{3} R (416\sqrt{13} - 1301) + 9 \right) \right]$$
$$\leqslant R . \qquad \text{(by (8))}$$

Thus, we have, from the definition (4) of $\mathcal{B}_R$, that $Tf \in \mathcal{B}_R$ and thus we have shown that $T : \mathcal{B}_R \to \mathcal{B}_R$.

Contraction condition We now show that the mapping is a contraction. We thus want to show that $d(Tf, Tg) \leqslant k\, d(f, g)$ for $f, g \in \mathcal{B}_R$ and $0 < k < 1$. Let $f, g \in \mathcal{B}_R$ and $\eta \in [-1, 1]$. Then, by using the definition of H in Lemma 3, we have

$$\begin{aligned}
&|(Tf)(\eta) - (Tg)(\eta)| = \\
&\left| \int_{-1}^{1} G(\eta, s)\mathcal{R} \left([f'(s)f''(s) - f(s)f'''(s)] - [g'(s)g''(s) - g(s)g'''(s)] \right) \right| ds \\
&\leqslant \int_{-1}^{1} |G(\eta, s)| \\
&\quad \times |H(s, f(s), f'(s), f''(s), f'''(s)) - H(s, g(s), g'(s), g''(s), g'''(s))| \, ds \\
&\leqslant \int_{-1}^{1} |G(\eta, s)| \sum_{i=1}^{4} L_i |f^{(i)}(s) - g^{(i)}(s)| \, ds && \text{(by Lemma 4)} \\
&\leqslant \frac{\beta_i}{\beta_0} \sum_{i=1}^{4} L_i d(f, g) \int_{-1}^{1} |G(\eta, s)| \, ds \\
&\qquad \text{(using } |f^{(i)}(s) - g^{(i)}(s)| \leqslant \tfrac{\beta_i}{\beta_0} d(f, g) \text{, not proved here)} \\
&\leqslant \left(\frac{\beta_i}{\beta_0} d(f, g) \sum_{i=0}^{3} L_i \right) \beta_0 && \text{(using Lemma 2)} \\
&= (L_0\beta_0 + L_1\beta_1 + L_2\beta_2 + L_3\beta_3) d(f, g) \\
&= |\mathcal{R}| \left[\tfrac{1}{72} \left(16R \left(208\sqrt{13} - 727 \right) + 416\sqrt{13} - 1301 \right) \right] d(f, g) .
\end{aligned}$$

For convenience define

$$\gamma := |\mathcal{R}| \left[\tfrac{1}{72} \left(16R \left(208\sqrt{13} - 727 \right) + 416\sqrt{13} - 1301 \right) \right] .$$

By very similar methods we can show that, in general,

$$|(\mathsf{T}f)^{(i)}(\eta) - (\mathsf{T}g)^{(i)}(\eta)| \leqslant \beta_i \left(L_0 + L_1 \frac{\beta_1}{\beta_0} + L_2 \frac{\beta_2}{\beta_0} + L_3 \frac{\beta_3}{\beta_0} \right) d(f, g).$$

Since $W_i = \beta_0/\beta_i$, we then have

$$W_i |(\mathsf{T}f)^{(i)}(\eta) - (\mathsf{T}g)^{(i)}(\eta)| \leqslant \gamma d(f, g)\,.$$

Since the above inequalities hold for all $\eta \in [-1, 1]$, it is also the case that

$$W_i \max_{\eta\in[-1,1]} |(\mathsf{T}f)^{(i)}(\eta) - (\mathsf{T}g)^{(i)}(\eta)| \leqslant \gamma d(f, g)\,,$$

for all $i \in \{0, 1, 2, 3\}$.

We now show our desired result. Consider, for all $\mathsf{T}f, \mathsf{T}g \in \mathcal{B}_R$,

$$\begin{aligned} d(\mathsf{T}f, \mathsf{T}g) &= \max_{i\in\{0,1,2,3\}} \{W_i \max_{\eta\in[-1,1]} |f^{(i)}(s) - g^{(i)}(s)|\} \\ &\leqslant \gamma d(f, g)\,. \end{aligned}$$

The operator T is a contraction mapping when $\gamma < 1$, which is precisely the second inequality (9).

We have thus satisfied all the necessary conditions to conclude, by Banach's FPT, that there exists a unique $f^* \in \mathcal{B}_R$ such that $\mathsf{T}f^* = f^*$. This $f^*(\eta)$ solves our BVP (1) and (2). ♠

5 Conclusion

In this work, the unique solution for the problem of laminar flow through a porous wall channel with a pair of walls with different permeabilities is guaranteed to exist for sufficiently low Reynolds number. Its existence was proven using Banach's fixed point theorem. Future work could involve approximating this solution via Picard iteration.

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